COMPLEX ANALYSIS
A Short Course

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Preface

- This book is primarily for the students and teachers of IIT Madras.
- This is based on a Core Course that I have given for the second semester students of M.Sc. (Mathematics) at IIT Madras. The audience included some B.Tech. students and a faculty member (Dr. Parag Ravindran) from Mechanical Engineering department.
- The contents of the book is in the line of the well-written, small book *Complex Function Theory*[1] by Donald Sarason. I fondly acknowledge some e-mail discussions that I had with Prof. Sarason during the time of giving the course.

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[1]Second Editin, Hindustan Book Agency ('trim' series), New Delhi, 2008.

1

Complex Plane

1.1 Complex Numbers

After having the real *field* $\mathbb{R}$, it is natural to look for a bigger field in which algebraic equations such as

$$x^2 + 1 = 0 \qquad (*)$$

has a solution. Of course, the + sign here must be the symbol for addition in the bigger field. Since two fields can be considered to be identical if there is a surjective isomorphism between then, it is enough to have a field which contains an isomorphic image of $\mathbb{R}$ and having required properties such as solution to algebraic equations. We shall define such a field with the intention of having a solution to the equation $(*)$.

Definition 1.1.1 The set $\mathbb{C}$ of complex numbers is the set of all ordered pairs (x, y) of real numbers with the following operations of *addition* and *multiplication*:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

$$(x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

$\Diamond$

The proof of the following theorem is left to the reader.

Theorem 1.1.1 *The following hold.*

(i) $\mathbb{C}$ *is a field with additive identity* $(0, 0)$ *and multiplicative identity* $(1, 0)$.

(ii) The map $\varphi : \mathbb{R} \to \mathbb{C}$ *defined by*

$$\varphi(x) = (x, 0), \qquad x \in \mathbb{R},$$

is a field isomorphism.

We observe that the multiplicative inverse of a nonzero complex number $z = (x, y)$ is given by

$$\left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2}\right).$$

Writing

$$i = (0, 1)$$

and for $x \in \mathbb{R}$,

$$\tilde{x} = (x, 0),$$

we observe that

$$i^2 = -\tilde{1},$$

and

$$\mathbb{C} = \{\tilde{x} + i\tilde{y} : x, y \in \mathbb{R}\}.$$

With the above notations, the addition and multiplication in $\mathbb{C}$ can be written as

$$\begin{aligned}(\tilde{x}_1 + i\tilde{y}_1) + (\tilde{x}_2 + i\tilde{y}_2) &= (\tilde{x}_1 + \tilde{y}_2) + i(\tilde{y}_1 + \tilde{y}_2) \\ &= (\widetilde{x_1 + y_2}) + i(\widetilde{y_1 + y_2}),\end{aligned}$$

$$\begin{aligned}(\tilde{x}_1 + i\tilde{y}_1).(\tilde{x}_2 + i\tilde{y}_2) &= (\tilde{x}_1\tilde{x}_2 - \tilde{y}_1\tilde{y}_2) + i(\tilde{x}_1\tilde{y}_2 + \tilde{x}_2\tilde{y}_1) \\ &= (\widetilde{x_1x_2 - y_1y_2}) + i(\widetilde{x_1y_2 + x_2y_1}).\end{aligned}$$

Throughout this course, we identify $\tilde{x}$ with x for every $x \in \mathbb{R}$, so that $\mathbb{C}$ is the set of all *numbers* of the form

$$a + ib \quad \text{with} \quad a, b \in \mathbb{R}.$$

Thus,

$$a + i0 = a, \qquad 0 + i1 = i \quad \text{and} \quad i^2 = -1,$$

and for nonzero $z = x + iy$,

$$\frac{1}{z} := z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

One of the important properties of this field is that, not only equation $(*)$ has a solution in $\mathbb{C}$, but every algebraic equation also has a solution. This is the so called *fundamental theorem of algebra* which we shall prove in the due course.

