Chapter 10

Elliptic Equations

10.1 Introduction

The mathematical modeling of steady state or equilibrium phenomena generally result in to elliptic equations. The best example is the steady diffusion of heat in any two-domain Ω bounded by $\partial\Omega$. In the absence of any sources, the governing equation is the Laplace's equation given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega \tag{10.1}$$

Due to the absence of time derivative terms in the equation (10.1), unlike the problems given in the earlier two chapters, these are pure boundary value problems. Therefore, boundary conditions alone (no initial conditions) have to be prescribed over the entire boundary $\partial\Omega$.

Depending on the nature of these boundary conditions, forced, natural or mixed type, the elliptic problems are classified as

1. **Dirichelt problem :** The differential equation along with fixed (forced) boundary conditions on the boundary, that is, $u = f(x, y)$ over $\partial\Omega$.

2. **Neumann problem :** The differential equation and derivative boundary conditions given by $\frac{\partial u}{\partial x_n} = f(x, y)$ over $\partial\Omega$, where x_n is the normal to $\partial\Omega$.

3. **Robin or Mixed problem :** The differential equation along with a combination of forced and natural boundary conditions given by $\alpha u + \beta\frac{\partial u}{\partial x_n} = f(x, y)$ over $\partial\Omega$, where α, β are constants.

10.2 Separation of Variables

Decomposing $u(x,y) = X(x)Y(y)$ result in to ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \tag{10.2}$$
$$\frac{d^2Y}{dy^2} + kY = 0 \tag{10.3}$$

where k is a constant. The solution u can be written as

Case i: k is positive, that is $k = \lambda^2$.

$$u(x,y) = (c_1e^{\lambda x} + c_2e^{-\lambda x}) \times (c_3 \cos \lambda y + c_4 \sin \lambda y) \tag{10.4}$$

Case ii: k is negative, that is $k = -\lambda^2$.

$$u(x,y) = (c_1 \cos \lambda x + c_2 \sin \lambda x) \times (c_3e^{\lambda y} + c_4e^{-\lambda y}) \tag{10.5}$$

Case iii: $k = 0$

$$u(x,y) = (c_1x + c_2) \times (c_3y + c_4) \tag{10.6}$$

10.3 Dirichlet Problem in a Rectangular Domain

Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega$ where $\Omega = (0,a) \times (0,b)$ with boundary conditions $u(x,0) = u(x,b) = 0, u(0,y) = 0$ and $u(a,y) = f(y)$.

Solution : Due to the homogenous nature of boundary conditions, in the y direction, that is, at $y = 0$ and $y = b$, non-trivial solution exists only for the case (prove that for the other two cases, the solution is identically zero)

$$u(x,y) = (c_1e^{\lambda x} + c_2e^{-\lambda x}) \times (c_3 \cos \lambda y + c_4 \sin \lambda y)$$

Now, applying the boundary conditions at $y = 0$ and $y = b$ on $(c_3 \cos \lambda y + c_4 \sin \lambda y)$ gives $c_3 = 0$ and $\sin \lambda b = 0$.

Therefore $\lambda_n = \frac{n\pi}{b}$, $n = 1, 2, \cdots$. (the other n are omitted because, $n = 0$ gives trivial solution and negative n only repeats the existing eigenfunctions with a minus sign).

Similarly, using the zero boundary condition at $x = 0$ on $(c_1e^{\lambda x} + c_2e^{-\lambda x})$ gives $c_1 = -c_2$. Now, using the superposition principle, the solution can be written as

$$u(x,y) = \sum_{n=1}^{\infty} A'_n(e^{\lambda_n x} - e^{-\lambda_n x}) \sin \lambda_n y = \sum_{n=1}^{\infty} A_n \sinh \lambda_n x \sin \lambda_n y$$

where $A_n = 2A'_n$.

Finally, the values of A_n can be computed using the non-zero boundary condition at $x = a$ in the following way:

At $x = a$ we have

$$u(a, y) = \sum_{n=1}^{\infty} A_n \sinh \lambda_n a \sin \lambda_n y = f(y)$$

The above is a Fourier sine series, therefore,

$$\begin{aligned} A_n \sinh \lambda_n a &= \frac{2}{b} \int_0^b f(y) \sin \lambda_n y \, dy \quad \text{or} \\ A_n &= \frac{2}{b \sinh \lambda_n a} \int_0^b f(y) \sin \lambda_n y \, dy \end{aligned}$$

The solution of the given problem, after substituting the values of λ_n, is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \tag{10.7}$$

$$A_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b} \, dy \tag{10.8}$$

Note :

1. The convergence of the series in the final solution, under certain conditions on f and f', is not included in the present Lecture notes.

