5 Partial Differential Equations

5.1 First Order PDE

Let

$$z = z(x, y) \tag{37}$$

be any surface which also may be written as

$$F(x, y, z) = z(x, y) - z = 0 \tag{38}$$

The vector

$$\mathbf{N} = z_x\mathbf{i} + z_y\mathbf{j} - \mathbf{k} \tag{39}$$

is normal to the surface (38). Let **V** be a vector given by

$$\mathbf{V} = a\mathbf{i} + b\mathbf{j} + g\mathbf{k} \tag{40}$$

which is orthogonal to **N**, then we have

$$\mathbf{V}.\mathbf{N} = az_x + bz_y - g = 0 \tag{41}$$

That is, **V** is tangent to the surface (also called as integral surface) (38) and lies in the tangent plane at every point of the surface. Therefore, geometrically, the first order partial differential equation (41) is the condition for any integral surface through a given point be tangent to the vector **V**.

Hence, if we start at any point (say at the initial condition) and move in the direction of **V** then we move along a curve lying entirely on the integral surface $F(x, y, z) = 0$. The curve is called a *characteristic* and the integral surface can be described in terms of these characteristic curves. Further, the first order PDE (41) is known as the Lagrange's form.

To find the solution of the first order PDE (41), consider the parametric form of the characteristic curve given by

$$\mathbf{P(t)} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \tag{42}$$

Then differentiation of (42) with respect to t gives the tangent vector to the characteristic which must belong to the tangent plane of the integral surface and it must be proportional to **V**. Therefore, we have

$$\frac{dx/dt}{a} = \frac{dy/dt}{b} = \frac{dz/dt}{g}$$

The same is also can be written as

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{g} \tag{43}$$

Equation (43) gives two ODE (one for characteristic and another for solution along the characteristic) which can be solved to find the solution of the first order PDE (41).

Solve the following initial value problems (known as Cauchy problems)

1. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + \sqrt{2}z \ = \ 0, \quad z(x,0) = x$

   Solution : The equation can be rewritten as $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \ = \ -\sqrt{2}z$ Therefore, the auxiliary equation is

$$\frac{dx}{1} \ = \ \frac{dy}{-1} \ = \ \frac{dz}{-\sqrt{2}z}$$

   First, consider $\frac{dx}{1} \ = \ \frac{dy}{-1} \ \Rightarrow \ x \ = \ -y + c_1 \ \Rightarrow \ c_1 = x \ + \ y$
   Similarly, if we consider $\frac{dx}{1} \ = \ \frac{dz}{-\sqrt{2}z}$,
   it gives $ln|z| \ = \ -\sqrt{2}x + c_2 \ \Rightarrow \ z \ = \ c_3e^{-\sqrt{2}x} \ \Rightarrow \ c_3 \ = \ ze^{\sqrt{2}x}$

   (*The meaning of the above statements is, along the characteristic* $x+y = c_1$, *the solution is* $z = c_3e^{-\sqrt{2}x}$ *or the solution can also be written as* $\phi(x+y, ze^{\sqrt{2}x}) = 0$ *where* ϕ *is an arbitrary function*)

   One can eliminate the constants c_1 and c_3 or the arbitrary functionϕ by using the given initial condition in the following way:

   Given $z(x,0) = x \ \Rightarrow \ c_1 = x$ and $c_3 = xe^{\sqrt{2}x} \ = \ c_1e^{\sqrt{2}c_1} \ \Rightarrow$

$$ze^{\sqrt{2}x} \ = \ (x+y)e^{\sqrt{2}(x+y)} \ \Rightarrow \ z = (x+y)e^{\sqrt{2}y}$$

   At $y = 0$, the above solution gives back

$$z = x$$

   and

$$z_x - z_y = e^{\sqrt{2}y} - e^{\sqrt{2}y} - (y+x)\sqrt{2}e^{\sqrt{2}y} = -\sqrt{2}z$$

   That is, $z = (x+y)e^{\sqrt{2}y}$, satisfies the given PDE and also the initial condition, therefore, it is the solution of the given problem.

2. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + \sqrt{2}z \ = \ 0, \quad z(x,0) = x + e^x$

   Solution : The equation can be rewritten as $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \ = \ -\sqrt{2}z$ Therefore, the auxiliary equation is

$$\frac{dx}{1} \ = \ \frac{dy}{-1} \ = \ \frac{dz}{-\sqrt{2}z}$$

   $\frac{dx}{1} \ = \ \frac{dy}{-1} \ \Rightarrow \ x \ = \ -y + c_1 \ \Rightarrow \ c_1 = x \ + \ y$
   $\frac{dx}{1} \ = \ \frac{dz}{-\sqrt{2}z} \ \Rightarrow \ ln|z| \ = \ -\sqrt{2}x + c_2 \ \Rightarrow \ z \ = \ c_3e^{-\sqrt{2}x} \ \Rightarrow \ c_3 \ = \ ze^{\sqrt{2}x}$

   Given $z(x,0) = x + e^x \ \Rightarrow \ c_1 = x$ and $c_3 = (x+e^x)e^{\sqrt{2}x} \ = \ (c_1 + e^{c_1})e^{\sqrt{2}c_1} \ \Rightarrow$

$$ze^{\sqrt{2}x} \ = \ \left((x+y) + e^{(x+y)}\right)e^{\sqrt{2}(x+y)} \ \Rightarrow \ z = \left((x+y) + e^{(x+y)}\right)e^{\sqrt{2}y}$$

   At $y = 0$, the above solution gives back

$$z = x + e^x$$

   and

$$z_x - z_y = e^{\sqrt{2}y}(1+e^{(x+y)}) - e^{\sqrt{2}y}(1+e^{(x+y)}) - \left((x+y) + e^{(x+y)}\right)\sqrt{2}e^{\sqrt{2}y} = -\sqrt{2}z$$

   That is, $z = \left((x+y) + e^{(x+y)}\right)e^{\sqrt{2}y}$, satisfies the given PDE and also the initial condition, therefore, it is the solution of the given problem.

3. $3\frac{\partial z}{\partial x} - 4\frac{\partial z}{\partial y} \;=\; 7-2z, \quad z(x,0) = e^x$
Solution : The auxiliary equation is

$$\frac{dx}{3} \;=\; \frac{dy}{-4} \;=\; \frac{dz}{7-2z}$$

$\frac{dx}{3} \;=\; \frac{dy}{-4} \;\Rightarrow\; 4x \;=\; -3y + c_1 \;\Rightarrow\; c_1 = 4x \;+\; 3y$
$\frac{dx}{3} \;=\; \frac{dz}{7-2z} \;\Rightarrow\; ln|7-2z| \;=\; c_2 - \frac{2x}{3} \;\Rightarrow\; z \;=\; \frac{1}{2}\left(7 - c_3 e^{-\frac{2x}{3}}\right)$
$c_3 \;=\; (7-2z)e^{\frac{2x}{3}}$

Given $z(x,0) = e^x$
$\Rightarrow \;\; c_1 = 4x$ and $c_3 = (7-2z)e^{\frac{2x}{3}} \;=\; (7-2e^{\frac{4x}{4}})e^{\frac{4x}{6}} \;=\; (7-2e^{\frac{c_1}{4}})e^{\frac{c_1}{6}}$

$$(7-2z)e^{\frac{2x}{3}} \;=\; (7-2e^{\frac{4x+3y}{4}})e^{\frac{4x+3y}{6}} \;\;\Rightarrow\;\; z = \frac{1}{2}\left[7 - e^{\frac{y}{2}}\left(7-2e^{\frac{4x+3y}{4}}\right)\right]$$

At $y=0$, the above solution gives back $z = e^x$ and

$$3z_x - 4z_y = e^{\frac{y}{2}}e^{\frac{4x+3y}{4}} - \frac{7}{4}e^{\frac{y}{2}} + \frac{5}{4}e^{\frac{y}{2}}e^{\frac{4x+3y}{4}} \;=\; 7-2z$$

That is, $z = \frac{1}{2}\left[7 - e^{\frac{y}{2}}\left(7-2e^{\frac{4x+3y}{4}}\right)\right]$, satisfies the given PDE and also the initial condition, therefore, it is the solution of the given problem.