1.2 Some Definitions and Properties

Definition 1.2.1 For a complex number $z = x + iy$, x is called the **real part** of z and is denoted by $\text{Re}(z)$, y is called the **imaginary part** of z and is denoted by $\text{Im}(z)$,

$$\bar{z} = x - iy$$

is called the **complex conjugate** of z, and the non-negative number

$$|z| = \sqrt{x^2 + y^2}$$

is called the **absolute value** or **modulus** of z. ♢

We observe that

$$\text{Re}(z) \leq |z|, \qquad \text{Im}(z) \leq |z|, \qquad \bar{z}z = |z|^2, \qquad \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Using the face that $|z|^2 = \bar{z}z$, it can be easily verified that

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad \forall\, z_1, z_2 \in \mathbb{C}.$$

The above inequality is called the **triangle inequality**.

1.2.1 Metric on $\mathbb{C}$

Theorem 1.2.1 *The function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by*

$$d(z_1, z_2) = |z_1 - z_2|, \qquad z_1, z_2 \in \mathbb{C}$$

is a metric on $\mathbb{C}$.

Hereafter any metric property of $\mathbb{C}$ is referred to the metric defined as in the above theorem. With respect to the above metric we have the following:

- A sequence (z_n) in $\mathbb{C}$ *converges* to $z \in \mathbb{C}$ if and only if

$$|z_n - z| \to 0 \quad \text{as} \quad n \to \infty.$$

- A point $z_0 \in \mathbb{C}$ is called an *interior point* of a set $A \subseteq \mathbb{C}$ if there exists $r > 0$ such that

$$B(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\} \subseteq A.$$

  The set $B(z_0, r)$ is called the *open ball* with centre z and radius r, usually denoted by $B(z, r)$.

- A subset G of $\mathbb{C}$ is *open* in $\mathbb{C}$ if and only if every point in G is an interior point of G.

- A point $z_0 \in \mathbb{C}$ is call a *boundary point* of a set $A \subseteq \mathbb{C}$ if every open ball containing z_0 contains some point of A and some point of its compliment, i.e., for every $r > 0$, $B(z_0, r) \cap A \neq \varnothing$ and $B(z_0, r) \cap A^c \neq \varnothing$.

- A subset S of $\mathbb{C}$ is *closed* in $\mathbb{C}$ it contains all its boundary points. It can be shown that a set

  $S \subseteq \mathbb{C}$ is closed if and only if $S^c := \mathbb{C} \setminus S$ is open if and only if for every sequence (z_n) in S,

  $$z_n \to z \Longrightarrow z \in S.$$

- A function $f : A \to \mathbb{C}$ defined on a subset A of $\mathbb{C}$ is *continuous* at a point $z_0 \in A$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$z \in A,\ |z - z_0| < \delta \Longrightarrow |f(z) - f(z_0)| < \varepsilon.$$

  Equivalently, f is continuous at a point $z_0 \in A$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$z \in A \cap B(z_0, \delta) \Longrightarrow f(z) \in B(f(z_0), \varepsilon).$$

- A function $f : A \to \mathbb{C}$ defined on a subset A of $\mathbb{C}$ has the *limit* $\zeta \in \mathbb{C}$ at a point $z_0 \in \mathbb{C}$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

  $$z \in A,\ 0 < |z - z_0| < \delta \Longrightarrow |f(z) - \zeta| < \varepsilon,$$

  and in that case we write

  $$\lim_{z \to z_0} f(z) = \zeta.$$

- A subset A of $\mathbb{C}$ is *bounded* if and only if there exists $\alpha > 0$ such that

  $$|z| \leq \alpha \qquad \forall z \in A.$$

- A subset A of $\mathbb{C}$ is *compact* if and only if every sequence in A has a convergent subsequence.

