Chapter 8

Hyperbolic Equations

Consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{8.1}$$

It has been shown in the earlier chapter that, two real characteristics exists for the equation (8.1) along which the given differential equation takes a simpler form given by

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{8.2}$$

where $\xi = x + at$ and $\eta = x - at$. By integrating on both sides of (8.2), it can solved to get $u(\xi, \eta) = \phi(\xi) + \psi(\eta)$ where ϕ and ψ are two arbitrary functions ($u_{\xi\eta} = 0 \Rightarrow u_\xi = F(\xi) \Rightarrow u = \phi(\xi) + \psi(\eta)$). Therefore, the solution of (8.1) can be written as

$$u = \phi(\xi) + \psi(\eta) \Rightarrow u(x, y) = \phi(x + at) + \psi(x - at) \tag{8.3}$$

8.1 D'Alembert's Solution

Include the initial conditions (for an initial value or Cauchy problem)

$$u(x, 0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}|_{t=0} = g(x) \tag{8.4}$$

and eliminate the arbitrary functions from (8.3) gives the solution of the wave equation in the domain $-\infty < x < \infty$, $t > 0$ with initial conditions (8.4) at $t = 0$. The D'Alembert's solution is given by

$$\begin{aligned}
u(x, t) &= \phi(x + at) + \psi(x - at) \\
u(x, 0) &= \phi(x) + \psi(x) = f(x) \\
\frac{\partial u}{\partial t} &= a\phi'(x + at) - a\psi'(x - at) \\
\frac{\partial u}{\partial t}(x, 0) &= a\phi'(x) - a\psi'(x) = g(x)
\end{aligned}$$

Integrating the last equation in the limits x_0 to x gives

$$\begin{aligned}\int_{x_0}^{x} \left(a\phi'(x) - a\psi'(x)\right)\, dx &= \int_{x_0}^{x} g(x)\, dx \\ \phi(x) - \psi(x) &= \phi(x_0) + \psi(x_0) + \frac{1}{a}\int_{x_0}^{x} g(x)\, dx\end{aligned}$$

We also have

$$\phi(x) + \psi(x) = f(x)$$

Solving these two equations for ϕ and ψ gives

$$\begin{aligned}\phi(x) &= \frac{\phi(x_0) + \psi(x_0) + f(x)}{2} + \frac{1}{2a}\int_{x_0}^{x} g(x)\, dx \\ \psi(x) &= -\frac{\phi(x_0) + \psi(x_0) - f(x)}{2} - \frac{1}{2a}\int_{x_0}^{x} g(x)\, dx\end{aligned}$$

Therefore

$$\begin{aligned}\phi(x+at) &= \frac{\phi(x_0) + \psi(x_0) + f(x+at)}{2} + \frac{1}{2a}\int_{x_0}^{x+at} g(x)\, dx \\ \psi(x-at) &= -\frac{\phi(x_0) + \psi(x_0) - f(x-at)}{2} - \frac{1}{2a}\int_{x_0}^{x-at} g(x)\, dx\end{aligned}$$

Finally, the D'Alembert's solution for the wave equation is

$$\begin{aligned}u(x,y) &= \phi(x+at) + \psi(x-at) \\ &= \frac{f(x+at) + f(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} g(x)\, dx \qquad (8.5)\end{aligned}$$

It is clear from the solution (8.5) that, the solution of the wave equation at any point (x,t) depends on f at the two points $(x-at,0)$ and $(x+at,0)$ and also on g between $x-at$ and $x+at$, as shown in the Figure (8.1), which is called the domain of dependence. That is, the region on which, the solution at any point say, $P(x,y)$ depends on is bounded by the lines $x-at=0$, $x+at=0$ and the x-axis. Therefore, any changes in the initial conditions outside this region can't influence the solution at the point P. Similarly, the domain influence of the point P is the region bounded by the extension of the lines $x-at=0$ and $x+at=0$ for all the later times of t.

