M.T.Nair March 26, 2012

MA-6110: Topics in Advanced Analysis

Assignment Sheet - IV

In the following X denotes a locally compact Hausdorff space.

1. Suppose K_1 and K_2 are disjoint nonempty compact subsets of X. Justify the statement: There exists f_1 and f_2 in $C_c(X)$ such that $f_i(x) = \begin{cases} 1, & x \in K_i, \\ 0, & x \in K_j. \end{cases}$ for $i, j = 1, 2$ and $i \neq j$.

2. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional and $\mu : 2^X \to [0, \infty]$ satisfy
$$\begin{aligned} \mu(V) &= \sup\{\Lambda f : f \prec V\} \quad \text{for open sets } V \subseteq X, \\ \mu(E) &= \inf\{\mu(V) : E \subseteq V\} \quad \text{for every } \ E \subseteq X. \end{aligned}$$
Prove the following:
   (a) If μ is countably sub-additive on the topology on X, then μ is countably sub-additive on 2^X.
   (b) If K is compact and $K \prec f$, then $\mu(K) \leq \Lambda f$.
   (c) If K is compact, then $\mu(K) < \infty$.
   (d) If V is an open set, then $\mu(V) = \sup\{\mu(K) : K \subseteq V,\ K \text{ compact}\}$.

3. Let μ be a measure on a σ-algebra $\mathcal{A} \supseteq \mathcal{B}_X$ and $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. Suppose μ and Λ satisfy
$$\mu(E) = \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\} \quad \forall\, E \subseteq X,$$
$$f \in C_c(X),\ V \text{ open and } \ f \prec V \Longrightarrow \Lambda f \leq \mu(V).$$
Show that for any $f \in C_c(X)$,
$$\Lambda f = \int_X f d\mu.$$

4. Let $\mathcal{A}$ be a σ-algebra containing $\mathcal{B}_X$. Prove that there exists only one measure μ on $\mathcal{A}$ satisfying the following properties:
$$\begin{aligned} \mu(E) &= \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\} \quad \forall\, E \in \mathcal{A}, \\ \mu(V) &= \sup\{\mu(K) : K \subseteq V,\ K \text{ compact}\} \quad \text{for open } V \subseteq X. \end{aligned}$$

5. Let $\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ be defined by $\Lambda f := \int_{a_f}^{b_f} f(x)dx, \ f \in C_c(\mathbb{R})$, where a_f and b_f are such that $\text{supp}(f) \subseteq [a_f, b_f]$, and let μ be the measure obtained as in Riesz representation theorem corresponding to the positive linear functional Λ. Is μ the Lebesgue measure? Why?

6. Let $X = \mathbb{N}$ with usual topology and $\Lambda f := \sum_{j=1}^{\infty} f(j)w_j$ for $f \in C_c(\mathbb{N})$, where (w_n) is a sequence of positive real numbers. Determine the σ-algebra and measure obtained as in Riesz representation theorem.

7. Let $\mathcal{M}$ and μ be as in Riesz representation theorem and X, in addition, be σ-compact. Prove the following:

   (a) For every $E \in \mathcal{M}$ and $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that $\mu(E) < \mu(K) + \epsilon$.

   (b) For every $E \in \mathcal{M}$ and $\epsilon > 0$, there exist an open set $V \supseteq E$ and a closed set $F \subseteq E$ such that
   $$\mu(V \setminus E) < \epsilon, \qquad \mu(E \setminus F) < \epsilon.$$

8. Let μ be as in Riesz representation theorem and $1 \leq p < \infty$. Let $\mathcal{S}_p$ be the set of all simple measurable functions in $L^p(\mu)$. Prove that $\mathcal{S}_p$ is dense in $L^p(\mu)$.

9. Prove that $C_c(X)$ is dense in $L^p(\mu)$ w.r.t. the norm $\|\cdot\|_p$ for $1 \leq p < \infty$.

10. Prove that $C_c(X)$ is dense in $C_0(X)$ w.r.t. the norm $\|\cdot\|_\infty$.