Theorem 1.2.2 *The set $\mathbb{C}$ is a complete metric space.*

Proof. Let (z_n) be a Cauchy sequence in $\mathbb{C}$. Writing $z_n = x_n + iy_n$ with $x_n, y_n \in \mathbb{R}$, we have

$$|x_n - x_m| \leq |z_n - z_m|, \quad |y_n - y_m| \leq |z_n - z_m|$$

for all $n, m \in \mathbb{N}$. Hence, (x_n) and $y_n)$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space with respect to the absolute-value metric, there exist $x, y \in \mathbb{R}$ such that $x_n \to x$ and $y_n \to y$. Then, writing $z = x + iy$, we have

$$|z_n - z|^2 = (x_n - x)^2 + (y_n - y)^2 \to 0.$$

Thus, (z_n) converges to z. ∎

1.2.2 Polar representation and n^{th}-roots

Let z be a nonzero complex number and let θ be the angle which the line segment joining 0 to z makes with the positive real axis, and $r = |z|$, the length of the line segment. Then it is clear from the geometry that

$$z = r(\cos\theta + i\sin\theta). \qquad (*)$$

Definition 1.2.2 The representation $(*)$ of a nonzero $z \in \mathbb{C}$ is called its **polar representation**, and an angle θ for which $(*)$ holds is called an **argumet** of z, denoted by $\arg(z)$. ◊

Note that each (r, θ) with $r > 0$ and $\theta \in \mathbb{R}$ represents a unique nonzero $z \in \mathbb{C}$ with the representation $(*)$, but a non zero $z \in \mathbb{C}$ has many polar representations, namely,

$$z = r(\cos\theta + i\sin\theta), \qquad \theta \in \{\theta_0 + 2\pi k : k \in \mathbb{Z}\},$$

where θ_0 is one of the angles for which $(*)$ holds.

We note that if $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

Thus, if $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{N}$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta) = r^n[\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi)], \qquad k \in \mathbb{Z}.$$

The above observation helps us in finding the n^{th} **roots** of complex numbers.

Let $\zeta = \rho(\cos\alpha + i\sin\alpha)$, $z = r(\cos\theta + i\sin\theta)$ and $n \in \mathbb{N}$. Then we have

$$z^n = \zeta \iff r = \rho^{1/n} \quad \text{and} \quad \theta = \frac{\alpha + 2k\pi}{n}, \qquad k \in \mathbb{Z}.$$

Thus, n-th roots of ζ are

$$z_k = \rho^{1/n}\left[\cos\left(\frac{\alpha + 2k\pi}{n}\right) + i\sin\left(\frac{\alpha + 2k\pi}{n}\right)\right], \qquad k \in \mathbb{Z}.$$

Note that for each $k \in \mathbb{Z}$

$$z_k = z_{k+n} \quad \text{for} \quad j \in \{1, \ldots, n\}.$$

Thus, the only distinct n-th roots of ζ are $z_1, z_2, \ldots, z_n$.

If $\zeta = 1$, then we have $\rho = 1$ and $\alpha = 0$, and in this case we see that the n-th roots of unity are

$$\omega, \omega^2, \ldots, \omega^n \quad \text{with} \quad \omega^n = 1,$$

where

$$\omega = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right).$$

1.2.3 Steriographic projection

Theorem 1.2.3 *The complex plane is homeomorphic with the set $S^2 \setminus \{(1,0,0)\}$, where S^2 is the unit sphere in $\mathbb{R}^3$ with centre as the origin, i.e.,*

$$S^2 = \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}.$$

Proof. Clearly, $\mathbb{C}$ is homeomorphic with the set

$$X = \{(x, y, 0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$$

with the homeomorphism being the map $x + iy \mapsto (x, y, 0)$. Hence, we find a surjective homeomorphism from X to $S^2 \setminus \{(1,0,0)\}$.

