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MA-6110: Topics in Advanced Analysis

Assignment Sheet - II

1. Prove that for $E \subseteq X$, $E \in \mathcal{A}$ if and only if χ_E is a measurable function.

2. Prove that if f is a real valued function on $(X, \mathcal{A})$ such that $\{x : f(x) \geq r\}$ for every rational number r, then f is measurable.

3. Prove that if f and g are real valued measurable functions on $(X, \mathcal{A})$, then the sets $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

4. Prove that if (f_n) is a sequence of measurable functions on $(X, \mathcal{A})$, then the set $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

5. Let $f : X \to [-\infty, \infty]$. Prove: If f is measurable, then f^+ and f^- are measurable. In case f is real valued, then f is measurable iff f^+ and f^- are measurable, and in that case $|f|$ is also measurable.

6. Let X be an uncountable set and $\mathcal{A} \subseteq 2^X$ such that $A \in \mathcal{A}$ if and only if either A or A^c is atmost countable. Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Show that

   (i) $\mathcal{A}$ is a σ-algebra and μ is a measure.

   (ii) Describe measurable functions on $(X, \mathcal{A})$ and their integrals w.r.t. μ.

7. Suppose $X = \{x_1, x_2, \ldots\}$ with the counting measure μ. If f is an extended real valued non-negative measurable function on X, then show that $\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i)$.

8. Suppose $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Then show that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

9. Suppose $X = \{x_1, \ldots, x_n\}$, and $w_1, \ldots, w_n$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x\in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X, $\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i$.

10. Suppose $X = \{x_1, x_2, \ldots\}$, and $w_1, w_2, \ldots$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x\in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X, $\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i$.

11. Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on $(X, \mathcal{A}, \mu)$ such that $f_1 \geq f_2 \geq \ldots$ and $f_n(x) \to f(x)$ for every $x \in X$. If $\int_X f d\mu < \infty$, then show that $\int_X f_n d\mu \to \int_X f d\mu$. Show that the condition that $\int_X f d\mu < \infty$ cannot be dropped.

12. If $f \in \mathcal{L}(\mu)$ such that $\int_X f \geq 0$, then show that

$$\int_X f = \int_X \text{Re} f \leq \int_X |f|.$$

13. Show that $\mathcal{L}(\mu)$ is a vector space over $\mathbb{C}$, and the map $f \mapsto \int_X f$ is a linear functional on $\mathcal{L}(\mu)$.

14. Show that the map $f \mapsto \int_X |f|$ is a semi-norm on the vector space $\mathcal{L}(\mu)$.

15. Show that the set $\mathcal{N} := \{f \in \mathcal{L}(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}(\mu)$, and the map $[f] \mapsto \int_X |f|$ is a norm on the quotient space $\mathcal{L}(\mu)/\mathcal{N}$.

16. If $f \in \mathcal{L}(\mu)$ such that $\left|\int_X f\right| = \int_X |f|$, then show that there exists $c \in \mathbb{C}$ such that $f(x) = c|f(x)|$ for almost all $x \in X$.

    Hint: Write $\int_X f$ as $\int_X f = \left|\int_X f\right| e^{i\theta}$.

17. Suppose f and g are complex measurable functions such that $f = 0$ a.e. on X and $f = 0$ a.e. on X. Show that $f + g = 0$ a.e. on X.

18. Suppose $f \in \mathcal{L}(\mu)$ such that $\int_E f = 0$ for all $E \in \mathcal{A}$. Show that $f = 0$ a.e.

    Hint: First observe that it is enough to prove for the case of real valued f, and then take $E = \{x \in X : f(x) \geq 0\}$ and show that $\int_X f^+ = 0$. Similarly show that $\int_X f^- = 0$.

19. Suppose E is a Lebesgue measurable subset of $\mathbb{R}^1$. Show that there exists a Borel subset A of $\mathbb{R}^1$ such that $\chi_A = \chi_E$ a.e.