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# MA-5340: Measure and Integration

## Assignment Sheet - I

1. Let $(a_n)$ and $(b_n)$ be sequences in $[-\infty, \infty]$. Prove the following:
   (i) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty}(a_n)$
   (ii) $a_n \le b_n \,\forall\, n \in \mathbb{N} \implies \liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Corresponding to a partition $P$ of $[a, b]$ let $L(P, f)$ and $U(P, f)$ denote the lower sum and upper sum. Show that for any two partitions $P$ and $Q$ of $[a, b]$, $L(P, f) \le U(Q, f)$.

3. Prove the following:
   (a) Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.
   (b) Every bounded function $f : [a, b] \to \mathbb{R}$ having atmost a finite number of discontinuities is Riemann integrable.
   (c) Every monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

4. Show the following:
   (a) If $(f_n)$ is a sequence of Riemann integrable functions on $[a, b]$ and if $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in [a, b]$, then it is *not necessary* that $f$ is Riemann integrable.
   (b) Even if $f$ in (a) is Riemann integrable , it is *not necessary* to have the convergence $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.

5. Show that if $(I_n)$ is a sequence of open intervals, then $m^*(\cup I_n) \le \sum_n \ell(I_n)$.

6. Show that every non-degenerate interval is an uncountable set.

7. There exist disjoint subsets $A_1$ and $A_2$ of $\mathbb{R}$ such that $m^*(A_1 \cup A_2) \ne m^*(A_1) + m^*(A_2)$ – Why?

8. Show that, for any any denumerable disjoint family $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}$, the equality $m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} m^*(A_n)$ need not hold.

9. Show that there exists $E \subseteq \mathbb{R}$ such that $E \notin \mathfrak{M}$.

10. Assuming that for any $a \in \mathbb{R}$, $(a, \infty) \in \mathfrak{M}$, show that every $G_\delta$ subset of $\mathbb{R}$ and every $F_\sigma$ subset of $\mathbb{R}$ belongs to $\mathfrak{M}$.

11. Prove that, in the definition of $m^*(E)$, $\mathcal{I}_E$ can be taken be the collection of all

    (a) countable family $\{I_n\}$ of intervals of the form $J_n = (a_n, b_n)$,
    (b) countable family $\{I_n\}$ of intervals of the form $J_n = [a_n, b_n)$,
    (c) countable family $\{I_n\}$ of intervals of the form $J_n = (a_n, b_n]$,
    (d) countable family $\{I_n\}$ of intervals of the form $J_n = [a_n, b_n]$,
    (e) sequences $(I_n)$ of intervals of the form $J_n = (a_n, b_n)$,
    (f) sequences $(I_n)$ of intervals of the form $J_n = [a_n, b_n)$,
    (g) sequences $(I_n)$ of intervals of the form $J_n = (a_n, b_n]$,
    (h) sequences $(I_n)$ of intervals of the form $J_n = [a_n, b_n]$,

    where with $a_n, b_n \in \mathbb{R}$.

12. Show that, if $E \subseteq A$ and $m^*(E) = 0$, then $m^*(A \cup E) = m^*(A)$.

13. From countable subadditivity of $m^*$, deduce that outer measure of every countable set is 0.

14. If $E$ is a subset of an interval $I$ such that $m^*(E) = 0$, then prove that $E^c$ is dense in $I$.

15. Suppose $\{A_1, \ldots, A_n\}$ is a disjoint family of subsets of $\mathbb{R}$ such that $A_1, \ldots, A_{n-1}$ belong to $\mathfrak{M}$. Then show that $m^*\Big(\bigcup_{i=1}^n A_i\Big) = \sum_{i=1}^n m^*(A_i)$.

16. Show that if $E \in \mathfrak{M}$, then show that $m(E) = \inf\{m(G) : G \text{ open } E \subseteq G\}$.

17. If $A, B \in \mathfrak{M}$ with $A \subseteq B$ and if $m(B) < \infty$, then show that $m(B \setminus A) = m(B) - m(A)$.

18. If $A, B \in \mathfrak{M}$, prove that $m(A \cup B) + m(A \cap B) = m(A) + m(B)$.

19. Let $E \subseteq \mathbb{R}$ be such that $m^*(E) < \infty$. Prove that the following are equivalent:

    (a) $E \in \mathfrak{M}$.
    (b) For every $\epsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m(G \setminus E) < \epsilon$.
    (c) For every $\epsilon > 0$, there exists a closed set $F \subseteq \mathbb{R}$ such that $F \subseteq E$ and $m(E \setminus F) < \epsilon$.
    (d) There exists a $G_\delta$-set $G \supseteq E$ such that $m(G \setminus E) = 0$.
    (e) There exists an $F_\sigma$-set $F \subseteq E$ such that $m(E \setminus F) = 0$.

20. If $E \subseteq \mathbb{R}$ such that with $m^*(E) < \infty$. Prove that the following are equivalent:

    (a) $E \in \mathfrak{M}$.
    (b) There exists a $G_\delta$ set $G \supseteq E$ such that $E = G \setminus E_0$, where $m^*(E_0) = 0$.
    (c) There exists an $F_\sigma$ set $F \subseteq E$ such that $E = F \cup E_0$, where $m^*(E_0) = 0$.