Note that the parametric representation of the straight line joining the *north pole* $u_0 = (1,0,0)$ with a point $u = (x, y, 0) \in X$ is given by

$$u_\lambda = (1 - \lambda)u_0 + \lambda u, \qquad \lambda \in \mathbb{R},$$

i.e.,

$$u_\lambda = (\lambda x, \lambda y, 1-\lambda), \qquad \lambda \in \mathbb{R}.$$

Clearly, for every $u = (x, y, 0) \in X$ there exists one and only $\lambda := \lambda_u$ such that $u_\lambda \in S^2$. We consider the map

$$u := (x, y, 0) \mapsto (\lambda_u x, \lambda_u y, 1-\lambda)$$

from X to $S^2 \setminus \{(0,0,1)\}$. Note that

$$u_\lambda \in S^2 \iff \lambda^2 x^2 + \lambda^2 y^2 + (1-\lambda)^2 = 1$$

if and only if $\lambda = 0$ or $\lambda(x^2 + y^2 + 1) - 2 = 0$. The point $\lambda = 0$ correspond to u_0. Hence,

$$\lambda_u = \frac{2}{x^2 + y^2 + 1}.$$

Thus the map

$$z := x + iy \mapsto \left(\frac{2x}{1+|z|^2}, \frac{2x}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2}\right)$$

is a bijective continuous function from $\mathbb{C}$ onto $S^2 \setminus \{(1,0,0)\}$ with its inverse

$$(\alpha, \beta, \gamma) \mapsto \frac{\alpha + i\beta}{1-\gamma},$$

which is also continuous. ∎

1.3 Problems

1. Show that $\mathbb{C}$ is a field under the addition and multiplication defined for complex numbers.

2. Show that the map $f : \mathbb{R} \to \mathbb{C}$ defined by $f(x) = (x, 0)$ is a field isomorphism.

3. For a nonzero complex number x, show that $z^{-1} = \bar{z}/|z|$.

4. Show that for z_1, z_2 in $\mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$.

5. Show that $d(z_1, z_2) := |z_1 - z_2|$ defines a metric on $\mathbb{C}$, and it is a complete metric.

6. Show that $|z_1 - z_2| \geq |z_1| - |z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

7. Suppose α, β, γ are nonzero complex numbers such that $|\alpha| = |\beta| = |\gamma|$.

   Show that $\alpha + \beta + \gamma = 0 \iff \dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma} = 0$.

8. Suppose z_1, z_2, z_3 are vertices of an equilateral triangle. Show that $z_1^2 + z_2^2 + z_3^2 + = z_1 z_2 + z_2 z_3 + z_3 z_1$.

9. Show that the equation of a straight line using complex variable z is given by $\bar{\alpha} z + \alpha \bar{z} + \gamma = 0$ for some $\alpha \in \mathbb{C}$ and $\gamma \in \mathbb{R}$.

10. If $|z| = 1$ and $z \neq 1$, then show that $\dfrac{1+z}{1-z} = ib$ for some $b \in \mathbb{R}$.

11. For $n \in \mathbb{N}$, derive a formula for the n^{th} root of a complex number z using its polar representation.

12. Let S^2 be the unit sphere in $\mathbb{R}^3$ with centre at the origin, i.e., $S^2 := \{(\alpha, \beta, \gamma) \in \mathbb{R}^3 : \alpha^2 + \beta^2 + \gamma^2 = 1\}$. Show that the *steriographic projection*

    $$z := x + iy \mapsto \left(\frac{2x}{1 + |z|^2}, \frac{2x}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right)$$

    is a bijective continuous function from $\mathbb{C}$ onto $S^2 \setminus \{(1, 0, 0)\}$ with its inverse

    $$(\alpha, \beta, \gamma) \mapsto \frac{\alpha + i\beta}{1 - \gamma},$$

    which is also continuous.

13. Show that the functions $z \mapsto \mathrm{Re}(z), \quad z \mapsto \mathrm{Im}(z), \quad z \mapsto |z|$ are continuous functions on $\mathbb{C}$.

