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ADJOINT OF UNBOUNDED OPERATORS ON BANACH SPACES

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Banach spaces considered below are over the field $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$. Let X be a Banach space. following Kato [2], X^* denotes the linear space of all continuous conjugate linear functionals on X. We shall denote

$$\langle f, x\rangle := f(x), \quad x \in X,\ f \in X^*.$$

On X^*,

$$f \mapsto \|f\| := \sup_{\|x\|=1} |\langle f, x\rangle|$$

defines a norm on X^*.

Definition 1. The space X^* is called the **adjoint space** of X.

Note that if $\mathbb{K} = \mathbb{R}$, then X^* coincides with the dual space X'. It can be shown, analogues to the case of X', that X^* is a Banach space. Let X and Y be Banach spaces, and $A : D(A) \subseteq X \to Y$ be a densely defined linear operator. Now, we st out to define adjoint of A as in Kato [2].

Theorem 2. *There exists a linear operator $A^* : D(A^*) \subseteq Y^* \to X^*$ such that*

$$\langle f, Ax\rangle = \langle A^* f, x\rangle \quad \forall\, x \in D(A),\ f \in D(A^*)$$

and for any other linear operator $B : D(B) \subseteq Y^* \to X^*$ satisfying

$$\langle f, Ax\rangle = \langle Bf, x\rangle \quad \forall\, x \in D(A),\ f \in D(B),$$

$D(B) \subseteq D(A^*)$ and B is a restriction of A^*.

Proof. Suppose $D(A)$ is dense in X. Let

$$S := \{f \in Y^* : x \mapsto \langle f, Ax\rangle \text{ continuous on } D(A)\}.$$

For $f \in S$, define $g_f : D(A) \to \mathbb{K}$ by

$$(g_f)(x) = \langle f, Ax\rangle \quad \forall\, x \in D(A).$$

Since $D(A)$ is dense in X, g_f has a unique continuous conjugate linear extension to all of X, preserving the norm. Let us denote this extension of g_f by $\tilde{g}_f$. Taking $D(A^*) = S$, define $A^* : D(A^*) \to X^*$ by $A^* f = \tilde{g}_f$. It can be seen that B is a linear operator, and it satisfies

$$\langle A^* f, x\rangle = \langle f, Ax\rangle \quad \forall\, x \in D(A),\ f \in D(A^*).$$

Now, suppose $B : D(B) \subseteq Y^* \to X^*$ is another linear operator such that

$$\langle Bf, x\rangle = \langle f, Ax\rangle \quad \forall\, x \in D(A),\, f \in D(B).$$

Note that if $f \in D(B)$, then

$$|\langle f, Ax\rangle| = |\langle Bf, x\rangle| \leq \|Bf\|\,\|x\| \quad \forall\, x \in D(A),$$

so that $x \mapsto \langle f, Ax\rangle$ is continuous on $D(A)$. Thus, $D(B) \subseteq S = D(A^*)$. Further, $f \in D(B) \subseteq D(A^*)$ implies

$$\langle Bf, x\rangle = \langle f, Ax\rangle = \langle A^* f, x\rangle \quad \forall\, x \in D(A).$$

Hence, $Bf = A^* f$ for all $f \in D(B)$, showing that B is a restriction of A^*. □

Definition 3. Let $A : D(A) \subseteq X \to Y$ be a densely defined linear operator. The operator A^* defined in Theorem 2 is called the **adjoint** of A.

Corollary 4. *If $D(A) = X$ and A is a bounded operator, then $A^* : Y^* \to X^*$ is the operator which satisfies*

$$(A^* f)(x) = f(Ax) \quad \forall\, f \in Y^*,\, x \in X,$$

and A^* is a bounded linear operator with $\|A^*\| = \|A\|$.

Theorem 5. *Suppose $A : D(A) \subseteq X \to Y$ is a densely defined operator. Then A^* is a closed operator.*

Proof. Let (f_n) in $D(A^*)$ such that $f_n \to f \in Y^*$ and $A^* f_n \to g \in X^*$. Since

$$\langle A^* f_n, x\rangle = \langle f_n, Ax\rangle \quad \forall\, x \in D(A),\, n \in \mathbb{N},$$

we have

$$\langle g, x\rangle = \langle f, Ax\rangle \quad \forall\, x \in D(A).$$

Hence, $g \in D(A^*)$ and $A^* f = g$. □

The following example shows that the domain of the adjoint need not be dense even if A is a closed operator.

