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MA-6110: Topics in Advanced Analysis

Assignment Sheet

1 Basic measure and integration

1. Let μ^* be an outer measure on X. Show that, if $\mu^*(A_1 \cup A_2) = \mu^*(A_1) + \mu^*(A_2)$ holds for any two disjoint subsets A_1 and A_2 of X, then $\mu^*\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty \mu^*(A_n)$ holds for any denumerable disjoint family $\{A_n\}_{n=1}^\infty$ of subsets of X.

2. Let X be a set and $\mu^* : 2^X \to [0, \infty]$ satisfies

   (a) $\mu^*(\varnothing) = 0$ and

   (b) $\mu^*\Big(\bigcup_n A_n\Big) \le \sum_n \mu^*(A_n)$ for every countable family $\{A_n\}$ in 2^X.

   Prove that

   (i) $\mathcal{A} := \{E \subseteq X : \forall\, A \subseteq X,\ \mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)\}$ is a σ-algebra on X,

   (ii) $\mu := \mu^*|_{\mathcal{A}}$ is a measure on $\mathcal{A}$.

3. Let μ^* be as in Problem 2. Prove that

   (a) $A \subseteq X, \quad \mu^*(A) = 0 \implies A \in \mathcal{A}$.

   (b) $A \in \mathcal{A}, \quad \mu(A) = 0, \quad E \subseteq A \implies E \in \mathcal{A}$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space and

   $$\widetilde{\mathcal{A}} := \{E \subseteq X : \exists A, B \in \mathcal{A} \text{ such that } A \subseteq E \subseteq B,\ \mu(B \setminus A) = 0\}.$$

   For $E \in \widetilde{\mathcal{A}}$, let $A, B \in \mathcal{A}$ be such that $A \subseteq E \subseteq B$ and $\mu(B \setminus A) = 0$. Define $\tilde{\mu}(E) = \mu(A)$.

   (a) Prove that $\widetilde{\mathcal{A}}$ is a σ-algebra.

   (b) Prove that, $\tilde{m}$ is a well-defined function on $\widetilde{\mathcal{A}}$ and it is a measure on $\widetilde{\mathcal{A}}$.

   (c) Prove that $\tilde{\mu}$ satisfies the property in Problem 3(b).

5. Recall that if $(X, \mathcal{A}, \mu)$ be a measure space and if (A_n) is a sequence in $\mathcal{A}$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(A_1) < \infty$, then $\mu(A_n) \to \mu(\bigcap_{k=1}^\infty A_k)$.
   Give an example to show that the condition "$\mu(A_1) < \infty$" cannot be dropped.

6. Let $(X, \mathcal{A})$ be a measurable space. Prove that for $E \subseteq X$, $E \in \mathcal{A}$ if and only if χ_E is a measurable function.

7. Prove that if f is a real valued function on a measurable space $(X, \mathcal{A})$ such that $\{x : f(x) \geq r\}$ for every rational number r, then f is measurable.

8. Prove that if f and g are real valued measurable functions on $(X, \mathcal{A})$, then the sets $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

9. Prove that if (f_n) is a sequence of measurable functions on $(X, \mathcal{A})$, then the set $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

10. Let $f : X \to [-\infty, \infty]$. Prove: If f is measurable, then f^+ and f^- are measurable. In case f is real valued, then f is measurable iff f^+ and f^- are measurable, and in that case $|f|$ is also measurable.

11. Let X be an uncountable set and $\mathcal{A} \subseteq 2^X$ such that $A \in \mathcal{A}$ if and only if either A or A^c is atmost countable. Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(A) = 0$ if A is countable and $\mu(A) = 1$ if A is uncountable. Show that

    (i) $\mathcal{A}$ is a σ-algebra and μ is a measure.

    (ii) Describe measurable functions on $(X, \mathcal{A})$ and their integrals w.r.t. μ.

12. Suppose $X = \{x_1, x_2, \ldots\}$ with the counting measure μ. If f is an extended real valued non-negative measurable function on X, then show that $\int_X f d\mu = \sum_{i=1}^{\infty} f(x_i)$.

13. Suppose $a_{ij} \geq 0$ for all $i, j \in \mathbb{N}$. Then show that $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

14. Suppose $X = \{x_1, \ldots, x_n\}$, and $w_1, \ldots, w_n$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x\in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X, $\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i$.