14. Show that $\lim\limits_{z \to 0} \dfrac{z}{|z|}$ does not exist.

2

Analytic Functions

In this chapter we do calculus of complex valued functions of a complex variable.

2.1 Differentiation

Let f be a complex valued function defined on a set $\Omega \subseteq \mathbb{C}$.

Definition 2.1.1 Let z_0 be an oneerior point of Ω. Then f is said to be **differentiable** at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and in that case the above limit is called the derivative of f at z_0, denoted by $f'(z_0)$. ◊

Thus the following are equivalent:

(i) f is differentiable at $z_0 \in \Omega$.
(ii)There exists $c \in \mathbb{C}$ such that for every $\varepsilon > 0$, there exists $\delta > 0$ satisfying

$$z \in \Omega, \quad 0 < |z - z_0| < \delta \Longrightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - c \right| < \varepsilon.$$

(iii) There exists $c \in \mathbb{C}$ such that

$$\frac{f(z) - f(z_0) - c(z - z_0)}{|z - z_0|} \to 0 \quad \text{as} \quad z \to z_0.$$

The equivalence in (iii) above shows that if f is differentiable at z_0, then $f(z)$ is approximately equal to $f(z_0) - c(z - z_0)$ whenever z is in some neighbourhood of z_0, which we write as

$$f(z) \simeq f(z_0) - c(z - z_0)$$

whenever z is in some neighbourhood of z_0.

The following theorem can be proved (*exercise*) using arguments similar to the real case of real valued functions of a real variable.

Theorem 2.1.1 *Let z_0 be an interior point of $\Omega \subseteq \mathbb{C}$. Then the following holds.*

(i) *If f differentiable at $z_0 \in \Omega$, then f is continuous at z_0.*

(ii) *If f and g are differentiable at $z_0 \in \Omega$, then $f+g$ and fg are differentiable at z_0, and*

$$(f+g)'(z_0) = f'(z_0)+g(z_0), \qquad (fg)'(z_0) = f'(z_0)g(z_0)+f(z_0)g'(z_0).$$

(iii) *If f and g are differentiable at $z_0 \in \Omega$ and if $g(z_0) \neq 0$, then f/g is differentiable at z_0, and*

$$\left(\frac{f}{g}\right)'(z_0) = \frac{g(z_0)f'(z_0) - g'(z_0)f(z_0)}{[g(z_0)]^2}.$$

(iv) *If f is differentiable at $z_0 \in \Omega$ and g is differentiable in a neighbourhood of $f(z_0)$, then $g \circ f$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(f(z_0)f'(z_0).$$

Now, let us write $f(z)$ as $u(z) + iv(z)$, where $u(z) = \mathrm{Re}f(z)$ and $v(z) = \mathrm{Im}f(z)$. Recall that f is differentiable at $z_0 \in \Omega$ if and only if there exists $c \in \mathbb{C}$ such that

$$\frac{R(z)}{|z - z_0|} \to 0 \quad \text{as} \quad z \to z_0,$$

where $R(z) = f(z) - f(z_0) - c(z - z_0)$. Writing

$$z = x + iy, \quad z_0 = x_0 + iy_0, \quad c = a + ib,$$

we have

$$\begin{aligned}
R(z) &= f(z) - f(z_0) - c(z - z_0) \\
&= [u(z) - u(z_0)] + i[v(z) - v(z_0)] - (a + ib)[(x - x_0) + i(y - y_0)] \\
&= [u(z) - u(z_0) - a(x - x_0) + b(y - y_0)] \\
&\quad + i[v(z) - v(z_0) - b(x - x_0) - a(y - y_0)] \\
&= R_1(z) + iR_2(z),
\end{aligned}$$

where

$$R_1(z) = u(z) - u(z_0) - [a(x - x_0) - b(y - y_0)],$$
$$R_2(z) = v(z) - v(z_0) - [b(x - x_0) + a(y - y_0)].$$