2. If the boundary conditions in the above problem are modified to $u(0, y) = u(a, y) = 0, u(x, 0) = 0$ and $u(x, b) = f(x)$ then the solution of the corresponding problem is

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \tag{10.9}$$

$$A_n = \frac{2}{a \sinh \frac{n\pi b}{a}} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx \tag{10.10}$$

10.3.1 Numerical Example

The faces of a thin square plate of length 24cm are perfectly insulated (to avoid any atmospherical effects). Find the temperature distribution on the plate if the side at $y = 24$ is kept at $20^o C$ and all the other three sides are kept $0^o C$.

The solution of the problem, from the discussion given above, is

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{24} \sinh \frac{n\pi y}{24} \\ A_n &= \frac{2}{24 \sinh n\pi} \int_0^{24} 20 \sin \frac{n\pi x}{24} \, dx = \frac{40}{n\pi \sinh n\pi} (1 - \cos n\pi) \end{aligned}$$

$$\Rightarrow\ u(x,y) = \frac{80}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\pi\sinh(2n-1)\pi}\sin\frac{(2n-1)\pi x}{24}\sinh\frac{(2n-1)\pi y}{24}$$

10.3.2 Problems to Workout

1. The faces of a thin square plate of length 2cm are perfectly insulated. Find the temperature distribution on the plate if the side at $y = 2$ is kept at $\sin \pi x$ and all the other three sides are kept 0^oC.

2. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega$ where $\Omega = (0,\pi)\times(0,\pi)$ satisfying the boundary conditions $u(x,0) = u(x,\pi) = 0$ along $0 \le x \le \pi$ and $u(0,y) = 0, u(\pi,y) = 10$ along $0 < y < \pi$.

3. The faces of a thin square plate of length 2cm are perfectly insulated. Find the temperature distribution on the plate if $u = 0$ at $x = 0$ and $x = a$, and the other two sides are insulated.

4. The faces of a thin square plate of unit length are perfectly insulated. Find the temperature distribution on the plate if the upper and lower sides of the plate are insulated, left side is kept at 0^oC and the right side is kept at $f(y)^oC$

5. Find the steady state temperature in a rectangular plate bounded $x = 0, x = 1$, y=0 and $y = \pi$. The edges $x = 0$ and $x = 1$ are insulated and the temperature along $y = 0$ is $\cos \pi x$ and along $y = \pi$ is 0.

6. **Neumann problem :** Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in \Omega$ where $\Omega = (0,a)\times(0,b)$ with boundary conditions $u_y(x,0) = u_y(x,b) = 0, u_x(0,y) = 0$ and $u_x(a,y) = f(y)$

10.4 Laplacian in Polar Coordinates

Taking the transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

we have

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\frac{y}{x}$$

Therefore,

$$r_x = \cos\theta, \quad r_y = \sin\theta, \quad \theta_x = -\frac{\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}$$

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \\
\frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \\
\frac{\partial^2}{\partial x^2} &= \cos\theta\left(\cos\theta\frac{\partial^2}{\partial r^2} - \frac{\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\right) \\
&+ \left(-\frac{\sin\theta}{r}\right)\left(\cos\theta\frac{\partial^2}{\partial r\partial\theta} - \sin\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \sin\theta\left(\sin\theta\frac{\partial^2}{\partial r^2} + \frac{\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\cos\theta}{r^2}\frac{\partial}{\partial\theta}\right) \\
&+ \left(\frac{\cos\theta}{r}\right)\left(\sin\theta\frac{\partial^2}{\partial r\partial\theta} + \cos\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial^2}{\partial\theta^2} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0 \qquad (10.11)
\end{aligned}$$

Taking $u(r,\theta) = R(r)T(\theta)$ and substituting in the polar form of the Laplace equation gives the ordinary differential equations

$$\begin{aligned}
r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR &= 0 \\
\frac{d^2R}{d\theta^2} + kT & \quad 0
\end{aligned}$$

