Chapter 9

Parabolic Equations

9.1 One - Dimensional Unsteady Diffusion with Homogenous Boundary Conditions

One dimensional unsteady diffusion is governed by the parabolic equation

$$\begin{array}{rlll} \text{PDE} & u_t = a^2 u_{xx} & 0 < x < L, t > 0 & \\ \text{Initial Condition} & u(x,0) = f(x) & 0 \leq x \leq L & (9.1) \\ \text{Boundary Conditions} & u(0,t) = u(L,t) = 0 & t > 0 & \end{array}$$

where $a^2 = \frac{K}{\sigma\rho}$ is the thermal diffusivity, K is the thermal conductivity, σ is the specific heat, ρ is the density. With this nomenclature, u represents the temperature distribution at any point x on a one dimensional object, say, a very thin rod of length L, and $f(x)$ is the initial temperature distribution over the thin rod which is maintained at 0^o at the end points for all times.

Following the method of separation of variables, choose

$$u(x,t) = X(x)\, T(t)$$

to get

$$\frac{1}{a^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Since the left hand side of the above equation is a function of t alone and similarly, the right hand side is a function of x alone, the equality enforces the constant nature of these two. That is

$$\frac{1}{a^2 T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = k$$

where k is a constant. Equivalently, we have

$$\frac{dT}{dt} - ka^2T = 0, \quad \frac{d^2X}{dx^2} - kX = 0$$

Further, the boundary conditions on u at $x = 0$ and $x = L$ gives

$$X(0)T(t) = X(L)T(t) = 0$$

that is, either X is zero at $x = 0$ and $x = L$ or $T \equiv 0$. Since the latter condition makes $u \equiv 0$ for all times, for a possibility of any non-zero solution choose X is zero at $x = 0$ and $x = L$.

1. **Case i:** $k = \lambda^2$ (a positive constant)
   Solving the ODE for X gives $X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$, which gives $X \equiv 0$ for the boundary conditions X is zero at $x = 0$ and $x = L$.

2. **Case ii:** $k = 0$
   Once again solving for X gives $X(x) = c_1 + c_2 x$, which again gives $X \equiv 0$ for the boundary conditions X is zero at $x = 0$ and $x = L$.

3. **Case iii:** $k = -\lambda^2$ (a negative constant)
   Solving the ODE for X gives $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$

   $$\begin{aligned} X(0) = 0 &\;\Rightarrow\; c_1 = 0 \\ X(L) = 0 &\;\Rightarrow\; c_2 \sin \lambda L = 0 \end{aligned}$$

   If $c_2 = 0$, then once again $X \equiv 0$, therefore, $\sin \lambda L = 0$ must be zero. That is,

   $$\lambda L = n\pi \quad \Rightarrow \quad \lambda_n = \frac{n\pi}{L}$$

   for $n = 0, \pm 1, \pm 2, \cdots$.

   Now using the fact that sine function is odd and $\sin 0 = 0$, we have

   $$\lambda_n = \frac{n\pi}{L} \qquad \text{for} \quad n = 1, 2, \cdots$$

Now, solving

$$\frac{dT}{dt} + a^2 \lambda_n^2 T = 0$$

for T gives

$$T_n(t) = A_n e^{-a^2 \lambda_n^2 t} \quad \text{for} \quad n = 1, 2, \cdots$$

where A_n are constants. Now, using the superposition principle, the solution for $u(x, t)$ can be written as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \, \sin \frac{n\pi x}{L} e^{-a^2 \lambda_n^2 t}$$

Finally, the initial conditions must be applied to find A_n in the solution. That is,

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} A_n \ \sin \frac{n\pi x}{L} e^{-a^2\lambda_n^2(0)} &= f(x) \\ \sum_{n=1}^{\infty} A_n \ \sin \frac{n\pi x}{L} &= f(x) \end{aligned}$$

The last equation is a Fourier sine series, therefore, A_n can obtained as

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

for $n = 1, 2, \cdots$.

Therefore, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \ \sin \frac{n\pi x}{L} e^{-a^2\lambda_n^2 t}, \qquad \lambda_n = \frac{n\pi}{L} \tag{9.2}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{9.3}$$

9.1.1 Numerical Example

1. Consider an insulated bar of $80cm$ length with ends kept at 0^oC (Use the thermal diffusivity of the material $a^2 = 1.158cm^2/sec$).

