M.T.Nair (IIT Madras) March 12, 2013

MA-5340: Measure and Integration

Assignment Sheet - III

In the following, $(X, \mathcal{A}, \mu)$ is a measure space.

1. Suppose φ is a non-negative simple measurable function and E is a measurable set. Let μ_E be the restriction of μ to the restricted σ-algebra $\mathcal{A}_E := \{A \subseteq E : A \in \mathcal{A}\}$. Show that $\int_E \varphi d\mu = \int_E \varphi d\mu_E$.

2. Let φ and ψ be non-negative simple measurable functions such that $\varphi \leq \psi$. Prove that $\int_X (\psi - \varphi) d\mu = \int_X \psi\, d\mu - \int_X \varphi\, d\mu$.

3. Prove that if $f \geq 0$ is a measurable function and $E \in \mathcal{A}$ is such that $\mu(E) = 0$, then $\int_X f = \int_{E^c} f$.

4. Suppose $X = \{x_1, x_2, \ldots\}$ with the counting measure μ. If f is an extended real valued non-negative measurable function on X, then show that
$$\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i).$$

5. Suppose $X = \{x_1, \ldots, x_n\}$, and $w_1, \ldots, w_n$ are non-negative reals. Let $w(x_i) = w_i$ for $i = 1, \ldots n$ and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,
$$\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i.$$

6. Suppose $X = \{x_1, x_2, \ldots\}$, and $w_1, w_2, \ldots$ are non-negative reals. Let $w(x_i) = w_i$ for $i \in \mathbb{N}$ and $\mu(E) = \sum_{x \in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X,
$$\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i.$$

7. Suppose $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Then show that
$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

8. Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on $(X, \mathcal{A}, \mu)$ such that $f_1 \geq f_2 \geq \ldots$ and $f_n(x) \to f(x)$ for every $x \in X$. If $\int_X f d\mu < \infty$, then show that $\int_X f_n d\mu \to \int_X f d\mu$. Show that the condition that $\int_X f d\mu < \infty$ cannot be dropped.

9. Show that the condition that $\int_X f d\mu < \infty$ in Fathou's lemma cannot be dropped.

10. If $f \in \mathcal{L}(\mu)$ such that $\int_X f \geq 0$, then show that

$$\int_X f = \int_X \mathrm{Re} f \leq \int_X |f|.$$

11. Show that $\mathcal{L}(\mu)$ is a vector space over $\mathbb{C}$, and the map $f \mapsto \int_X f$ is a linear functional on $\mathcal{L}(\mu)$.

12. Show that the map $f \mapsto \int_X |f|$ is a semi-norm on the vector space $\mathcal{L}(\mu)$.

13. Show that the set $\mathcal{N} := \{f \in \mathcal{L}(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}(\mu)$, and the map $[f] \mapsto \int_X |f|$ is a norm on the quotient space $\mathcal{L}(\mu)/\mathcal{N}$.

14. If $f \in \mathcal{L}(\mu)$ such that $\left|\int_X f\right| = \int_X |f|$, then show that there exists $c \in \mathbb{C}$ such that $f(x) = c|f(x)|$ for almost all $x \in X$.

15. Suppose f and g are complex measurable functions such that $f = 0$ a.e. on X and $f = 0$ a.e. on X. Show that $f + g = 0$ a.e. on X.

16. Suppose $f \in \mathcal{L}(\mu)$ such that $\int_E f = 0$ for all $E \in \mathcal{A}$. Show that $f = 0$ a.e.

    Hint: First observe that it is enough to prove for the case of real valued f, and then take $E = \{x \in X : f(x) \geq 0\}$ and show that $\int_X f^+ = 0$. Similarly show that $\int_X f^- = 0$.