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MA-5340: Measure and Integration

Assignment Sheet - II

In the following, $(X, \mathcal{A})$ is a measurable space.

1. Suppose $(X, \mathcal{A})$ is a measurable space and $X_0 \in \mathcal{A}$. Then show that

   (i) $\mathcal{A}_0 := \{E \subseteq X_0 : E \in \mathcal{A}\}$ is a σ-algebra on X_0, and

   (ii) $\mathcal{A}_0 = \{A \cap X_0 : A \in \mathcal{A}\}$.

2. Suppose $\{\mathcal{A}_\alpha : \alpha \in \Lambda\}$ is a family of σ-algebras on a set X. Prove that $\bigcap_{\alpha\in\Lambda} \mathcal{A}_\alpha$ is also a σ-algebra.

3. Every Borel subset of $\mathbb{R}$ is Lebesgue measurable - Why?

4. Let $(X, \mathcal{A}, \mu)$ is a measure space. Prove that, if α is a non-negative real number, then $E \mapsto \alpha\mu(E)$, $E \in \mathcal{A}$ is a also measure on $(X, \mathcal{A})$.

   More, generally, if $\mu_1, \ldots, \mu_k$ are measures on a measurable space $(X, \mathcal{A})$ and if $\alpha_1, \ldots, \alpha_k$ are non-negative real numbers, then prove that $E \mapsto \sum_{i=1}^k \mu_i(E)$ is a measure on $(X, \mathcal{A})$.

5. Let X be a set. A function $\mu^* : 2^X \to [0, \infty]$ is called an **outer measure** on X if

   (i) $\mu^*(\varnothing) = 0$,

   (ii) $A \subseteq B \implies \mu^*(A) \le \mu^*(B)$, and

   (iii) for every countable family $\{A_n\}$ in 2^X, $\mu^*\Big(\bigcup_n A_n\Big) \le \sum_n \mu^*(A_n)$.

   Prove that $\mathcal{A} := \{E \subseteq X : \forall\, A \subseteq X,\ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$ is a σ-algebra on X, and $\mu := \mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$.

6. Lebesgue measure on $\mathbb{R}$ is complete - Why?

7. Prove that $\mathfrak{M}$ is the completion of the Borel σ-algebra $\mathfrak{B}_1$.

8. Let X be an uncountable set and $\mathcal{A} \subseteq 2^X$ such that $A \in \mathcal{A}$ if and only if either A or A^c is atmost countable. Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Show that $\mathcal{A}$ is a σ-algebra and μ is a measure.

9. Prove that for $E \subseteq X$, $E \in \mathcal{A}$ if and only if χ_E is a measurable function.

10. Let X be a measurable space, and Y and Z be topological spaces. If $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous, then prove that $g \circ f : X \to Z$ is measurable.

11. Let X be a measurable space and Y be a set and $f : X \to Y$. Prove:

(i) $\mathcal{S} := \{E \subseteq Y : f^{-1}(E) \in \mathcal{A}\}$ is a σ-algebra on Y.

(i) If Y is a topological space and f is measurable, then $\mathcal{S}$ contains the Borel σ-algebra on Y.

12. Let f be a real valued function on a measurable space X. Prove that f is Borel (resp. Lebesgue) measurable if and only if $\{x \in X : f(x) > a\}$ is a Borel (resp. Lebesgue) measurable set for every $a \in \mathbb{R}$.

13. Prove that if f is a real function on $(X, \mathcal{A})$ such that $\{x : f(x) \geq r\}$ is measurable for every rational number r, then f is measurable.

14. Let X be a measurable space and Y be a topological space. Prove that $f : X \to Y$ is measurable if and only if for every Borel set A in Y, $f^{-1}(A)$ is measurable in X.

15. Prove that if f and g are measurable on $(X, \mathcal{A})$, then the sets $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

16. Prove that if (f_n) is a sequence of measurable functions on $(X, \mathcal{A})$, then the set $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

17. Let $(X, \mathcal{A})$ be a measurable space and $f : X \to [-\infty, \infty]$. Prove:

(i) If f is measurable, then f^+ and f^- are measurable.

(ii) In case f is real valued, then f is measurable iff f^+ and f^- are measurable, and in that case $|f|$ is also measurable.

18. Let $(X, \mathcal{A}, \mu)$ be as in Problem 8. Describe measurable functions on $(X, \mathcal{A})$ and their integrals w.r.t. μ.

19. Let $(X, \mathcal{A}, \mu)$ be a measurable space and $f_n : X \to [-\infty, \infty]$ be measurable for each $n \in \mathbb{N}$. Then prove that the functions $\sup_n f_n$, $\inf_n f_n$, $\limsup_n f_n$, $\liminf_n f_n$ are measurable functions.

20. Let $(X, \mathcal{A}, \mu)$ be a measurable space and (f_n) be a sequence of real valued measurable functions on X. Prove that if $\sum_{n=1}^{\infty} f_n(x)$ converges for each $x \in X$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$, $x \in X$, then f is measurable.