   (a) Find the temperature distribution u if the initial temperature distribution is given by $100\sin\frac{\pi x}{80}{}^oC$.

   (b) How long will it take for the maximum temperature in the bar to drop to 50^0C.

   (c) If the initial temperature distribution is changed to $100\sin\frac{3\pi x}{80}{}^oC$ with all the other data same as above, how much time it will take for the maximum temperature in the bar to drop to 50^0C.

   Solution : For the given conditions we have $a^2\lambda_n^2 = a^2(\frac{n\pi}{80})^2 = .001785n^2$, the solution of the problem is given by

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} A_n \ \sin\frac{n\pi x}{80}\ e^{-.001785n^2 t} \\ A_n &= \frac{2}{80}\int_0^{80} f(x)\sin\frac{n\pi x}{80}\,dx = \frac{2}{80}\int_0^{80} 100\sin\frac{\pi x}{80}\ \sin\frac{n\pi x}{80}\,dx \\ A_1 &= 100\,\frac{2}{80}\,\frac{80}{2} = 100 \\ & \quad (A_2 = A_3 = \cdots = 0) \quad \text{from the orthogonality of the sine function } n = 1 \\ \Rightarrow u(x,t) &= 100\sin\frac{\pi x}{80}\ e^{-.001785t} \quad \text{is the required solution.} \end{aligned}$$

For the maximum temperature to drop to 50^oC, due to symmetry of the solution, the maximum temperature must happen at $x = 40$ or one can argue that, $\sin\frac{\pi x}{80}$ must be 1 for maximum of u to happen, therefore, we have

$$50 = 100e^{-.001785n^2t} \Rightarrow t = \frac{ln(0.5)}{.001785} = 388sec = 6.5min$$

If the initial condition is changed to $100\sin\frac{3\pi x}{80}{}^oC$, then from the orthogonality condition, the integral will survive for $n = 3$, that is $A_3 = 100$ and all other coefficients of the infinite series are zero, therefore the solution is

$$u(x,t) = 100\sin\frac{3\pi x}{80}\ e^{-3^2\ .001785t} = u(x,t) = 100\sin\frac{\pi x}{80}\ e^{-.01607t}$$

Therefore, the time required for the maximum temperature to drop to 50^oC is

$$50 = 100\ e^{-.01607t} \quad \Rightarrow \quad t = \frac{ln(0.5)}{.01607} = 43sec$$

Observation: The drop in the temperature will be faster for the initial temperatures with larger n.

2. Find the temperature $u(x,t)$ if the initial temperature distribution is given by

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < 40 \\ 80 - x & \text{if} \quad 40 < x < 80 \end{cases}$$

Solution: A_n is given by

$$\begin{aligned}
A_n &= \frac{2}{80}\int_0^{40} x\sin\frac{n\pi x}{80}\,dx + \frac{2}{80}\int_{40}^{80}(80-x)\sin\frac{n\pi x}{80}\,dx \\
A_n &= \frac{2}{80}\left[\left(-\frac{80}{n\pi}x\cos\frac{n\pi x}{80}\right)_0^{40} + \frac{80}{n\pi}\int_0^{40}\cos\frac{n\pi x}{80}\,dx\right] \\
&+ \frac{2}{80}\left[\left(-\frac{80}{n\pi}(80-x)\cos\frac{n\pi x}{80}\right)_{40}^{80} - \frac{80}{n\pi}\int_{40}^{80}\cos\frac{n\pi x}{80}\,dx\right] \\
&= \frac{2}{80}\,\frac{80}{n\pi}\left[\int_0^{40}\cos\frac{n\pi x}{80}\,dx - \int_{40}^{80}\cos\frac{n\pi x}{80}\,dx\right] \\
&= \frac{2}{80}\,\frac{80}{n\pi}\,\frac{80}{n\pi}\left[\left(\sin\frac{n\pi x}{80}\right)_0^{40} - \left(\sin\frac{n\pi x}{80}\right)_{40}^{80}\right] \\
&= \frac{2}{80}\,\frac{80}{n\pi}\,\frac{80}{n\pi}\left[2\sin\frac{n\pi}{2}\right] = \frac{320}{n^2\,\pi^2}\ \sin\frac{n\pi}{2} \\
&= 0 \qquad \text{if } n \text{ is even} \\
&= \frac{320}{n^2\,\pi^2} \qquad \text{if } n \text{ is } 1,5,9,\cdots \\
&= -\frac{320}{n^2\,\pi^2} \qquad \text{if } n \text{ is } 3,7,11\cdots
\end{aligned}$$