8.1.1 Numerical Examples

1. Find the vertical displacement $u(x,t)$ in a plane of an infinitely long elastic string which is started from rest and having an initial displacement $K\sin\pi x$, where K is a constant (Assume $a^2 = \frac{T}{\rho} = 1$, where T is the tension and ρ is the

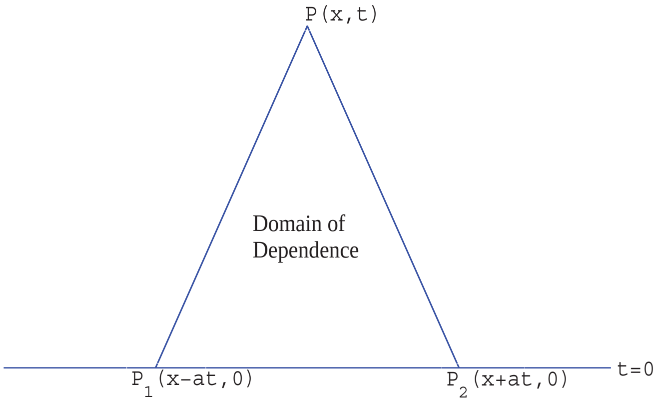

Figure 8.1: Domain of dependence

density).

Solution: By D'Alembert's solution, we have $f(x) = K\sin\pi x$ and $g(x) = 0$, therefore, the solution is

$$u(x,t) = \frac{K}{2}\left(\sin\pi(x+t) + \sin\pi(x-t)\right) = K\sin\pi x\cos\pi t$$

2. Find the vertical displacement $u(x,t)$ in a plane of an infinitely long elastic string which is started with a constant velocity g and having an initial displacement $K\sin\pi x$, where K is a constant (Assume $a^2 = \frac{T}{\rho} = 1$, where T is the tension and ρ is the density).

   Solution: By D'Alembert's solution, we have $f(x) = K\sin\pi x$ and $g(x) = g$, therefore, the solution is

$$u(x,t) = \frac{K}{2}\left(\sin\pi(x+t) + \sin\pi(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g\,dx = K\sin\pi x\cos\pi t + gt$$

8.1.2 Problems to workout

Solve the wave equation using the following initial conditions $x \in (-\infty, \infty)$

1. $f(x) = Kx(1-x)$ and $g(x) = 0$
2. $f(x) = K(x - x^3)$ and $g(x) = 0$
3. $f(x) = K(1 - \cos 2\pi x)$ and $g(x) = 0$
4. $f(x) = 0$ and $g(x) = -4xe^{-x^2}$

8.2 Fourier Series based solutions for Hyperbolic Equations

Hyperbolic equations defined in finite domains with boundary conditions, that is initial-boundary value hyperbolic equations can be solved using a method called Separation of variables in which series solutions can be obtained with the help of Fourier series. Before looking at such problems, in the next section, we look at some of the important concepts, which we need in our discussion, from Fourier series.

8.2.1 Fourier Series

Due to the orthogonal nature of $\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$, in the interval 0 to L, given by

$$\int_0^L \sin \frac{n\pi x}{L} \, \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \end{cases} \tag{8.6}$$

$$\int_0^L \cos \frac{n\pi x}{L} \, \cos \frac{m\pi x}{L} \, dx = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n \\ L & m = n = 0 \end{cases} \tag{8.7}$$

any piecewise smooth function f on the interval $-L \leq x \leq L$ can be expressed as

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{8.8}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \, dx \tag{8.9}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \, \cos \frac{n\pi x}{L} \, dx \tag{8.10}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \, \sin \frac{n\pi x}{L} \, dx \tag{8.11}$$

for $n = 1, 2, \ldots$.

