M.T.Nair January 19, 2012

MA-6110: Topics in Advanced Analysis

Assignment Sheet - I

1. Let (a_n) and (b_n) be sequences in $[-\infty, \infty]$. Prove the following:

   (i) $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty}(a_n)$

   (ii) $a_n \leq b_n \,\forall\, n \in \mathbb{N} \implies \liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Corresponding to a partition P of $[a, b]$ let $L(P, f)$ and $U(P, f)$ denote the lower sum and upper sum. Show that for any two partitions P and Q of $[a, b]$, $L(P, f) \leq U(Q, f)$.

3. Prove the following:

   (a) Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

   (b) Every bounded function $f : [a, b] \to \mathbb{R}$ having atmost a finite number of discontinuities is Riemann integrable.

   (c) Every monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

4. Give an example of a sequence (f_n) of Riemann integrable functions on $[a, b]$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in [a, b]$ for some $f : [a, b] \to \mathbb{R}$, but f is Riemann integrable.

5. Show that if (I_n) is a sequence of open intervals, then $m^*(\bigcup I_n) \leq \sum_n \ell(I_n)$.

6. If $E \subseteq \mathbb{R}$, then prove that for every $\epsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m^*(G) \leq m^*(E) + \epsilon$.

7. Show that every non-degenerate interval is an uncountable set.

8. Show that the relation $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$ need not hold for every disjoint subsets A_1 and A_2 of $\mathbb{R}$.

9. Show that, if $m^*(A_1 \cup A_2) = m^*(A_1) + m^*(A_2)$ holds for any two disjoint subsets A_1 and A_2 of $\mathbb{R}$, then $m^*\left(\bigcup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty m^*(A_n)$ holds for any denumerable disjoint family $\{A_n\}_{n=1}^\infty$ of subsets of $\mathbb{R}$.

10. Prove that there exists $E \subseteq \mathbb{R}$ such that $E \notin \mathcal{M}$.

11. Assuming that $(a, \infty) \in \mathcal{M}$ for every $a \in \mathbb{R}$, show that every G_δ subset of $\mathbb{R}$ and every F_σ subset of $\mathbb{R}$ belongs to $\mathcal{M}$.

12. If $E \in \mathcal{M}$ with $m(E) < \infty$ then show that for every $\epsilon > 0$, there exists an open set $G \subseteq \mathbb{R}$ such that $G \supseteq E$ and $m(G \setminus E) < \epsilon$.

13. If $E \in \mathcal{M}$ with $m(E) < \infty$ then show that there exists a G_δ-set $G \supseteq E$ and F_σ-set $F \subseteq E$ such that $m(G \setminus E) = 0$ and $m(E \setminus F) = 0$.

14. Suppose $(X, \mathcal{A})$ is a measurable space and $X_0 \in \mathcal{A}$. Then show that

    (i) $\mathcal{A}_0 := \{E \subseteq X_0 : E \in \mathcal{A}\}$ is a σ-algebra on X_0, and

    (ii) $\mathcal{A}_0 = \{A \cap X_0 : A \in \mathcal{A}\}$.

15. Let X be a set and $\mu^* : 2^X \to [0, \infty]$ satisfies

    (a) $\mu^*(\varnothing) = 0$ and

    (b) $\mu^*\Big(\bigcup_n A_n\Big) \leq \sum_n \mu^*(A_n)$ for every countable disjoint family $\{A_n\}$ in 2^X.

    Prove that

    (i) $\mathcal{A} := \{E \subseteq X : \forall\, A \subseteq X,\ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$ is a σ-algebra on X,

    (ii) $\mu := \mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$.

16. Let μ^* be as in Problem 15. Prove that

$$A \subseteq X, \quad \mu^*(A) = 0 \implies A \in \mathcal{A}.$$

17. Let μ be as in Problem 15. Prove that

$$A \in \mathcal{A}, \quad \mu(A) = 0, \quad E \subseteq A \implies E \in \mathcal{A}.$$

18. Show that Lebesgue measure has the property described in Problem 16.

    This property is called **completeness of** μ.

19. Let $(X, \mathcal{A}, \mu)$ be a measure space and

$$\widetilde{\mathcal{A}} := \{E \subseteq X : \exists A, B \in \mathcal{A} \text{ such that } A \subseteq E \subseteq B,\ \mu(B \setminus A) = 0\}.$$

    For $E \in \widetilde{\mathcal{A}}$, let $A, B \in \mathcal{A}$ be such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Define $\tilde{\mu}(E) = \mu(A)$.

    (i) Prove that $\widetilde{\mathcal{A}}$ is a σ-algebra.

    (ii) Prove that, $\tilde{m}$ is a well-defined function on $\widetilde{\mathcal{A}}$ and it is a measure on $\widetilde{\mathcal{A}}$.

    (iii) Prove that $\tilde{\mu}$ is a complete measure (as defined in Problem Prob-gen-meas-2.)

20. Recall that if $(A, \mathcal{A}, \mu)$ be a measure space and if (A_n) is a sequence in $\mathcal{A}$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(\bigcap_{k=1}^{\infty} A_k)$. Give an example to show that the condition "$\mu(A_1) < \infty$" cannot be dropped.