15. Suppose $X = \{x_1, x_2, \ldots\}$, and $w_1, w_2, \ldots$ are non-negative reals. For $x \in X$, define $w(x) = w_j$ whenever $x = x_j$, and $\mu(E) = \sum_{x\in E} w(x)$ for $E \subseteq X$. Show that μ is a measure on $(X, 2^X)$, and for every extended real valued non-negative measurable function f on X, $\int_X f d\mu = \sum_{i=1}^{n} f(x_i) w_i$.

    Let $(X, \mathcal{A}, \mu)$ be a measure space.

16. Suppose (f_n) is a sequence of extended real valued non-negative measurable functions on X such that $f_1 \geq f_2 \geq \ldots$ and $f_n(x) \to f(x)$ for every $x \in X$. If $\int_X f d\mu < \infty$, then show that $\int_X f_n d\mu \to \int_X f d\mu$. Show that the condition that $\int_X f d\mu < \infty$ cannot be dropped.

17. Let $\mathcal{L}(\mu)$ be the vector space of all measurable $f : X \to \mathbb{C}$ such that $\int_X |f| d\mu < \infty$. Show that

(a) $\mathcal{N} := \{f \in \mathcal{L}(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}(\mu)$, and

(b) $[f] \mapsto \int_X |f|$ is a norm on the quotient space $\mathcal{L}(\mu)/\mathcal{N}$.

18. Suppose f and g are complex measurable functions such that $f = 0$ a.e. on X and $f = 0$ a.e. on X. Show that $f + g = 0$ a.e. on X.

19. Suppose $f \in \mathcal{L}(\mu)$ such that $\int_E f = 0$ for all $E \in \mathcal{A}$. Show that $f = 0$ a.e.

    Hint: First observe that it is enough to prove for the case of real valued f, and then take $E = \{x \in X : f(x) \geq 0\}$ and show that $\int_X f^+ = 0$. Similarly show that $\int_X f^- = 0$.

20. Suppose E is a Lebesgue measurable subset of $\mathbb{R}^1$. Show that there exists a Borel subset A of $\mathbb{R}^1$ such that $\chi_A = \chi_E$ a.e.

21. For complex valued measurable functions f on X and for $1 \leq p \leq \infty$, let

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p d\mu\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \inf\{c > 0 : |f| \leq c \quad \text{a.e.}\} & \text{if } p = \infty. \end{cases}$$

    Let $\mathcal{L}^p(\mu)$ be the set of all complex valued measurable functions f on $(X, \mathcal{A}, \mu)$ such that $\|f\|_p < \infty$. Show that

    (a) $\mathcal{L}^p(\mu)$ is a vector space,

    (b) $f \mapsto \|f\|_p$ is a semi-norm on $\mathcal{L}^p(\mu)$.

    (c) $\mathcal{N} := \{f \in \mathcal{L}^p(\mu) : \int_X |f| = 0\}$ is subspace of the vector space $\mathcal{L}^p(\mu)$,

    (d) $[f] \mapsto \int_X |f|$ is a norm on the quotient space $L^p(\mu) := \mathcal{L}(\mu)/\mathcal{N}$.

    (e) $L^p(\mu)$ is a Banach space w.r.t. $\|\cdot\|_p$.

22. Realize the spaces $\mathcal{L}^p(\mu)$ and $L^p(\mu)$ in the following cases:

    (a) $X = \mathbb{N}$, $X = \mathbb{Z}$ with counting measure on 2^X.

    (b) $X = \{1, \ldots, k\}$ with counting measure on 2^X.

    (c) $X = [0, 1]$ with Lebesgue measure.

23. If $\mu(X) < \infty$, then show that for $1 \leq p \leq r \leq \infty$,

    (a) $L^\infty(\mu) \subseteq L^r(\mu) \subseteq L^p(\mu) \subseteq L^1(\mu)$, and

    (b) if $X = \mathbb{N}$ or $\mathbb{Z}$, then $L^\infty(\mu) \supseteq L^r(\mu) \supseteq L^p(\mu) \supseteq L^1(\mu)$.