For $k = -\lambda^2, 0, \lambda^2$ (k is negative, zero and positive), the solution u is

$$\begin{aligned}
u(r,\theta) &= (c_1\cos\lambda\log r + c_2\sin\lambda\log r) \times (c_3e^{\lambda\theta} + c_4e^{-\lambda\theta}) \\
&= (c_1\log r + c_2) \times (c_3\theta + c_4) \\
&= (c_1r^{\lambda} + c_2r^{-\lambda}) \times (c_3\cos\lambda\theta + c_4\sin\lambda\theta)
\end{aligned}$$

respectively.

10.5 Dirichlet Interior Problem

Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0, 0 \le \theta \le 2\pi, r < a$$

subjected to the boundary conditions $u(a,\theta) = f(\theta)$ for $0 \le \theta \le 2\pi$.

Solution : Since for Dirichlet interior problem, $r = 0$ is also a part of the domain at which $\log r$ is not defines, therefore, the required solution can be obtained only from $k = \lambda^2$, that is,

$$u(r,\theta) = (c_1r^{\lambda} + c_2r^{-\lambda}) \times (c_3\cos\lambda\theta + c_4\sin\lambda\theta)$$

Further, the solution must be periodic with period 2π, therefore

$$\begin{aligned}c_3 \cos \lambda\theta + c_4 \sin \lambda\theta &= c_3 \cos \lambda(\theta + 2\pi) + c_4 \sin \lambda(\theta + 2\pi)\\ c_3(\cos \lambda\theta - \cos \lambda(\theta + 2\pi)) + c_4(\sin \lambda\theta - \sin \lambda(\theta + 2\pi)) &= 0\\ 2 \sin \lambda\pi \,(c_3 \sin(\lambda\theta + \lambda\pi) + c_4 \cos(\lambda\theta + \lambda\pi)) &= 0\end{aligned}$$

Therefore, $\sin \lambda\pi = 0, \quad \lambda\pi = n\pi \quad \Rightarrow \lambda = n, n = 0, 1, 2, \cdots.$

Using the superposition principle, the solution can be written as

$$u(r, \theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-n}) \times (c_n \cos n\theta + d_n \sin n\theta)$$

Further since the solution must be finite at $r = 0$ implies d_n must be zero (for outer problem wherein the domain is defined over $r > 1$, c_n has to be zero to make the solution finite).

Using $d_n = 0$ and renaming the constants will give

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

Now using the given boundary condition gives

$$f(\theta) = \sum_{n=0}^{\infty} a^n(A_n \cos n\theta + B_n \sin n\theta)$$

which is a full Fourier series hence the coefficients are

$$\begin{aligned}A_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta\\ A_n &= \frac{1}{a^n}\frac{2}{2\pi}\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta\\ B_n &= \frac{1}{a^n}\frac{2}{2\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta\end{aligned}$$

10.5.1 Numerical Example

Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 \le \theta \le 2\pi, r < 1$$

subjected to the boundary conditions $u(1, \theta) = 10 \cos^2 \theta$ for $0 \le \theta \le 2\pi$.

Solution : The solution is

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta)$$

with the coefficients

$$
\begin{aligned}
A_0 &= \frac{1}{2\pi}\int_0^{2\pi} 5(1+\cos 2\theta)\, d\theta = 5 \\
A_n &= \frac{2}{2\pi}\int_0^{2\pi} 5(1+\cos 2\theta)\cos n\theta\, d\theta = 0 \quad (n \neq 2) \;\; \& \;\; A_2 = 5 \\
B_n &= \frac{2}{2\pi}\int_0^{2\pi} 5(1+\cos 2\theta)\sin n\theta\, d\theta = 0
\end{aligned}
$$

Therefore, $u(r,\theta) = 5 + 5r^2\cos 2\theta \;=\; 5(1 + r^2\cos 2\theta)$.