Some Convergence Results

1. The, fourier series, (right hand side of (8.8)), converges to
   (a) the periodic extension of $f(x)$ if f is continuous at x
   (b) the average of the two limits, that is $\frac{1}{2}(f(x+) + f(x-))$ at the points of jump discontinuity
2. The Fourier series of $f(x)$ is continuous and converge to $f(x)$ for $-L \leq x \leq L$ if and only if $f(x)$ is continuous and $f(-L) = f(L)$. The corresponding results for cosine and sine series (the series obtained with even and odd extensions of $f(x)$ in $0 \leq x \leq L$, respectively) are

(a) The Fourier cosine series of $f(x)$ is continuous and converge to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous

(b) The Fourier sine series of $f(x)$ is continuous and converge to $f(x)$ for $0 \le x \le L$ if and only if $f(x)$ is continuous and $f(0) = f(L) = 0$

3. A Fourier series that is continuous can be differentiated term by term if $f'(x)$ is piecewise smooth. Therefore, if $f(x)$ is piecewise smooth then $f(x)$ can be differentiated term by term if $f(-L) = f(L)$.

   Counter Example: $2\sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^{n+1} \sin \frac{n\pi x}{L}$ is the Fourier sine series of x in the interval $0 \le x < L$ however, the series with its termwise derivatives $2\sum_{n=1}^{\infty}(-1)^{n+1} \cos \frac{n\pi x}{L}$ is not a cosine series of $f(x) = 1$ (the cosine series of 1 is 1 itself).

4. If $f'(x)$ is piecewise smooth, then the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

5. If $f'(x)$ is piecewise smooth, then the Fourier sine series of a continuous function $f(x)$ can be differentiated term by term if $f(0) = f(L) = 0$.

From the results given above, since the solutions of the wave equation are twice differentiable (space direction), therefore, Fourier series along with term by term differentiation exists for the solutions of Hyperbolic equations with homogeneous boundary conditions.

8.2.2 Fixed Oscillations of a Thin Homogeneous String

A thin perfectly flexible homogenous string under uniform tension is tied at the ends $x = 0$ and $x = L$. The string is pulled in a planar region and released. In the absence of any external forces, the problem is modeled by

$$
\begin{array}{rll}
\text{PDE} & u_{tt} = a^2 u_{xx} & 0 < x < L, t > 0 \\
\text{Initial Conditions} & u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 \le x \le L \\
\text{Boundary Conditions} & u(0,t) = u(L,t) = 0 & t > 0
\end{array}
\tag{8.12}
$$

Computation of u : To separate the variables, assume

$$u(x,t) = X(x)T(t)$$

Substituting in the differential equation gives

$$\frac{1}{a^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Since the left hand side of the above equation is a function of t alone and similarly, the right hand side is function of x alone, therefore, the equality of these two enforces the constant nature of these two. That is

$$\frac{1}{a^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = k$$

where k is a constant. Equivalently, we have

$$\frac{d^2T}{dt^2} - ka^2T = 0, \quad \frac{d^2X}{dx^2} = kX = 0$$

Further, the boundary conditions on u at $x = 0$ and $x = L$ gives

$$X(0)T(t) = X(L)T(t) = 0$$

that is, either X is zero at $x = 0$ and $x = L$ or $T \equiv 0$. Since the latter condition makes $u \equiv 0$ for all times, for a possibility of any non-zero solution choose X is zero at $x = 0$ and $x = L$.

1. **Case i:** $k = \lambda^2$ (a positive constant)
   Solving the ODE for X gives $X(x) = c_1e^{\lambda x} + c_2e^{-\lambda x}$, which gives $X \equiv 0$ for the boundary conditions X is zero at $x = 0$ and $x = L$.

2. **Case ii:** $k = 0$
   Once again solving for X gives $X(x) = c_1 + c_2x$, which again gives $X \equiv 0$ for the boundary conditions X is zero at $x = 0$ and $x = L$.