Thus,

$$\frac{R(z)}{|z - z_0|} \to 0 \iff \frac{R_1(z)}{|z - z_0|} \to 0 \quad \& \quad \frac{R_2(z)}{|z - z_0|} \to 0$$

if and only if u and v are differentiable as a functions of two real variables at (x_0, y_0), and

$$a = \frac{\partial u}{\partial x}(x_0, y_0), \quad -b = \frac{\partial u}{\partial y}(x_0, y_0),$$

$$b = \frac{\partial v}{\partial x}(x_0, y_0), \quad a = \frac{\partial v}{\partial y}(x_0, y_0),$$

i.e.,

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \quad \& \quad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0),$$

and in that case

$$\begin{aligned} f'(z_0) = a + ib &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) \\ &= \frac{\partial v}{\partial y}(x_0, y_0) - i\frac{\partial u}{\partial y}(x_0, y_0). \end{aligned}$$

Thus, we have proved the following theorem.

Theorem 2.1.2 *The function f is differentiable at $z_0 \in \Omega$ if and only if its real part u and imaginary part v are differentiable at (x_0, y_0) and u_x, u_y, v_x, v_y satisfy the equations*

$$u_x(z_0) = v_y(z_0), \qquad u_y(z_0) = -v_x(z_0), \tag{$*$}$$

and in that case

$$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

Equations in $(*)$ are called the **Cauchy-Riemann equations**, or in short **CR-equations**.

Now, recalling from a sufficient condition for differentiability of a real valued function of two variables, we have the following sufficient condition of differentiability of f at $z_0 \in \Omega$.

Theorem 2.1.3 *Let $f = u + iv$ and $z_0 = x_0 + iy_0 \in \Omega$. Suppose u_x, u_y, v_x, v_y exist in a neighbourhood of z_0 and are continuous at (x_0, y_0), and suppose they satisfy the CR-equations at (x_0, y_0). Then f is differentiable at $z = x_0 + iy_0$, and*

$$f'(z_0) = u_x + iv_x.$$

EXAMPLE 2.1.1 Let us find out points at which some of the simple functions are differentiable.

(i) Let $f(z) = x$. In this case we have $u(x, y) = x$ and $v(x, y) = 0$. Hence, $u_x = 1$, $u_y = 0$, $v_x = 0 = v_y$ at every point. Since u and v do not satisfy CR-equations, f is not differentiable at any $z \in \mathbb{C}$.

(ii) Let $f(z) = \bar{z} = x - iy$. In this case we have $u(x, y) = x$ and $v(x, y) = -y$. Hence, $u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$ at every point. Again CR-equations are not satisfied at any point. Hence, f is not differentiable at any $z \in \mathbb{C}$.

(iii) Let $f(z) = |z|^2 = x^2 + y^2$. In this case we have $u(x, y) = x^2 + y^2$ and $v(x, y) = 0$. Hence, $u_x = 2x$, $u_y = 2y$, $v_x = 0 = v_y$ at every point. Thus, partial derivatives of u and v exist and are continuous on $\mathbb{R}^2$, and CR-equations are satisfied only at $(0, 0)$. Hence, f is differentiable only at 0.

(iv) Let $f(z) = \bar{z}^2 = (x^2 - y^2) - i2xy$. In this case we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Hence, $u_x = 2x$, $u_y = -2y$, $v_x = -2y$, $v_y = -2x$ at every point. Thus, partial derivatives of u and v exist and are continuous on $\mathbb{R}^2$, and CR-equations are satisfied only at $(0, 0)$. Hence, f is differentiable only at 0. □

CR-equations in polar coordinates:

Let $f = u + iv$ where $u = \text{Re} f$ and $v = \text{Im} f$. For $z = x + iy$ with $x = \text{Re} z$ and $y = \text{Im} z$, writing

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}(y/x), \quad x \neq 0,$$

and considering u and v as we have the following:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x},$$
$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