9.1.2 Problems to Workout

1. Find $u(x,t)$, satisfying $u_t = 2u_{xx}$, $u(x,0) = \sin 2x$, $u(0,t) = u(\pi,t) = 0$.

2. Find $u(x,t)$, satisfying $u_t = 2u_{xx}$, $u(x,0) = 25$, $u(0,t) = u(3,t) = 0$ and $|u|$ is bounded.

3. Find $u(x,t)$, satisfying $u_t = 2u_{xx}$, $u(x,0) = 25x$, $u(0,t) = u(4,t) = 0$ and $|u|$ is bounded.

4. Find $u(x,t)$, satisfying $u_t = 2u_{xx}$, $u(x,0) = 5\sin 4\pi x - 3\sin 8\pi x + 2\sin 10\pi x$, $u(0,t) = u(3,t) = 0$ and $|u|$ is bounded.

5. Find $u(x,t)$, satisfying $u_t = 4u_{xx}$, $u(x,0) = 10\sin 2\pi x - 6\sin 4\pi x$, $u(0,t) = u(3,t) = 0$ and $|u|$ is bounded.

6. Find $u(x,t)$, satisfying $u_t = 4u_{xx}, 0 < x < 2, t > 0$, $u(x,0) = \begin{cases} x & 0 < x \le 1 \\ 2-x & 1 \le x < 2 \end{cases}$, $u(0,t) = u(2,t) = 0$ and $|u|$ is bounded.

9.2 Bar with Insulated Ends

$$\begin{array}{rll} \text{PDE} & u_t = a^2 u_{xx} & 0 < x < L, t > 0 \\ \text{Initial Condition} & u(x,0) = f(x) & 0 \le x \le L \\ \text{Boundary Conditions} & u_x(0,t) = u_x(L,t) = 0 & t > 0 \end{array} \tag{9.4}$$

Solution : Once again if we choose $u(x,t) = X(x)T(t)$ then substituting u in the given PDE gives two ODE

$$\frac{dT}{dt} - ka^2T = 0, \quad \frac{d^2X}{dx^2} - kX = 0, \qquad k \text{ is a constant}$$

but with conditions on x as $u_x(0,t) = u_x(L,t) = 0$. The problem ends up with trivial solutions with $k \ge 0$, therefore choose $k = -\lambda^2$ to get $X(x) = c_1 \cos \lambda x + c_2 \sin \lambda x$. Now, applying the boundary conditions gives

$$\begin{aligned} X_x(0) &= \lambda(-c_1 \sin(0) + c_2 \cos(0)) = 0 \quad \Rightarrow \quad c_2 = 0 \\ X_x(L) &= -\lambda c_1 \sin \lambda L = 0 \sin \lambda L = 0 \quad \Rightarrow \quad \lambda L = n\pi, \quad n = 0, \pm 1, \pm 2, \cdots \\ &\Rightarrow \lambda_n = \frac{n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \cdots \end{aligned}$$

Therefore,

$$x_n(x) = \cos \frac{n\pi x}{L}, \quad n = 0, 1, 2, \cdots$$

since cos is an even function. The eigenfunctions are

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L} \, e^{-a^2\lambda_n^2 t}$$

Now using the initial condition $u(x,0) = f(x)$ gives

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L} = f(x)$$

Using the Fourier cosine series give

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx \quad n = 0,1,2,\cdots$$

9.2.1 Numerical Example

Find the temperature $u(x,t)$ if the initial temperature distribution is given by

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < 40 \\ 80 - x & \text{if} \quad 40 < x < 80 \end{cases}$$

for an insulated ends of a rod.