Optional Problems

4. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \;=\; z, \quad z(y,y) = e^y$
Solution : The auxiliary equation is

$$\frac{dx}{1} \;=\; \frac{dy}{1} \;=\; \frac{dz}{z}$$

$\frac{dx}{1} \;=\; \frac{dy}{1} \;\Rightarrow\; x \;=\; y + c_1 \;\Rightarrow\; c_1 = x \;-\; y$
$\frac{dx}{1} \;=\; \frac{dz}{z} \;\Rightarrow\; ln|z| \;=\; x - c_2 \;\Rightarrow\; z \;=\; e^{x-c_2} \;= c_3 e^x \;\Rightarrow\; c_3 \;=\; ze^{-x}$

Given $z(y,y) = e^y \;\;\Rightarrow\;\; c_1 = 0$ and $c_3 = e^y e^{-y} = 1$. In this case, the initial condition is given along one of the characteristic. In such cases, the initial condition can't be given arbitrary and the solution to the problem exists only if it is compatible with the solution of the differential equation. If we use $c_3 = 1$, since the initial condition and the solution $z = e^x$ are compatible, therefore the solution exits and is equal to the e^x for the given initial value problem.

5. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \;=\; 1, \quad z(y,y) = e^y$
Solution : The auxiliary equation is

$$\frac{dx}{1} \;=\; \frac{dy}{1} \;=\; \frac{dz}{1}$$

$\frac{dx}{1} \;=\; \frac{dy}{1} \;\Rightarrow\; x \;=\; y + c_1 \;\Rightarrow\; c_1 = x \;-\; y$
$\frac{dx}{1} \;=\; \frac{dz}{1} \;\Rightarrow\; x \;=\; z + c_2 \;\Rightarrow\; z \;=\; x - c_2 \;\&\; c_2 \;=\; x - z$

Given $z(y,y) = e^y \;\;\Rightarrow\;\; c_1 = 0$ and $c_2 = y - e^y \;\;\Rightarrow\;\; z \;=\; x - y + e^y$. The obtained z satisfies the initial condition but not he given differential equation, therefore solution doesn't exists for the given initial value problem.

6. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1, \quad z(y,y) = y+7$
Solution : The auxiliary equation is

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{1}$$

$\frac{dx}{1} = \frac{dy}{1} \Rightarrow x = y + c_1 \Rightarrow c_1 = x - y$
$\frac{dx}{1} = \frac{dz}{1} \Rightarrow x = z + c_2 \Rightarrow z = x - c_2$ & $c_2 = x - z$

Given $z(y,y) = y+7 \Rightarrow c_1 = 0$ and $c_2 = -7 \Rightarrow z = x+7$. The obtained z satisfies the initial condition and the given differential equation, therefore $z = x+7$ is a solution for the given initial value problem. Similarly, $z = y+7$ is also a solution can be obtained by starting with $\frac{dy}{1} = \frac{dz}{1}$. Therefore, $z = \frac{k_1(x+7)+k_2(y+7)}{k_1+k_2}, k_1, k_2 \in \mathbb{R}$ satisfies the given initial value problem hence it has infinitely many solutions.

7. $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2 - z, \quad z(x,0) = \sin(x)$

8. $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = y - z, \quad z(x,0) = x^2$

9. $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1, \quad z(1,y) = e^{-y}$

10. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z, \quad z(1,y) = y^2$

11. $x\frac{\partial z}{\partial x} + y^{-1}\frac{\partial z}{\partial y} = 1, \quad z(x,0) = 5 - x$

12. $\frac{\partial z}{\partial x} + \frac{1}{2y}\frac{\partial z}{\partial y} = 2, \quad z(x,0) = \sin x - 2$