Now, $r^2 = x^2 + y^2$ and $\tan\theta = y/x$ so that

$$2r\frac{\partial r}{\partial x} = 2x, \qquad 2r\frac{\partial r}{\partial y} = 2y,$$

$$\sec^2\theta\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}, \qquad \sec^\theta\frac{\partial \theta}{\partial y} = \frac{1}{x},$$

i.e.,

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r},$$

$$\frac{x^2+y^2}{x^2}\frac{\partial \theta}{\partial x} = -\frac{y}{x^2}, \qquad \frac{x^2+y^2}{x^2}\frac{\partial \theta}{\partial y} = \frac{1}{x},$$

i.e.,

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \qquad \frac{\partial r}{\partial y} = \frac{y}{r}, \qquad \frac{\partial \theta}{\partial x} = -\frac{y}{r^2}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

Now,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}, \qquad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

Thus,

$$\frac{\partial u}{\partial x} = \frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta}, \qquad \frac{\partial u}{\partial y} = \frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta}. \tag{1}$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{x}{r}\frac{\partial v}{\partial r} - \frac{y}{r^2}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial y} = \frac{y}{r}\frac{\partial v}{\partial r} + \frac{x}{r^2}\frac{\partial v}{\partial \theta}. \tag{2}$$

Recall that the CR-equations in Cartesian coordinates are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, (1) – (2) give

$$\frac{x}{r}\frac{\partial u}{\partial r} - \frac{y}{r^2}\frac{\partial u}{\partial \theta} = \frac{y}{r}\frac{\partial v}{\partial r} + \frac{x}{r^2}\frac{\partial v}{\partial \theta}, \tag{5}$$

$$\frac{y}{r}\frac{\partial u}{\partial r} + \frac{x}{r^2}\frac{\partial u}{\partial \theta} = \frac{x}{r}\frac{\partial v}{\partial r} - \frac{y}{r^2}\frac{\partial v}{\partial \theta}. \tag{6}$$

The equations (3) – (4) imply that

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}, \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

These are the CR-equations in polar coordinates.

2.2 Holomorphic or Analytic Functions

Definition 2.2.1 Let $\Omega \subseteq \mathbb{C}$.

(i) A function $f : \Omega \to \mathbb{C}$ is said to be **analytic** at a point $z_0 \in \Omega$ if there exists $r > 0$ such that $B(z_0, r) \subseteq \Omega$ and f is differentiable at every point in $B(z_0, r)$.

(ii) A function $f : \Omega \to \mathbb{C}$ is said to be **holomorphic** or **analytic** on $\Omega_0 \subseteq \Omega$ if f is analytic at every point in Ω_0.

$\diamondsuit$

- If $f : \Omega \to \mathbb{C}$ is analytic on $\Omega_0 \subseteq \Omega$, then Ω_0 is an open set.
- $f : \Omega \to \mathbb{C}$ is not analytic at a point $z_0 \in \Omega$ if and only if for every $r > 0$, there exists $\zeta \in B(z_0, r) \cap \Omega$ such that f is not differentiable at ζ.

Definition 2.2.2 A complex valued function defined and analytic on the entire complex plane is called an **entire function**. $\diamondsuit$

- $f : \Omega \to \mathbb{C}$ is analytic at a point $z_0 \in \Omega$ if it is analytic on some open set containing z_0.
- $f : \Omega \to \mathbb{C}$ is analytic on Ω if and only if $u := \mathrm{Re}(f)$ and $v := \mathrm{Im}(f)$ have continuous first partial derivatives in Ω and they satisfy the CR-equations at every point in Ω.

Remark 2.2.1 In the subject of complex analysis, it is very common to say a function

f is analytic at a point $z_0 \in \mathbb{C}$

to mean that f is defined in an open neigbourhood of z_0 and f is analytic at z_0.

