M.T.Nair March 2012

MA-6110: Topics in Advanced Analysis

Assignment Sheet - III

1. For complex valued measurable functions f on a measure space $(X, \mathcal{A}, \mu)$ and for $1 \le p \le \infty$, let

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p d\mu \right)^{1/p} & \text{if } 1 \le p < \infty \\ \inf\{c > 0 : |f| \le c \quad \text{a.e.}\} & \text{if } p = \infty. \end{cases}$$

Let $\mathcal{L}^p(\mu)$ be the set of all complex valued measurable functions f on $(X, \mathcal{A}, \mu)$ such that $\|f\|_p < \infty$. Show that

(i) $\mathcal{L}^p(\mu)$ is a vector space,

(ii) $f \mapsto \|f\|_p$ is a semi-norm on $\mathcal{L}^p(\mu)$.

(iii) $\mathcal{N} := \{f \in \mathcal{L}^p(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}^p(\mu)$,

(iv) $[f] \mapsto \int_X |f|$ is a norm on the quotient space $L^p(\mu) := \mathcal{L}(\mu)/\mathcal{N}$.

(v) $L^p(\mu)$ is a Banach space w.r.t. $\|\cdot\|_p$.

2. Realize the spaces $\mathcal{L}^p(\mu)$ and $L^p(\mu)$ in the following cases:

(a) $X = \mathbb{N}$, $X = \mathbb{Z}$ with counting measure on 2^X.

(b) $X = \{1, \ldots, k\}$ with counting measure on 2^X.

(c) $X = [0, 1]$ with Lebesgue measure.

3. If $\mu(X) < \infty$, then show that for $1 \le p \le r \le \infty$, $L^\infty(\mu) \subseteq L^r(\mu) \subseteq L^p(\mu) \subseteq L^1(\mu)$, and if $X = \mathbb{N}$ or $\mathbb{Z}$, then $L^\infty(\mu) \supseteq L^r(\mu) \supseteq L^p(\mu) \supseteq L^1(\mu)$.

4. Show that every Cauchy sequence in $L^p(\mu)$ for $1 \le p < \infty$ has a subsequence which converges a.e. to a function in $L^p(\mu)$. In particular, if f_n and f are in $L^p(\mu)$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$, then a subsequence (f_{k_n}) which converges a.e. to f.

5. Let $\mathcal{S}$ be the set of all step functions on $\mathbb{R}^1$. Show that $\mathcal{S}$ is dense in $L^p(\mathbb{R}^1)$ for $1 \le p < \infty$.

6. Let $\varphi : (a, b) \to \mathbb{R}$. Show that the following conditions on φ are equivalent:

(i) $\varphi((1-\lambda)x + \lambda y)`(1-\lambda)\varphi(x) + \lambda\varphi(y) \quad \forall\, x, y \in (a, b)$ and $\forall\, \lambda \in (0, 1)$.

(ii) $\frac{\varphi(t)-\varphi(s)}{t-s} \le \frac{\varphi(u)-\varphi(t)}{u-t} \quad \forall s, t, u$ such that $a < s < t < u < b$.

7. (Jensen's inequality) Suppose that $(\Omega, \mathcal{A}, \mu)$ be a probability measure space and $f \in L^1(\mu)$ is real valued with $a < f(x) < b$ for all $x \in \Omega$. If $\varphi : (a, b) \to \mathbb{R}$ is convex, the show that

$$\varphi\left(\int_\Omega f d\mu\right) = \int_X (\varphi \circ f) d\mu.$$

8. If $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, then show that $\int_\mathbb{R} |f(x+h) - f(x)|^p dx \to 0$ as $h \to 0$.

9. Let $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$. For each $y \in \mathbb{R}$, let $f_y(x) := f(x - y)$ a.a. $x \in \mathbb{R}$. Show that the map $f_y \in L^p(\mathbb{R})$ for all $y \in \mathbb{R}$, and

   (i) the map $f \mapsto \|f\|_p$ is a linear isometry from $L^p(\mathbb{R})$ into itself,

   (ii) $y \mapsto f_y$ is uniformly continuous from $\mathbb{R}$ into $L^p(\mathbb{R})$.

10. For $f, g \in L^1(\mathbb{R})$, show that

   (i) the function $(x, y) \mapsto f(x - y)g(y)$ is measurable on $\mathbb{R}^2$, and let $(f * g)(x) := \int_\mathbb{R} f(x - y)g(y)dy$ is well-defined a.e. on $\mathbb{R}$,

   (ii) $f * f \in L^1(\mathbb{R})$,

   (iii) $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

11. For $f \in L^1(\mathbb{R})$, let $\hat{f}(t) := \int_\mathbb{R} f(x)e^{-itx}dx$. Show that $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$.