Example 6. Consider the Banach space ℓ^1 and

$$D := \{(\alpha_n) \in \ell^1 : (n\alpha_n) \in \ell^1\}.$$

Define

$$A(\alpha_n) = (n\alpha_n), \quad (\alpha_n) \in D.$$

Then A is a closed densely define operator: Since $c_{00} \subseteq D$, it follows that D is dense in ℓ^1. To see that A is a closed operator, note first that A is surjective and bounded below:

$$(\beta_n) \in \ell^1 \implies (\alpha_n) = (\beta_n/n) \in \ell^1 \in D,\; A(\alpha_n) = (\beta_n).,$$

$$\|A(\alpha_n)\|_1 = \sum_n n|\alpha_n| \geq \sum_n |\alpha_n| \geq \|(\alpha_n)\|_1.$$

Hence, A^{-1} is continuous so that it is closed, and hence its inverse, which is A, is also a closed operator.

But, the domain of A^* is not dense: For $(\beta_n) \in \ell^\infty$,

$$(\beta_n) \in D(A^*) \iff \exists (\gamma_n) \in \ell^1 \text{ such that } \langle (\gamma_n), (\alpha_n)\rangle = \langle (\beta_n), A(\alpha_n)\rangle \quad \forall \quad (\alpha_n) \in D.$$

Note that

$$\langle (\beta_n), A(\alpha_n)\rangle = \langle (\beta_n), (n\alpha_n)\rangle = \sum_n n\alpha_n\beta_n.$$

Thus,

$$\langle (\gamma_n), (\alpha_n)\rangle = \langle (\beta_n), A(\alpha_n)\rangle \iff \sum_n \alpha_n\gamma_n = \sum_n n\alpha_n\beta_n.$$

Hence, taking $(\alpha_n) = e_j$,

$$(\beta_n) \in D(A^*) \implies \gamma_j = j\beta_j \quad \forall\, j \in \mathbb{N}.$$

Thus,

$$D(A^*) \subseteq \{(\beta_n) \in \ell^\infty : (n\beta_n) \in \ell^1\} \subseteq c_0.$$

Hence, $D(A^*)$ is not dense.

The following is a modified form of a theorem in Kato ([2], Theorem 5.29).

Theorem 7. *Suppose $A : D(A) \subseteq X \to Y$ is a closed densely defined operator and Y is a reflexive space. Then $A^* : D(A^*) \subseteq Y^* \to X^*$ is a closed densely defined operator.*

Proof. By Theorem 1, A^* is a closed operator. Hence it remains to proof that $D(A^*)$ is dense. Suppose $D(A^*)$ is not dense in Y^*. Then there exists $\varphi \in Y^{**}$ such that

$$\|\varphi\| = 1, \quad \varphi(f) = 0 \quad \forall\, f \in D(A^*).$$

Since Y is reflexive, there exists $y_0 \in Y$ such that

$$\|y_0\| = \|\varphi\|, \quad \varphi(f) = f(y_0) \quad \forall\, f \in Y^*.$$

In particular,

$$\|y_0\| = 1, \quad f(y_0) = 0 \quad \forall\, f \in D(A^*).$$

Now, $(0, y_0) \notin G(A)$. Since $G(A)$ is a closed subspace of $X \times Y$, $\exists F \in (X \times Y)^*$ such that $F(0, y_0) \neq 0$ and $F(x, Ax) = 0$ for all $x \in D(A)$. Let $f(y) = F(0, y)$. Then $f \in Y^*$ and for $x \in D(A)$,

$$\langle f, Ax\rangle = f(Ax) = F(0, Ax) = F(x, Ax) - F(x, 0) = -F(x, 0) = \langle g, x\rangle,$$

where g dfined by $g(x) = -F(x, 0)$ belongs to X^*. Hence, $f \in D(A^*)$. But, then by (i), $f(y_0) = 0$. This is a contradiction since $f(y_0) = F(0, y_0) \neq 0$. □

It can be easily proved (see, eg. Nair [1]) that

- If $A : D(A) \subseteq X \to Y$ is a closed operator which is also continuous, then $D(A)$ is closed in X.