24. Show that if (f_n) is a Cauchy sequence in $L^p(\mu)$ for $1 \leq p < \infty$, then there exists $f \in L^p(\mu)$ and a subsequence (f_{k_n}) such that $f_{k_n} \to f$ a.e.

2 Signed measures

Let $(X, \mathcal{A})$ be a measurable space.

1. If μ is a measure on $(X, \mathcal{A})$ and $f : X \to \mathbb{R}$ is measurable such that $\int_X f\, d\mu$ is well-defined, show that $f \mapsto \nu(E) = \int_E f\, d\mu$ is a signed measure.

2. If ν is a signed measure on $(X, \mathcal{A})$ and $A \in \mathcal{A}$ is a positive set, then show that $E \mapsto \nu(E \cap A)$ is a (positive) measure on $(X, \mathcal{A})$.

3. Let ν be a signed measure on $(X, \mathcal{A})$ and $\{E_n : n \in \mathbb{N}\}$ is a disjoint family in $\mathcal{A}$ such that $|\nu(\bigcup_{n=1}^{\infty} E_n)| < \infty$, then show that $\sum_{n=1}^{\infty} \nu(E_n)$ is absolutely convergent.

4. Let ν be a signed measure on $(X, \mathcal{A})$. If $A \in \mathcal{A}$ is such that $|\nu(A)| < \infty$, then show that $|\nu(B)| < \infty$ for evert $B \in \mathcal{A}$ with $B \subseteq A$.

5. Let ν be a signed measure on $(X, \mathcal{A})$. If $E_n \in \mathcal{A}$ is such that $E_n \supseteq E_{n+1}$ for every $n \in \mathbb{N}$ and $|\nu(E_1)| < \infty$, show that $\nu(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \nu(E_n)$.

6. Show that signed measures ν_1 and ν_2 are mutually singular[1] if and only if there exists a measurable decomposition (A, B) of X such that

$$\nu_1(E) = \nu_1(E \cap A) \quad \text{and} \quad \nu_2(E) = \nu_2(E \cap B) \quad \forall\, E \in \mathcal{A}.$$

7. Show that measures μ_1 and μ_2 are mutually singular if and only if there exists a measurable decomposition (A, B) of X such that $\mu_1(B) = 0 = \mu_2(A)$.

8. Let ν be as in Problem 1. Find Hahn-decomposition and Jordan decomposition of ν.

9. Show that ν is finite[2] (resp. σ-finite[3]) if and only if $|\nu|$ is finite (resp. σ-finite).

10. Show that ν_1 and ν_2 on a measurable space $(X, \mathcal{A})$ are mutually singular if and only if there exists a measurable decomposition (A, B) of X such that ν_1 is concentrated[4] on B and ν_2 is concentrated on A.

11. Let ν be a signed measure and (A, B) be a Hahn decomposition of ν. Show that

    (a) A is positive for ν and B is negative for ν,

    (b) ν^+ is concentrated on A and ν^- is concentrated on B.

    (c) $\nu^+ \perp \nu^-$.

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[1] ν_1 and ν_2 are said to be **mutually singular**, written as $\nu_1 \perp \nu_2$, if there exists a measurable decomposition (A, B) of X such that $\nu_1(E \cap B) = 0 = \nu_2(E \cap A)$ for all $E \in \mathcal{A}$.

[2] ν is said to be **finite** if $|\nu(X)| < \infty$.

[3] ν is said to be **σ-finite** if $X = \bigcup_{n=1}^{\infty} X_n$ with $X_n \in \mathcal{A}$ and $|\nu(X_n)| < \infty\ \forall n \in \mathbb{N}$.

[4] ν is said to be **concentrated** on a set $A \in \mathcal{A}$ if $\nu(E) = \nu(E \cap A)$ for all $E \in \mathcal{A}$.

3 Measure on a locally compact Hausdorff space

In the following, unless otherwise specified, X denotes a locally compact Hausdorff space. Also, for compact K, open V in X, and $f \in C_c(X)$ we denote

- $K \prec f$ iff $0 \leq f \leq 1$ and $f = 1$ on K and
- $f \prec V$ iff $0 \leq f \leq 1$ and $\text{supp} f \subseteq V$.