3. **Case iii:** $k = -\lambda^2$ (a negative constant)
   Solving the ODE for X gives $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$

   $$\begin{aligned} X(0) = 0 \quad &\Rightarrow \quad c_1 = 0 \\ X(L) = 0 \quad &\Rightarrow \quad c_2 \sin \lambda L = 0 \end{aligned}$$

   If $c_2 = 0$, then once again $X \equiv 0$, therefore, $\sin \lambda L = 0$ must be zero. That is,

   $$\lambda L = n\pi \quad \Rightarrow \quad \lambda = \frac{n\pi}{L}$$

   for $n = 0, \pm 1, \pm 2, \cdots$.

   Now using the facts that sine function is odd and $\sin 0 = 0$, we have

   $$\lambda = \frac{n\pi}{L} \qquad \text{for} \quad n = 1, 2, \cdots$$

Now, solving for T gives (after replacing λ with $\frac{n\pi}{L}$)

$$T(t) = A \cos \frac{n\pi at}{L} + B \sin \frac{n\pi at}{L} \quad \text{for} \quad n = 1, 2, \cdots$$

where A and B are constants. Now, using the superposition principle, the solution for $u(x, t)$ can be written as

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \cos \frac{n\pi at}{L} + B_n \sin \frac{n\pi at}{L} \right)$$

where A_n and B_n for $n = 0, \pm 1, \pm 2, \cdots$ are constants. Finally, the initial conditions must be applied to find the remaining constants in the solution. That is,

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi a(0)}{L} + B_n \sin\frac{n\pi a(0)}{L}\right) &= f(x) \\ \sum_{n=1}^{\infty} A_n \, \sin\frac{n\pi x}{L} &= f(x) \end{aligned}$$

The last equation is a Fourier sine series, therefore, A_n can obtained as

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

for $n = 1, 2, \cdots$.

The second initial condition is $u_t(x,0) = g(x)$, that is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n\pi a}{L}\,\sin\frac{n\pi x}{L}\left(-A_n\sin\frac{n\pi a(0)}{L} + B_n\cos\frac{n\pi a(0)}{L}\right) &= g(x) \\ \sum_{n=1}^{\infty}\frac{n\pi a}{L}\,\sin\frac{n\pi x}{L}\,B_n &= g(x) \\ \sum_{n=1}^{\infty} B_n\,\frac{n\pi a}{L}\,\sin\frac{n\pi x}{L} &= g(x) \\ \sum_{n=1}^{\infty} B_n^*\sin\frac{n\pi x}{L} &= g(x), \qquad \text{where } B_n^* = B_n\,\frac{n\pi a}{L} \end{aligned}$$

Again solving the Fourier sine series gives

$$B_n^* = \frac{n\pi a}{L}\,B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

for $n = 1, 2, \cdots$

The required solution is

$$u(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi at}{L} + B_n\sin\frac{n\pi at}{L}\right) \tag{8.13}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{8.14}$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \tag{8.15}$$

8.2.3 Problems to Workout

1. Show that the above solution satisfies the given initial and boundary conditions.

2. Solve the wave equation with homogeneous boundary conditions and the following initial conditions $f(x) = \sin^3 \frac{\pi x}{2}$ and $g(x) = 0$ (take $a = 1$ in the PDE).

3. Assuming $g(x) = 0$, show that the above solution gives the D'Alemberts solution of a suitable problem

4. Solve the wave equation with homogeneous boundary conditions and the following initial conditions $f(x) = \begin{cases} \frac{2k}{L}x & if\ 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & if\ \frac{L}{2} < x < L \end{cases}$, $g(x) = 0$ (take $a = 1$ in the PDE).

5. Solve the wave equation with homogeneous boundary conditions, $L = \pi$, $a = 1$, and with the following initial deflection and velocities

   (a) $f(x) = k(\sin x - \frac{1}{2}\sin 2x)$, $g(x) = 0$

   (b) $f(x) = .1x(\pi^2 - x^2)$, $g(x) = 0$

   (c) $f(x) = 0$, $g(x) = \begin{cases} 0.01x & if\ 0 \le x \le \frac{\pi}{2} \\ 0.01(\pi - x) & if\ \frac{\pi}{2} \le x \le L \end{cases}$,