Hence, together with closed graph theorem, we obtain

Theorem 8. *Suppose $A : D(A) \subseteq X \to Y$ is a closed densely defined operator. Then A is continuous if and only if $D(A) = X$, and in that case $D(A^*) = Y^*$ and $A^* : Y^* \to X^*$ is continuous.*

Remark 9. (i) Analogous definitions and results hold if we take dual spaces in place of adjoint spaces.

(ii) Suppose X is a Hilbert space. For $f \in X^*$, let $\tilde{f}(x) = \overline{f(x)}$. Then we see that $\tilde{f} \in X'$, and hence by Riesz representation theorem, there exists a unique $u \in X$ such that

$$\tilde{f}(x) = \langle x, u\rangle_X \quad \forall\, x \in X.$$

Thus,

$$\langle f, x\rangle = f(x) = \langle u, x\rangle_X \quad \forall\, x \in X.$$

Hence, for every $f \in X^*$, there exists a unique $z_f \in X$ such that $\langle f, x\rangle = \langle z_f, x\rangle_X$ for all $x \in X$, and the map $f \mapsto z_f$ is a surjective linear isometry.

Suppose $A : D(A) \subseteq X \to Y$ is a densely defined operator between Hilbert spaces X and Y. Then, in view of Remark 9 (ii),

$$\langle z_{A^*f}, x\rangle_X = \langle A^*f, x\rangle = \langle f, Ax\rangle = \langle z_f, Ax\rangle_Y \quad \forall\, x \in D(A),\ f \in D(A^*).$$

For $u \in Y$, let $f_u \in Y^*$ be defined by $f_u(y) = \langle u, y\rangle$, $y \in Y$. Then $z_{f_u} = u$. Thus, we obtain

$$\langle u, Ax\rangle_Y = \langle z_{f_u}, Ax\rangle_Y = \langle f_u, Ax\rangle = \langle A^*f_u, x\rangle = \langle z_{A^*f_u}, x\rangle_X$$

for all $u \in Y$ such that $f_u \in D(A^*)$ and $x \in D(A)$. Let us define a linear operator $B : D(B) \subseteq Y \to X$ such that

$$D(B) = \{u \in Y :\ f_u \in D(A^*)\},$$

and

$$Bu = z_{A^*f_u}.$$

Note that

$$\begin{aligned}\{u \in Y :\ f_u \in D(A^*)\} &= \{u \in Y :\ x \mapsto \langle f_u, Ax\rangle \text{ continuous}\}\\ &= \{u \in Y :\ x \mapsto \langle u, Ax\rangle_Y \text{ continuous}\}\end{aligned}$$

Thus,

$$D(B) = \{u \in Y : x \mapsto \langle u, Ax\rangle_Y \text{ continuous}\}$$

and

$$\langle u, Ax\rangle_Y = \langle Bu, x\rangle_X \quad \forall\, x \in D(A),\, u \in D(B).$$

Definition 10. If X and Y are Hilbert spaces, then the operator $B : D(B) \subseteq Y \to X$ with

$$D(B) = \{u \in Y : x \mapsto \langle u, Ax\rangle_Y \text{ continuous}\}$$

and

$$\langle u, Ax\rangle_Y = \langle Bu, x\rangle_X \quad \forall\, x \in D(A),\, u \in D(B)$$

is called the **adjoint** of A.

If we denote by $J_X : X^* \to X$ the map which takes $f \in X^*$ to its Riesz representer z_f, then we have

$$Bu = z_{A^* f_u} = J_X(A^* f_u) = J_X A^* J_Y^{-1} u \quad \forall\, u \in D(B).$$

Thus

$$\begin{aligned} B &= J_X A^* J_Y^{-1} \quad \text{on} \quad D(B), \\ A^* &= J_X^{-1} B J_Y \quad \text{on} \quad D(A^*). \end{aligned}$$

In view of the above observations, (abusing the notation) we use the notation A^* for B also.

References

[1] M.T. Nair, *Functional Analysis: A First Course*, Prentice-Hall of India, New Delhi, 2002 (Third Print, 2010).

[2] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, New York, 1976.

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