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MA-5340: Measure and Integration

Assignment Sheet - IV

In the following, $(X, \mathcal{A}, \mu)$ is a measure space.

1. Suppose E is a Lebesgue measurable subset of $\mathbb{R}^1$. Show that there exists a Borel subset A of $\mathbb{R}^1$ such that $\chi_{_A} = \chi_{_E}$ a.e.

2. Suppose f is a real valued Lebesgue measurable function on $\mathbb{R}^1$. Show that there exists a Borel measurable function g on $\mathbb{R}^1$ such that $f = g$ a.e.

3. For complex valued measurable functions f on a measure space $(X, \mathcal{A}, \mu)$ and for $1 \le p \le \infty$, let

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p d\mu\right)^{1/p} & \text{if } 1 \le p < \infty \\ \inf\{c > 0 : |f| \le c \quad \text{a.e.}\} & \text{if } p = \infty. \end{cases}$$

   Let $\mathcal{L}^p(\mu)$ be the set of all complex valued measurable functions f on $(X, \mathcal{A}, \mu)$ such that $\|f\|_p < \infty$. Show that

   (i) $\mathcal{L}^p(\mu)$ is a vector space,

   (ii) $f \mapsto \|f\|_p$ is a semi-norm on $\mathcal{L}^p(\mu)$.

   (iii) $\mathcal{N} := \{f \in \mathcal{L}^p(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}^p(\mu)$,

   (iv) $[f] \mapsto \int_X |f|$ is a norm on the quotient space $L^p(\mu) := \mathcal{L}(\mu)/\mathcal{N}$.

   (v) $L^p(\mu)$ is a Banach space w.r.t. $\|\cdot\|_p$.

4. Realize the spaces $\mathcal{L}^p(\mu)$ and $L^p(\mu)$ in the following cases:

   (a) $X = \mathbb{N}$, $X = \mathbb{Z}$ with counting measure on 2^X.

   (b) $X = \{1, \ldots, k\}$ with counting measure on 2^X.

   (c) $X = [0, 1]$ with Lebesgue measure.

5. If $\mu(X) < \infty$, then show that for $1 \le p \le r \le \infty$, $L^\infty(\mu) \subseteq L^r(\mu) \subseteq L^p(\mu) \subseteq L^1(\mu)$, and if $X = \mathbb{N}$ or $\mathbb{Z}$, then $L^\infty(\mu) \supseteq L^r(\mu) \supseteq L^p(\mu) \supseteq L^1(\mu)$.

6. Show that every Cauchy sequence in $L^p(\mu)$ for $1 \le p < \infty$ has a subsequence which converges a.e. to a function in $L^p(\mu)$.

7. Let $\mathcal{S}$ be the set of all step functions on $\mathbb{R}^1$. Show that $\mathcal{S}$ is dense in $L^p(\mathbb{R}^1)$ for $1 \le p < \infty$.

8. Let $\mathcal{S}_{a,b}$ be the set of all step functions on $[a, b]$. Show that $\mathcal{S}_{a,b}$ is dense in $L^p[a, b]$ for $1 \leq p < \infty$.

9. Prove that $C[a, b]$ is dense in $L^p[a, b]$ (with respect to $\|\cdot\|_p$).

10. Suppose $f : [a, b] \to \mathbb{R}$ is absolutely continuous. Prove that

    (a) f is continuous,

    (b) f is of bounded variation.

11. Give an example of a function which is

    (a) continuous, but not absolutely continuous,

    (b) of bounded variation, but not absolutely continuous,

12. Let $f \in \mathcal{L}^1[a, b]$ and $g : [a, b] \to \mathbb{F}$ be defined by $g(x) = \int_a^x f dm$, $x \in [a, b]$. Prove that g is absolutely continuous,

13. Quoting relevant results, prove: A function $g : [a, b] \to \mathbb{F}$ is absolutely continuous if and only if there exists an integrable function $f : [a, b] \to \mathbb{F}$ such that $g(x) - g(a) = \int_a^x f\, dm$ $x \in [a, b]$, and in that case $g' = f$ a.e.

    In the following: $(X_1, \mathcal{A}_1, \mu_1)$ **and** $(X_2, \mathcal{A}_2, \mu_2)$ **measure spaces.**

14. Prove that for every $(x, y) \in X_1 \times X_2$, $E_x \in \mathcal{A}_2$ and $E^y \in \mathcal{A}_1$.

15. Let $\mathcal{S}$ be the class of all $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$ such that the functions $x \mapsto \mu_2(E_x)$ and $y \mapsto \mu_1(E^y)$ are measurable with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and

$$\int_{X_1} \mu_2(E_x) d\mu_1 = \int_{X_2} \mu_1(E^y) d\mu_2.$$

    Prove:

    (a) $\mathcal{S}$ contains all elementary sets, and

    (b) $\mathcal{S}$ is closed under finite disjoint unions of its members.

    (c) If μ_1 and μ_2 are finite measures, then $\mathcal{S}$ is a monotone class.

16. Let μ_1 and μ_2 be finite measures. prove:

    (a) For every $E \in \mathcal{A}_1 \otimes \mathcal{A}_2$, the functions $x \mapsto \mu_2(E_x)$ and $y \mapsto \mu_1(E^y)$ are measurable with respect to $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and

$$\int_{X_1} \mu_2(E_x) d\mu_1 = \int_{X_2} \mu_1(E^y) d\mu_2.$$

(b) $(\mu_1 \times \mu_2)(E) := \int_{X_1} \mu_2(E_x) d\mu_1$ defines a measure on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

17. Show that if μ_1 and μ_2 are complete measures, then $\mu_1 \times \mu_2$ need not be complete.

18. Is $m \times m$ on $(\mathbb{R} \times \mathbb{R}, \mathfrak{M} \otimes \mathfrak{M})$ complete? Why?

19. Is $m \times m$ on $(\mathbb{R} \times \mathbb{R}, \mathfrak{M} \otimes \mathfrak{M})$ same as the Lebesgue measure m_2 on $\mathbb{R}^2$?

20. Let $X_1 = X_2 = \mathbb{R}$ and $\mathcal{A}_1 = \mathcal{A}_2 = \mathfrak{M}$ and $\mu_1 = \mu_2 = m$, the Lebesgue measure. Corresponding to a measurable function $f : \mathbb{R} \to [0, \infty)$, let

$$E_f := \{(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Prove the following[1]:

(a) E_f is measurable with respect to $\mathfrak{M} \otimes \mathfrak{M}$.

(b) $(m \times m)(E_f) = \int_{\mathbb{R} \times \mathbb{R}} f d(m \times m)$.

(c) Graph of f is measurable and has zero measure.

[1] See de Barra, Page 184.