8.2.4 Forced Vibrations

$$\begin{array}{rll} \text{PDE} & u_{tt} - a^2 u_{xx} = F(x,t) & 0 < x < L, t > 0 \\ \text{Initial Conditions} & u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 \le x \le L \\ \text{Boundary Conditions} & u(0,t) = u(L,t) = 0 & t > 0 \end{array} \tag{8.16}$$

Define $u = u_1 + u_2$ such that u_1 is the solution of (8.12) and u_2 is the solution of the problem

$$\begin{array}{rll} \text{PDE} & u_{tt} - a^2 u_{xx} = F(x,t) & 0 < x < L, t > 0 \\ \text{Initial Conditions} & u(x,0) = 0,\ u_t(x,0) = 0 & 0 \le x \le L \\ \text{Boundary Conditions} & u(0,t) = u(L,t) = 0 & t > 0 \end{array} \tag{8.17}$$

then u satisfies the given equation and also the initial and boundary conditions. Since u_1 is already known from the earlier problem (problem of free vibrations of the string), it is complete if we compute u_2. Let assume (from the understanding of the problem of the free vibrations)

$$u_2(x,t) = \sum_{n=1}^{\infty} \phi_n(t) \sin \frac{n\pi x}{L} \tag{8.18}$$

(8.17) already satisfies the boundary conditions of (8.16) and it will satisfy the initial conditions if, $\phi_n(0) = \phi_n'(0) = 0$, $n = 1, 2, \cdots$. Substituting (8.17) in the differential equation gives

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{d^2\phi_n(t)}{dt^2} + \frac{n^2\pi^2a^2}{L^2}\phi_n(t) \right) \sin\frac{n\pi x}{L} &= F(x,t) \\
\sum_{n=1}^{\infty} \left(\frac{d^2\phi_n(t)}{dt^2} + \omega_n^2\phi_n(t) \right) \sin\frac{n\pi x}{L} &= F(x,t), \quad \omega_n = \frac{n\pi a}{L}
\end{aligned}$$

Multiplying with $\sin\frac{k\pi x}{L}$ on both sides and then integrating in the limits $x \in (0, L)$ gives (using the orthogonal property)

$$\begin{aligned}
\left(\frac{d^2\phi_k(t)}{dt^2} + \omega_k^2\phi_k(t) \right) \frac{2}{L} &= \int_0^L F(x,t)\sin\frac{k\pi x}{L}\,dx \\
\frac{d^2\phi_k(t)}{dt^2} + \omega_k^2\phi_k(t) &= \bar{F}_k(t)
\end{aligned}$$

where $\bar{F}_k(t) = \frac{L}{2}\int_0^L F(x,t)\sin\frac{k\pi x}{L}\,dx$ subject to the boundary conditions $\phi_k(0) = \phi_k'(0) = 0$ for $k = 1, 2, \cdots$. Solving this problem with variation of parameters gives the solution

$$\phi_k(t) = \frac{1}{\omega_k}\int_0^t \bar{F}_k(s)\ \sin(\omega_k(t-s))\,ds$$

Therefore, by superposition principle, the solution u_2 is given by

$$u_2(x,t) = \sum_{n=1}^{\infty} \left\{ \frac{1}{\omega_k}\int_0^t \bar{F}_k(s)\ \sin(\omega_k(t-s))\,ds \right\} \sin\frac{n\pi x}{L}$$

Finally, the solution of the given forced vibrations problem is given by

$$\begin{aligned}
u(x,t) = u_1 + u_2 &= \sum_{n=1}^{\infty} \{A_n\cos\omega_n t + B_n\sin\omega_n t\}\sin\frac{n\pi x}{L} \\
&+ \sum_{n=1}^{\infty} \left\{ \frac{1}{\omega_k}\int_0^t \bar{F}_k(s)\ \sin(\omega_k(t-s))\,ds \right\} \sin\frac{n\pi x}{L}
\end{aligned}$$

where

$$\begin{aligned}
A_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \\
B_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx
\end{aligned}$$