MA2030: Linear Algebra and Numerical Analysis
Assignment-3

January – May 2012

In the following V, W, V_1, V_2, V_3 denote vector spaces over $\mathbb{F}$, which is $\mathbb{R}$ or $\mathbb{C}$.

1. State with reason whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ in each of the following is a linear transformation:

   (a) $T(x_1, x_2) = (1, x_2)$, (b) $T(x_1, x_2) = (x_1, x_2^2)$
   (c) $T(x_1, x_2) = (\sin(x_1), x_2)$ (d) $T(x_1, x_2) = (x_1, 2 + x_2)$

2. Check whether the functions T in the following are linear transformations:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + y, x + y^2)$.
   (b) $T : C^1[0, 1] \to \mathbb{R}$ defined by $T(u) = \int_0^1 [u(t)]^2 dt$.
   (c) $T : C^1[-1, 1] \to \mathbb{R}^2$ defined by $T(u) = (\int_{-1}^1 u(t)dt, u'(0))$.
   (d) $T : C^1[0, 1] \to \mathbb{R}$ defined by $T(u) = \int_0^1 u'(t)dt$.

3. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations. Show that the function $T : V_1 \to V_3$ defined by $Tx = T_2(T_1 x)$, $x \in V_1$, is a linear transformation.

   [T is called the *composition* of T_2 and T_1, and is usually denoted by T_2T_1.]

4. If $T_1 : C^1[0, 1] \to C[0, 1]$ is defined by $T_1(u) = u'$, and $T_2 : C[0, 1] \to \mathbb{R}$ is defined by $T_2(v) = \int_0^1 v(t)dt$, then find T_2T_1.

5. Let V_1, V_2, V_3 be finite dimensional vector spaces, and let E_1, E_2, E_3 be bases of V_1, V_2, V_3 respectively. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear transformations. Show that $[T_2T_1]_{E_1,E_3} = [T_2]_{E_2,E_3}[T_1]_{E_1,E_2}$.

6. If $T_1 : \mathcal{P}_n[0, 1] \to \mathcal{P}_n[0, 1]$ is defined by $T_1(u) = u'$, and $T_2 : \mathcal{P}_n[0, 1] \to \mathbb{R}$ is defined by $T_2(v) = \int_0^1 v(t)dt$, then find $[T_1]_{E_1,E_2}$, $[T_2]_{E_2,E_3}$, and $[T_2T_1]_{E_1,E_3}$, where $E_1 = E_2 = \{1, t, t^2, \ldots, t^n\}$ and $E_3 = \{1\}$.

7. Justify the statement: A linear transformation $T_1 : V_1 \to V_2$ is bijective iff there exists a linear transformation $T_2 : V_2 \to V_1$ such that $T_1T_2 : V_2 \to V_2$ is the identity transformation on V_2 and $T_2T_1 : V_1 \to V_1$ is the identity transformation on V_1.

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 5)$. Find $T(2, 3)$.

9. Does there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1, 0, 2) = (1, 1)$ and $T(1/2, 0, 1) = (0, 1)$?

10. Let V_1 and V_2 be vector spaces with $dimV_1 = n < \infty$. Let $\{u_1, \ldots, u_n\}$ be a basis of V_1 and $\{v_1, \ldots, v_n\} \subset V_2$. Let $T : V_1 \to V_2$ be the linear transformation with $T(u_j) = v_j$ for $j = 1, \ldots, n$. Show that

    (a) T is one-one if and only if $\{v_1, \ldots, v_n\}$ is linearly independent.

    (b) T is onto if and only if $\text{Span}(\{v_1, \ldots, v_n\}) = V_2$.

11. Show that if V_1 and V_2 are finite dimensional vector spaces of the same dimension, then the there exists a bijective linear transformation from V_1 to V_2.

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

    Find the matrix representation of T with respect to the basis given in each of the following.

    (a) $E_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $E_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

    (b) $E_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$, $E_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

    (c) $E_1 = \{(1, 1, -1), (-1, 1, 1), (1, -1, 1)\}$, $E_2 = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)$

13. Let $T : \mathcal{P}^3 \to \mathcal{P}^2$ be defined by $T(a_0 + a_1t + a_2t^2 + a_3t^3) = a_1 + 2a_2t + 3a_3t^2$. Find the matrix representation of T with respect to the basis given in each of the following.

    (a) $E_1 = \{1, t, t^2, t^3\}$, $E_2 = \{1 + t, 1 - t, t^2\}$

    (b) $E_1 = \{1, 1 + t, 1 + t + t^2, t^3\}$, $E_2 = \{1, 1 + t, 1 + t + t^2\}$

    (c) $E_1 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$, $E_2 = \{t^2, t, 1\}$

14. Let $T : \mathcal{P}^2 \to \mathcal{P}^3$ be defined by $T(a_0 + a_1t + a_2t^2) = (a_0t + \frac{a_1}{2}t^2 + \frac{a_2}{3}t^3)$. Find the matrix representation of T with respect to the basis given in each of the following.

    (a) $E_1 = \{1 + t, 1 - t, t^2\}$, $E_2 = \{1, t, t^2, t^3\}$,

    (b) $E_1 = \{1, 1 + t, 1 + t + t^2\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, t^3\}$,

    (c) $E_1 = \{t^2, t, 1\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$,

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