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Problems for Assignment Test

1. A bounded metric on a linear space cannot induce a norm - Why?

2. A nonzero subspace of a normed linear space can not be bounded - Why?

3. Let $X = C[0,1]$ with norm $\|x\|_1 := \int_0^1 |x(t)\, dt$. Show that X is not a Banach space. Does it have a denumerable (Hamel) basis ? – Justify the answer.

4. Are the norms $\|x\|_\infty := \max_{0\le t\le 1} |x(t)|$ and $\|x\|_1 := \int_0^1 |x(t)|dt$ defined on $C[0,1]$ equivalent ? – Justify the answer.

5. On the space $C[0,1]$, is the norm $\|x\|_1 := \int_0^1 x(t)dt$, $x \in C[0,1]$, induced by an inner product? Why?

6. If Y is a finite dimensional subspace of a normed linear space X, then prove that there is an $x \in X$ with $\|x\| = 1$ and $\inf_{y\in Y} \|x - y\| = 1$.

7. Let X be a normed liner space and $f \in X'$. Show that, if $N(f)$ is a complete subspace, then X is also a complete space, i.e., a Banach space.

8. Let $X = C[0,1]$ with norm $\|\cdot\|_1$ defined by $\|x\|_1 := \int_0^1 |x(t)|dt$. Show that the operators $A : X \to X$ defined on X by

$$(Ax)(t) = \int_0^t x(s)ds, \quad (Bx)(t) = tx(t), \quad 0 \le t \le 1,$$

is continuous. Find $\|A\|$.

9. Let $X_1 = C[0,1]$ with norm $\|x\|_1 := \displaystyle\int_0^1 |x(t)|dt$ and $X_2 = C[0,1]$ with norm $\|x\|_\infty := \displaystyle\max_{0\le t\le 1} |x(t)|$. Then the identity operator from X_1 to X_2 is not continuous - Why?

10. Show that, on every infinite dimensional normed liner space, there exists at least one discontinuous linear functional.

11. Suppose X_0 is a closed subspace of a normed linear space X and $\eta : X \to X/X_0$ is the canonical mapping, i.e., $\eta(x) = x + X_0$, $x \in X$. Show that η is a bounded linear operator with $\|\eta\| \le 1$, and η is onto.

12. Let X and Y be normed linear spaces and $A : X \to Y$ be a linear operator. For $f \in Y'$, let $g_f : X \to \mathbb{K}$ be defined by $g_f(x) = f(Ax)$ for all $x \in X$. Show that,

    (i) g_f is a linear functional for every $f \in Y'$,

    (ii) if $A \in \mathcal{B}(X,Y)$, then $g_f \in X'$ for every $f \in Y'$, and

(iii) the map $A' : Y' \to X'$ defined by

$$A'f = g_f, \quad f \in Y',$$

is a bounded linear operator and $\|A'\| = \|A\|$.

The operator A' defined above is called the **dual** *of A.*

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13. Let H be a Hilbert space and $T : H \to H$ be a linear operator. Show that if there exists a linear operator $T_0 : H \to H$ such that $\langle Tx, y\rangle = \langle x, T_0 y\rangle$ for every $x, y \in H$, then T_0 is a bounded operator and $\|T_0\| = \|T\|$.

14. Let X be a Hilbert space and for $f \in X'$, let $u_f \in X$ be the unique element in X such that $f(x) = \langle x, u_f\rangle$ for all $x \in X$. For $f, g \in X'$, let $\langle f, g\rangle' := \langle u_g, u_f\rangle$. Show that

    (i) $\langle \cdot, \cdot\rangle'$ is an inner product on X', and

    (ii) X' is a Hilbert space w.r.t. the inner product $\langle \cdot, \cdot\rangle'$, and $\|f\|' = \|f\|$ for all $f \in X'$.

15. Let X and Y be Hilbert spaces and $T : X \to Y$ be a bounded linear operator. For $y \in H$, let $f_y : X \to \mathbb{K}$ be such that $f_y(x) := \langle Tx, y\rangle$, $x \in X$.

    (i) Show that $f_y \in X'$.

    Let the $u_y \in X$ be the unique element such that $f_y(x) = \langle x, u_y\rangle$, $x \in X$. Define adjoint $T^* : Y \to X$ be defined by $T^* y = u_y$.

    (ii) Show that T^* is a bounded linear operator.

