MA2030: Linear Algebra and Numerical Analysis
Assignment-5

January – May 2012

1. Recall the notation:

   $\mathcal{P}_n([0,1],\mathbb{R})$ is the space of all polynomial functions from $[0,1]$ to $\mathbb{R}$.

   $\mathcal{R}([0,1],\mathbb{R})$ is the space of all Riemann integrable functions from $[0,1]$ to $\mathbb{R}$.

   $C^1([0,1],\mathbb{R})$ is the space of all continuously differentiable functions from $[0,1]$ to $\mathbb{R}$.

   Check whether each of the following is an inner product on the given vector space V:

   (a) $\langle x,y\rangle = x_1y_1$ for $x=(x_1,x_2)$, $y=(y_1,y_2)$ in $V=\mathbb{R}^2$.

   (b) $\langle x,y\rangle = x_1y_1$ for $x=(x_1,x_2)$, $y=(y_1,y_2)$ in $V=\mathbb{C}^2$.

   (c) $\langle x,y\rangle = \sum_{j=0}^{n} x(t_j)y(t_j)$ for x,y in $V=\mathcal{P}_n([0,1],\mathbb{R})$, where $t_1,\ldots,t_n$ are distinct points in $[0,1]$.

   (d) $\langle x,y\rangle = \int_0^1 x(t)y(t)\,dt$ for x,y in $V=\mathcal{R}([0,1],\mathbb{R})$.

   (e) $\langle A,B\rangle = \text{trace}(A+B)$ for A,B in $V=\mathbb{R}^{2\times 2}$.

   (f) $\langle A,B\rangle = \text{trace}(A^TB)$ for A,B in $V=\mathbb{R}^{3\times 3}$.

   (g) $\langle x,y\rangle = \int_0^1 x'(t)y'(t)\,dt$ for x,y in $V=C^1([0,1],\mathbb{R})$.

   (h) $\langle x,y\rangle = x(0)y(0)+\int_0^1 x'(t)y'(t)\,dt$ for x,y in $V=C^1([0,1],\mathbb{R})$.

2. Let V_1 and V_2 be inner product spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. On $V=V_1\times V_2$, define

   $$\langle (x_1,x_2),(y_1,y_2)\rangle_V := \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2 \quad \text{for all } (x_1,x_2),(y_1,y_2)\in V.$$

   Show that $\langle\cdot,\cdot\rangle_V$ is an inner product on V.

3. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be inner products on a vector space V. Show that

   $$\langle x,y\rangle := \langle x,y\rangle_1 + \langle x,y\rangle_2 \quad \text{for all } x,y\in V$$

   defines another inner product on V.

4. Let $T: V\to\mathbb{F}^n$ be a linear isomorphism (bijective linear transformation). Show that

   $$\langle x,y\rangle_T := \langle Tx,Ty\rangle_{\mathbb{F}^n} \quad \text{for all } x,y\in V,$$

   defines an inner product on V. Here, $\langle\cdot,\cdot\rangle_{\mathbb{F}^n}$ is the standard inner product on $\mathbb{F}^n$.

5. Let V be an inner product space over $\mathbb{C}$. Prove that for all $x,y\in V$, $\text{Re}\langle ix,y\rangle = -\text{Im}\langle x,y\rangle$.

6. Let V_1 and V_2 be inner product spaces. Let $T: V_1\to V_2$ be a linear transformation. Prove that for all $(x,y)\in V_1\times V_2$,

   $$\langle Tx,Ty\rangle = \langle x,y\rangle \text{ if and only if } \|Tx\| = \|x\|.$$

   [Notice that both the inner products are denoted by $\langle\cdot,\cdot\rangle$.
   Hint: For the only if part, use $\langle T(x+y),T(x+y)\rangle = \langle x+y,x+y\rangle$ and Problem 5.]

7. (a) If the scalar field is $\mathbb{R}$, then show that the converse of the Pythagoras theorem holds, that is, if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, then $x \perp y$.

   (b) If the scalar field is $\mathbb{C}$, then show that the converse of Pythagoras theorem need not be true. [Hint: Take $V = \mathbb{C}$ with standard inner product and $x = \alpha$, $y = i\,\beta$ for nonzero real numbers $\alpha, \beta \in \mathbb{R}$.]

8. Prove that every orthonormal set in an inner product space is linearly independent.

9. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of an inner product space V. Prove the following:

   (a) For every $x \in V$,
$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \qquad \|x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2.$$

   (b) For every $x, y \in V$,
$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, u_i \rangle \langle u_i, y \rangle.$$

10. Let V be an inner product space over $\mathbb{F}$ with $\dim(V) = n$. Consider the inner product space $\mathbb{F}^n$ with the standard inner product. Prove that there exists a linear isometry from V onto $\mathbb{F}^n$, i.e., a surjective linear operator $T : V \to \mathbb{F}^n$ with $\|T(x)\| = \|x\|$, for all $x \in V$.

    [Notice that both the norms are denoted by $\|\cdot\|$.]

11. For x, y in an inner product space V, show that $x + y \perp x - y$ if and only if $\|x\| = \|y\|$.

12. Let S be a nonempty subset of an inner product space V. Show that

    (a) $S^\perp$ is a subspace of V.

    (b) $V^\perp = \{0\}$, $\{0\}^\perp = V$.

    (c) $S \subset S^{\perp\perp}$.

    (d) If S is a basis of V, then $S^\perp = \{0\}$.

    (e) If V is finite dimensional and V_0 is a subspace of V, then $V_0^{\perp\perp} = V_0$.

13. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of an inner product space V. Define the linear functionals $f_1, \ldots, f_n$ by $f_j(x) = \langle x, u_j \rangle$, for $x \in V$. Show that every linear functional $f : V \to \mathbb{F}$ can be written as
$$f = \sum_{j=1}^{n} f(u_j) f_j.$$

14. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of an inner product space V. Show that for every linear functional $f : V \to \mathbb{F}$, there exists a unique $y \in V$ such that
$$f(x) = \langle x, y \rangle \qquad \forall\, x \in V.$$

    [Hint: Use Problem 13]