Solution : The solution of the problem is

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\; e^{-a^2\lambda^2 t}$$

where

$$\begin{aligned}
A_0 &= \frac{1}{80}\int_0^{40} x\,dx \;+\; \frac{1}{80}\int_{40}^{80}(80-x)\,dx \;=\; 20 \\
A_n &= \frac{2}{80}\int_0^{40} x\cos\frac{n\pi x}{80}\,dx \;+\; \frac{2}{80}\int_{40}^{80}(80-x)\cos\frac{n\pi x}{80}\,dx \\
A_n &= \frac{2}{80}\left[\left(\frac{80}{n\pi}x\sin\frac{n\pi x}{80}\right)_0^{40} - \frac{80}{n\pi}\int_0^{40}\sin\frac{n\pi x}{80}\,dx\right] \\
&+ \frac{2}{80}\left[\left(\frac{80}{n\pi}(80-x)\sin\frac{n\pi x}{80}\right)_{40}^{80} + \frac{80}{n\pi}\int_{40}^{80}\sin\frac{n\pi x}{80}\,dx\right] \\
&= \frac{2}{80}\left[\left(\frac{80}{n\pi}40\sin\frac{n\pi}{2}\right) + \frac{80}{n\pi}\frac{80}{n\pi}\left(\cos\frac{n\pi x}{80}\right)_0^{40}\right] \\
&+ \frac{2}{80}\left[-\left(\frac{80}{n\pi}40\sin\frac{n\pi}{2}\right) - \frac{80}{n\pi}\frac{80}{n\pi}\left(\cos\frac{n\pi x}{80}\right)_{40}^{80}\right] \\
&= \frac{2}{80}\,\frac{80}{n\pi}\,\frac{80}{n\pi}\left[\left(\cos\frac{n\pi x}{80}\right)_0^{40} - \left(\cos\frac{n\pi x}{80}\right)_{40}^{80}\right] \\
&= \frac{160}{n^2\pi^2}\;\left[\cos\frac{n\pi}{2} - 1 - \cos n\pi + \cos\frac{n\pi}{2}\right] = \frac{160}{n^2\pi^2}\;\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]
\end{aligned}$$

In this case, unlike in the earlier case where in $u \to 0$ as $n \to \infty$, $u \to 20$ as $n \to \infty$ due to the insulation of the end points of the rod (heat can't escape from the end points).

9.2.2 Problems to Workout

1. Determine the steady-state solution of $u_t = u_{xx}, \quad u(0,t) = T_0, u_x(1,t) = 0, t > 0$, where T_0 is a constant.

2. Find $u(x,t)$, satisfying $u_t = u_{xx}, \quad u(x,0) = x, \quad u_x(0,t) = u_x(1,t) = 0$ and $|u|$ is bounded.

3. Find $u(x,t)$, satisfying $u_t = u_{xx}, \quad u(x,0) = x - x^2, \quad u_x(0,t) = u_x(1,t) = 0$ and $|u|$ is bounded.

4. Find $u(x,t)$, satisfying $u_t = u_{xx}, \quad u(x,0) = x - x^2, \quad u(0,t) = T_0, \; u_x(1,t) = 0$ and $|u|$ is bounded.

9.3 Heat Equation with non-homogenous Boundary Conditions

$$\begin{array}{rll} \text{PDE} & u_t = a^2 u_{xx} & 0 < x < L, t > 0 \\ \text{Initial Condition} & u(x,0) = f(x) & 0 \le x \le L \\ \text{Boundary Conditions} & u(0,t) = T_1, u(L,t) = T_2 & t > 0 \end{array} \tag{9.5}$$

where T_1 and T_2 are some constants.

Solution : If $u(x,t) = \phi(x,t) + \psi(x)$ is the solution of the given heat equation such that $\psi(x)$ is the solution of the corresponding steady state equation $\psi_{xx} = 0$ with non-homogeneous boundary conditions $\psi(0,t) = T_1, \psi(L,t) = T_2$ then ϕ is the solution of $u_t = a^2 u_{xx}, u(x,0) = f(x) - \psi(x), u(0,t) = 0, u(L,t) = 0$.

Solving the steady equation for ψ gives, $\psi(x) = ax + b$, applying the boundary conditions $\psi(0,t) = T_1, \psi(L,t) = T_2$, gives $b = T_1$ and $a = \frac{T_2 - T_1}{L}$, that is

$$\psi(x) = \frac{T_2 - T_1}{L}\, x + T_1$$

The ϕ is the solution of

$$u_t = 2u_{xx}, u(x,0) = f(x) - \left(\frac{T_2 - T_1}{L}\, x + T_1 \right), u(0,t) = 0, u(L,t) = 0$$

Therefore,

$$\begin{aligned} \phi(x,t) &= \sum_{n=1}^{\infty} A_n \, \sin \frac{n\pi x}{L} e^{-a^2 \lambda_n^2 t}, \qquad \lambda_n = \frac{n\pi}{L} \\ A_n &= \frac{2}{L} \int_0^L \left[f(x) - \left(\frac{T_2 - T_1}{L}\, x + T_1 \right) \right] \sin \frac{n\pi x}{L} \, dx \end{aligned}$$