16. Suppose X is a Hilbert space and $\mathcal{S}(X) = \{A \in \mathcal{B}(X) : A = A^*\}$. Show that

    (i) $\mathcal{S}(X)$ is a closed subset of $\mathcal{B}(X)$,

    (ii) if $\mathbb{K} = \mathbb{R}$, then $\mathcal{S}(X)$ is a closed subspace of $\mathcal{B}(X)$.

    Operators in $\mathcal{S}(X)$ are called **self adjoint** operators.

17. Let X be Hilbert space and $A : X_0 \subseteq X \to X$ be a linear operator such that, for some $c > 0$, $|\langle Ax, x\rangle| \geq c\|x\|^2$ for all $x \in X$. Show that A is injective, $R(A)$ dense in X, and $A^{-1} : R(A) \to X$ has a unique continuous extension to all of X.

18. Suppose A in Problem 17 is a closed operator. Then show that A is bijective.

19. Give an example of a bounded operator which is not a closed operator. Let X be a Banach space, X_0 be a subspace of X and $A : X_0 \to X$ be a bounded linear operator. Show that A is a closed operator if and only if X_0 is a closed subspace of X.

20. If a linear functional on a normed linear space X is a closed operator, then it is continuous - Why?

21. If X is a non-zero normed linear space, then its dual space is also non-zero space-Why?

22. Prove that, if X is a normed linear space then for any $x \in X$, $\|x\| = \sup\{|f(x)| : f \in X', \|f\| = 1\}$.

23. Let X be a normed linear space . Show that $T : X \to X''$ defined by $(Tx)(f) = f(x),\ x \in X,\ f \in X'$, is a linear isometry.

24. If f is a nonzero continuous linear functional on a normed linear space X, then show that $\|f\| = 1/d$, where $d = \inf\{\|x\| : f(x) = 1\}$.

25. Let X be an inner product space and (e_n) be an orthonormal sequence in X. Show that for every $x \in X$, $\langle x, e_n\rangle \to 0$ as $n \to \infty$.

26. Let X be a Banach space and $A \in B(X)$ be injective. Show that the operator $A^{-1} : \mathcal{R}(A) \to X$ is continuous if and only if $\mathcal{R}(A)$ is closed.

27. If X is a Hilbert space and (e_n) is an orthonormal sequence in X, then show that for every $x \in X$, the series $\sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$ converges to an element $y \in X$, and $x - y$ is orthogonal to span $\{e_n : n = 1, 2, \ldots\}$.

28. Let H_1 and H_2 be Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Show that $T = 0$ if and only if $T^*T = 0$.

29. Let H be a complex Hilbert space and $T \in B(H)$ be a normal operator. Let $x \in H$ and $\lambda \in \mathbb{C}$. Show that $Tx = \lambda x$ if and only if $T^*x = \bar{\lambda}x$.

30. Let H be a Hilbert space and E be an orthonormal basis of H. Show that for every $x \in H$, the set $E_x := \{u \in E : \langle x, u\rangle \neq 0\}$ is a countable set, and $x = \sum_{u \in E_x} \langle x, u\rangle u$.

31. Show that a sequence (x_n) in a normed linear space X is bounded if and only if $(f(x_n))$ is bounded for every continuous linear functional f on X.

32. Prove that, if (T_n) be a sequence bounded operators in $\mathcal{B}(X)$, then $(\|T_n\|)$ is bounded if and only if $(|f(T_n x)|)$ is bounded for all $x \in X$ and for all $f \in Y'$.

33. If (α_n) is a sequence of scalars such that $\sum_{i=1}^{\infty} \alpha_i \xi_i$ converges for every $(\xi_n) \in \ell^1$, then $\sum_{n=1}^{\infty} |\alpha_j|$ converges.

34. Let X and Y be Banach spaces and $T : X \to Y$ be a one–one bounded linear operator. Show that $T^{-1} : \mathcal{R}(T) \to X$ is continuous if and only if $\mathcal{R}(T)$ is closed.

35. If $A : H \to H$ is a linear operator on a Hilbert space H such that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for every $x, y \in H$, then A is a bounded operator.

36. For $x = (a_1, a_2, a_3, \ldots) \in \ell^\infty$, let $Tx = (a_1, a_2/2, a_3/3, \ldots)$. Show that $T : \ell^\infty \to \ell^\infty$ is is continuous, one–one with its range not closed. Is $T^{-1} : \mathcal{R}(T) \to \ell^\infty$ continuous ? – Justify the answer.

37. The dual of ℓ^∞ is not linearly isometric with ℓ^1 – Why ?