10.5.2 Problems to Workout

Find u satisfying $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 \leq \theta \leq 2\pi, r < 1$ subjected to the boundary conditions

1. $f(\theta) = \sin^3\theta$
2. $f(\theta) = \begin{cases} \theta & \frac{-\pi}{2} < \theta < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$
3. $f(\theta) = \begin{cases} -\theta & -\pi < \theta < 0 \\ \theta & 0 < \theta < \pi \end{cases}$

10.6 Neumann Interior Problem

Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 \leq \theta \leq 2\pi, r < a$$

subjected to the boundary conditions $\frac{\partial u}{\partial n}(a,\theta) = \frac{\partial u}{\partial r}(a,\theta) = f(\theta)$ for $0 \leq \theta \leq 2\pi$.

Solution : Following the same computations given in Dirichlet interior problem, we get (since no change in any of the conditions until the application of the boundary conditions at $r = a$)

$$
\begin{aligned}
u(r,\theta) &= \sum_{n=0}^{\infty} r^n (A_n\cos n\theta + B_n \sin n\theta) \\
\frac{\partial u}{\partial n} &= \frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} n r^{n-1}(A_n \cos n\theta + B_n\sin n\theta)
\end{aligned}
$$

Now using the given boundary condition gives

$$f(\theta) = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty} n a^{n-1}(A_n\cos n\theta + B_n \sin n\theta)$$

which is once again a full Fourier series hence the coefficients can be written as

$$\begin{aligned} A_n &= \frac{1}{na^{n-1}}\frac{2}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \\ B_n &= \frac{1}{na^{n-1}}\frac{2}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta \end{aligned}$$

Therefore, the solution of the Neumann interior problem is

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \tag{10.12}$$

where

$$A_n = \frac{1}{na^{n-1}}\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \tag{10.13}$$

$$B_n = \frac{1}{na^{n-1}}\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta \tag{10.14}$$

Notice that, in this case, the solution can differ by an arbitrary constant A_0.

10.7 Semicircular Domain

Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0, 0 \le \theta \le \pi, r < a$$

subjected to the boundary conditions $u(a,\theta) = f(\theta)$ for $0 \le \theta \le \pi$, $u(r,0) = u(r,\pi) = 0$.

Solution : We have

$$u(r,\theta) = (c_1 r^{\lambda} + c_2 r^{-\lambda}) \times (c_3 \cos\lambda\theta + c_4 \sin\lambda\theta)$$

Applying the conditions at $\theta = 0$ and π gives $c_3 = 0$ and $\lambda = n$ for $n = 1, 2, \cdots$. Using the superposition principle, the solution can be written as

$$u(r,\theta) = \sum_{n=1}^{\infty} (a_n r^n + b_n r^{-n}) \sin n\theta$$

Further since the solution must be finite at $r = 0$ implies d_n must be zero. Using $d_n = 0$ and renaming the constants will give

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

Now using the given boundary condition gives

$$f(\theta) = \sum_{n=0}^{\infty} A_n a^n \sin n\theta$$

which is a full Fourier sine series hence the coefficients are

$$A_n = \frac{1}{a^n}\frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta \, d\theta$$

10.7.1 Numerical Example

Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 \le \theta \le \pi, r < 1$$

satisfying the boundary conditions $u(r,0) = u(r,\pi) = 0$, $u(1,\theta) = 10\theta(\pi - \theta)$ for $0 \le \theta \le 2\pi$.

Solution : The solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta$$

$$A_n = \frac{1}{a^n}\frac{2}{\pi}\int_0^{\pi} 10\theta(\pi - \theta)\sin n\theta \, d\theta = \frac{40}{\pi n^2 a^n}(\cos n\pi - 1)$$

10.7.2 Problems to Workout

1. Find u satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0, 0 \le \theta \le \pi, r < 1$$

   satisfying the conditions $u(r,0) = u(r,\pi) = 0$ and

   (a) $f(\theta) = 100\sin^3\theta$

   (b) $f(\theta) = T_0$, where T_0 is a constant

2. $u(r,\theta)$ is a function satisfying

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

   in an **semicircular annulus** defined by $0 \le \theta \le \pi, a < r < b$. If its value along the boundary $r = a$ is $\theta(\frac{\pi}{2} - \theta)$ and is zero on the remaining part of the boundary then prove that

$$u(r,\theta) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(\frac{r}{b})^{4n-2} - (\frac{b}{r})^{4n-2}}{(\frac{a}{b})^{4n-2} - (\frac{b}{a})^{4n-2}} \frac{\sin(4n-2)\theta}{(2n-1)^3}$$