9.3.1 Numerical Example

1. Solve $u_t = 2u_{xx}, \quad u(x,0) = 25, \quad u(0,t) = 10, u(3,t) = 40$ and $|u|$ is bounded.

   Solution : If $u(x,t) = \phi(x,t) + \psi(x)$ is the solution of the given heat equation such that $\psi(x)$ is the solution of the corresponding steady state equation then

   $$\psi(x) = \frac{40-10}{3}x + 10$$

   Further, ϕ must be the solution of

   $$u_t = 2u_{xx}, \quad u(x,0) = 25 - (10x+10), u(0,t) = 0, u(L,t) = 0$$

   Therefore

   $$\begin{aligned} \phi(x,t) &= \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{3} e^{-2\lambda^2 t}, \qquad \lambda = \frac{n\pi}{3} \\ A_n &= \frac{2}{3}\int_0^3 [15-10x]\sin\frac{n\pi x}{3}\,dx = \frac{30}{n\pi}(1+\cos n\pi) = \frac{30}{n\pi}(1+(-1)^n) \end{aligned}$$

   Finally, the solution of the given problem is

   $$u(x,t) = 10x + 10 + \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{3} e^{-2\lambda^2 t}, \qquad \lambda = \frac{n\pi}{3}, \qquad A_n = \frac{30}{n\pi}(1+(-1)^n)$$

2. The ends of a rod of 10 cm length have the temperature at 30^oC and 70^oC until steady state prevails. Then the temperature of the ends are changed to 40^oC and 60^oC. Find the temperature distribution in the rod at any time t (Assume the thermal diffusivity of the rod as 1).

   Solution : The steady state temperature distribution is $u_s = ax + b$, with boundary conditions $u_s(0) = 30$ and $u_s(10) = 70$, therefore, $u_s = 4x + 30$.
   u_s becomes the initial temperature distribution of the unsteady problem, that is, $u(x,0) = 4x + 30$.

   If $u(x,t) = \phi(x,t) + \psi(x)$ is the solution of the given heat equation with non-homogenous equation such that $\psi(x)$ is the solution of the corresponding steady state equation then

   $$\psi(x) = \frac{60-40}{10}x + 40$$

   Further, ϕ must be the solution of

   $$u_t = u_{xx}, \quad u(x,0) = (4x + 30 - (2x+40)), u(0,t) = 0, u(10,t) = 0$$

Therefore

$$\begin{aligned}
\phi(x,t) &= \sum_{n=1}^{\infty} A_n \,\sin\frac{n\pi x}{10} e^{-\lambda_n^2 t}, \qquad \lambda_n = \frac{n\pi}{10} \\
A_n &= \frac{2}{10}\int_0^{10} [2x-10]\sin\frac{n\pi x}{10}\,dx \;=\; -\frac{2}{n\pi}(1+10\cos n\pi)
\end{aligned}$$

Finally, the solution of the given problem is

$$u(x,t) = 2x+40+\sum_{n=1}^{\infty} A_n \,\sin\frac{n\pi x}{10} e^{-\lambda_n^2 t}, \quad \lambda_n = \frac{n\pi}{10}, \quad A_n = -\frac{2}{n\pi}(1+10\cos n\pi)$$

9.3.2 Problems to Workout

1. Find $u(x,t)$, satisfying $u_t = u_{xx}$, $u(x,0) = 0$, $u(0,t) = 20, u(1,t) = 10$ and $|u|$ is bounded.

2. The ends of a rod of 10 cm length with insulated sides has its end A and B maintained at temperatures have the temperature at 50^oC and 100^oC, respectively, until steady state conditions prevails. Then the temperature of A is suddenly raised to 90^oC and at the same time the that at$B$ is lowered to 60^oC. Find the temperature distribution in the rod at any time t (Assume the thermal diffusivity of the rod as 1).

3. The temperature at one end of a rod of length 50 cm, with insulated sides, is kept at 0^oC and that the other end is kept at 100^oC until steady state conditions prevails. The two ends are then suddenly insulated so that the temperature gradient is zero at the ends thereafter. Find the temperature distribution in the rod at any time t (Assume the thermal diffusivity of the rod as 1).