MA2030: Linear Algebra and Numerical Analysis
Assignment-4

January – May 2012

In the following V, W, V_1, V_2 denote finite dimensional vector spaces over $\mathbb{F}$, which is $\mathbb{R}$ or $\mathbb{C}$.

1. Let $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively.
   Let $F_1 = \{f_1, \ldots, f_n\}$ be the dual basis of $\mathcal{L}(V_1, \mathbb{F})$ with respect to E_1.
   For $i = 1, \ldots, n$; $j = 1, \ldots, m$, let $T_{ij} : V_1 \to V_2$ be defined by $T_{ij}(x) = f_j(x)v_i$ for $x \in V_1$.
   Show that $\{T_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$ is a basis of $\mathcal{L}(V_1, V_2)$.

2. Let $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively. Show the following:

   (a) If $T \in \mathcal{L}(V_1, V_2)$, then T is one-one iff columns of $[T]_{E_1,E_2}$ are linearly independent.

   (b) If $T \in \mathcal{L}(V_1, V_2)$, then T is not one-one iff $det[T]_{E_1,E_2} = 0$.

   (c) If $\{g_1, \ldots, g_m\}$ is the ordered dual basis of $\mathcal{L}(V_2, \mathbb{F})$ with respect to the basis E_2 of V_2, then for every $T \in \mathcal{L}(V_1, V_2)$, $[T]_{E_1,E_2} = (g_i(Tu_j))$.

   (d) If $T_1, T_2, T \in \mathcal{L}(V_1, V_2)$ and $\alpha \in \mathbb{F}$, then

   $$[T_1 + T_2]_{E_1,E_2} = [T_1]_{E_1,E_2} + [T_2]_{E_1,E_2} \quad \text{and} \quad [\alpha T]_{E_1,E_2} = \alpha [T]_{E_1,E_2}.$$

   (e) Suppose $\{M_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$ is a basis of $\mathbb{F}^{m \times n}$. Let $T_{ij} \in \mathcal{L}(V_1, V_2)$ be the linear transformation with $[T_{ij}]_{E_1,E_2} = M_{ij}$. Then

   $$\{T_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\} \text{ is a basis of } \mathcal{L}(V_1, V_2).$$

3. Let V_1 and V_2 be finite dimensional vector spaces and $T : V_1 \to V_2$ be a linear transformation. Give reasons for the following:

   (a) $\text{rank}(T) \le \dim V_1$.

   (b) T is onto implies $\dim V_2 \le \dim V_1$.

   (c) T is one-one implies $\dim V_1 \le \dim V_2$.

   (d) T is one-one if and only T is onto.

   (e) $\dim(V_1) > \dim(V_2)$ implies T is not one-one.

   (f) $\dim(V_1) < \dim(V_2)$ implies T is not onto.

4. Let V be the vector space of real valued functions on $\mathbb{R}$ which have derivatives of all orders. Let $T : V \to V$ be the differential operator: $Tx = x'$. What is $N(T)$?

5. Let $T : V \to V$ be a linear operator such that $T^2 = T$. Let I denote the identity operator. Prove that $R(T) = N(I - T)$ and $N(T) = R(I - T)$.

6. Find bases for the null space $N(T)$ and the range space $R(T)$ of the linear transformation T in each of the following:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 - x_2, 2x_2)$,

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 + x_2, 0, 2x_3 - x_2)$,

   (c) $T : \mathbb{R}^{n \times n} \to \mathbb{R}$ defined by $T(A) = tr(A)$.
   (Recall: $tr(A)$, the trace of a square matrix A, is the sum of its diagonal elements.)

7. Let $T : V_1 \to V_2$ be a linear transformation, where $\dim(V_1) < \infty, \dim(V_2) < \infty$. Prove the following:

   (a) There exists a subspace V_0 of V_1 such that $V_1 = N(T) + V_0$ and $V_0 \cap N(T) = \{0\}$.

   (b) If V_0 is as in (a) and $\{v_1, \ldots, v_k\}$ is a basis of V_0, then $R(T) = \text{span}\{Tv_1, \ldots, Tv_k\}$.

8. Let $T : V \to V$ be a linear transformation. Prove the following:

   (a) If T is a bijection and $0 \neq \lambda \in \mathbb{F}$, then λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.

   (b) If λ is an eigenvalue of T then λ^k is an eigenvalue of T^k.

   (c) If λ is an eigenvalue of T and $\alpha \in \mathbb{F}$, then $\lambda + \alpha$ is an eigenvalue of $T + \alpha I$.

   (d) If $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ for some $a_0, a_1, \ldots, a_k$ in $\mathbb{F}$, and if λ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T) := a_0 I + a_1 T + \ldots + a_k T^k$.

9. Let T_1 and T_2 be linear operators on V, λ is an eigenvalue of T_1 and μ is an eigenvalue of T_2. Is it necessary that $\lambda\mu$ an eigenvalue of T_1T_2? Why? What is wrong with the following statement?

   $\lambda\mu$ an eigenvalue of T_1T_2 because, if $T_1 x = \lambda x$ and $T_2 x = \mu x$, then
   $T_1T_2 x = T_1 \mu x = \mu T_1 x = \mu\lambda x$.

10. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.

11. Let V be finite dimensional, $T \in \mathcal{L}(V)$ and E be a basis of V. Suppose $[T]_{E,E}$ is a an upper triangular, or a lower triangular, or a diagonal matrix. Prove that the diagonal entries of $[T]_{E,E}$ are the eigenvalues of T.

12. Let V be a finite dimensional vector space with $\dim(V) = n$ and $T \in \mathcal{L}(V)$. Prove that T is a diagonalizable if and only if T has a basis consisting of eigenvectors.

13. Which of the following linear transformation T is diagonalizable? If it is diagonalizable, find the basis E and the matrix $[T]_{E,E}$.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ x_1 + x_2 - x_3,\ x_1 - x_2 + x_3)$.

   (b) $T : \mathcal{P}_3 \to \mathcal{P}_3$ such that $T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Te_1 = 0, \quad Te_2 = e_1, \quad Te_3 = e_2$.

   (d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Te_1 = e_2, \quad Te_2 = e_3, \quad Te_3 = 0$.

   (e) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $Te_1 = e_3, \quad Te_2 = e_2, \quad Te_3 = e_1$.

14. Show that the linear transformation $T : \underline{\mathbb{R}}^3 \to \underline{\mathbb{R}}^3$ corresponding to each of the following matrix is diagonalizable. Also find a basis of eigenvectors of T for $\underline{\mathbb{R}}^3$.

$$(a) \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 1/2 & -1/2 & 1 \end{bmatrix} \qquad (b) \begin{bmatrix} 3 & -1/2 & -3/2 \\ 1 & 3/2 & 3/2 \\ -1 & -1/2 & 5/2 \end{bmatrix}$$

15. Check whether the linear transformation $T : \underline{\mathbb{R}}^3 \to \underline{\mathbb{R}}^3$ corresponding to each of the following matrix is diagonalizable. If diagonalizable, find a basis of eigenvectors for the space $\underline{\mathbb{R}}^3$:

$$(a) \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \qquad (b) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (c) \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$