1. Let X be a Hausdorff space and K be compact in X. Show that for every $x \in X \setminus K$, there exist disjoint open sets U and V such that $x \in U$ and $K \subseteq V$.
2. Let K be compact in X and U be open in X such that $K \subseteq U$. Then there exists an open set V such that $\overline{V}$ compact and $K \subseteq V \subseteq \overline{V} \subseteq U$.
3. Show that
   (a) If $\{f_\alpha : \alpha \in J\}$ is a family of l.s.c. functions then the function $f := \sup_{\alpha \in J} f_\alpha$ is also l.s.c.
   (b) If $\{g_\alpha : \alpha \in J\}$ is a family of u.s.c. function then the function $g := \inf_{\alpha \in J} g_\alpha$ is also u.s.c.
4. Suppose K_1 and K_2 are disjoint nonempty compact subsets of X. Justify the statement: There exists f_1 and f_2 in $C_c(X)$ such that
$$f_1(x) = \begin{cases} 1, & x \in K_1, \\ 0, & x \in K_2. \end{cases} \qquad f_2(x) = \begin{cases} 1, & x \in K_2, \\ 0, & x \in K_1. \end{cases}$$
5. Let $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional and $\mu : 2^X \to [0, \infty]$ satisfy
$$\begin{aligned} \mu(V) &= \sup\{\Lambda f : f \prec V\} \quad \text{for open sets } V \subseteq X, \\ \mu(E) &= \inf\{\mu(V) : E \subseteq V\} \quad \text{for every } \ E \subseteq X. \end{aligned}$$
   Prove the following:
   (a) If μ is countably sub-additive on the topology on X, then μ is countably sub-additive on 2^X.
   (b) If K is compact and $K \prec f$, then $\mu(K) \leq \Lambda f$.
   (c) If K is compact, then $\mu(K) < \infty$.
   (d) If V is an open set, then $\mu(V) = \sup\{\mu(K) : K \subseteq V,\ K \text{ compact}\}$.

6. Let μ be a measure on a σ-algebra $\mathcal{A} \supseteq \mathcal{B}_X$ and $\Lambda : C_c(X) \to \mathbb{C}$ be a positive linear functional. Suppose μ and Λ satisfy

$$\mu(E) = \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\} \quad \forall E \subseteq X,$$

$$f \in C_c(X),\ V \text{ open and } f \prec V \Longrightarrow \Lambda f \leq \mu(V).$$

Show that for any $f \in C_c(X)$,

$$\Lambda f = \int_X f d\mu.$$

7. Let $\mathcal{A}$ be a σ-algebra containing $\mathcal{B}_X$. Prove that there exists only one measure μ on $\mathcal{A}$ satisfying the following properties:

$$\begin{aligned} \mu(E) &= \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\} \quad \forall E \in \mathcal{A}, \\ \mu(V) &= \sup\{\mu(K) : K \subseteq V,\ K \text{ compact}\} \quad \text{for open } V \subseteq X. \end{aligned}$$

8. Let $\Lambda : C_c(\mathbb{R}) \to \mathbb{C}$ be defined by $\Lambda f := \int_{a_f}^{b_f} f(x)dx$, $f \in C_c(\mathbb{R})$, where a_f and b_f are such that $\text{supp}(f) \subseteq [a_f, b_f]$, and let μ be the measure obtained as in Riesz representation theorem corresponding to the positive linear functional Λ. Is μ the Lebesgue measure? Why?

9. Let $X = \mathbb{N}$ with usual topology and $\Lambda f := \sum_{j=1}^{\infty} f(j)w_j$ for $f \in C_c(\mathbb{N})$, where (w_n) is a sequence of positive real numbers. Determine the σ-algebra and measure obtained as in Riesz representation theorem.

10. Let $\mathcal{M}$ and μ be as in Riesz representation theorem and X, in addition, be σ-compact. Prove the following:

   (a) For every $E \in \mathcal{M}$ and $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that $\mu(E) < \mu(K) + \epsilon$.

   (b) For every $E \in \mathcal{M}$ and $\epsilon > 0$, there exist an open set $V \supseteq E$ and a closed set $F \subseteq E$ such that

$$\mu(V \setminus E) < \epsilon, \qquad \mu(E \setminus F) < \epsilon.$$