Usually, a function is given in terms of certain expression, and in that case, the domain of definition of f is taken to be the largest subset of $\mathbb{C}$ in which the expression makes sense. For example, consider the expression

$$f(z) = \frac{1}{z}.$$

In this case, the domain of definition of f is taken to be $\Omega := \mathbb{C}\setminus\{0\}$, and f is not analytic at $z_0 = 0$, since $0 \notin \Omega$. The function

$$f(z) = \bar{z}$$

is not analytic at any point in $\mathbb{C}$, since f is defined on $\mathbb{C}$, but f is not differentiable at any $z_0 \in \mathbb{C}$. ◊

Definition 2.2.3 A set $\Omega_0 \subseteq \mathbb{C}$ is called the **domain of analyticity** of a function f if Ω_0 is the largest open set in which f is analytic. ◊

For example, the domain of analyticity of $f(z) = 1/z$ is $\mathbb{C} \setminus \{0\}$, whereas the domain of analyticity of $f(z) = \bar{z}$ is $\varnothing$.

Definition 2.2.4 A point $z_0 \in \mathbb{C}$ is a **singularity** of an analytic function f if z_0 is not in the domain of analyticity of f. ◊

Definition 2.2.5 A point $z_0 \in \mathbb{C}$ is an **isolated singularity** of a function f if the domain of analyticity of f contains a deleted neigbourhood of z_0. ◊

EXAMPLE 2.2.1 1. For $a_1, a_1, \ldots, a_n$ in $\mathbb{C}$, let

$$f(z) := a_0 + a_1 z + \ldots + a_n z^n, \qquad z \in \mathbb{C}.$$

Then f is an entire function.

2. The function $z \mapsto 1/(1-z)$ is analytic on $\mathbb{C} \setminus \{0\}$.

3. The function f defined by $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic on the unit disc $\Omega := \{z \in \mathbb{C} : |z| < 1\}$.

□

2.2.1 Analytic extension

Definition 2.2.6 Suppose Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is an analytic function. Let $\widetilde{\Omega}$ be an open set such that $\widetilde{\Omega} \supseteq \Omega$. We say that f has an **analytic extension** to $\widetilde{\Omega}$ if there exists an analytic function g on $\widetilde{\Omega}$ which is an extension of f, i.e., $g : \widetilde{\Omega} \to \mathbb{C}$ is analytic and $g(z) = f(z)$ for all $z \in \Omega$. ◊

EXAMPLE 2.2.2 1. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ and $f(z) = \dfrac{1}{1-z}$, $z \in \Omega$. Then f is analytic on Ω, and has extension to the open set $\widetilde{\Omega} := \mathbb{C}\setminus\{1\}$. Clearly, in this case, $g(z) = 1/(1-z)$ for all $z \in \widetilde{\Omega}$.

2. The function f defined by $f(z) = \sum_{n=0}^{\infty} z^n$ is analytic on the unit disc $\Omega := \{z \in \mathbb{C} : |z| < 1\}$, and cannot be extended to a bigger open set.

□

2.2.2 Geometric representations

We know that a real valued function of a real variable can be represented by its graph, as a subset of the plane. Such a representation is not possible for a complex valued function of a real variable, as we require at least four *real dimensions*. But, we can represent them as mappings of $\mathbb{C}$ into itself as which specify changes taking place for certain figures such as straight lines and circles. Let consider a few simple examples.

EXAMPLE 2.2.3 Consider the function

$$z \mapsto az$$

for some nonzero $a \in \mathbb{C}$. Let us consider the following cases.

Case (i) $a \in \mathbb{R}$ and $a > 0$.

We see that this maps scales the figures in the plane - if $a > 1$, then it takes a circle to a bigger circle, and if $a < 1$, then it takes a circle to a smaller circle. More precisely, consider a circle

$$C: \quad |z - z_0| = r. \tag{$*$}$$

Writing $\zeta = az$, we have

$$|z - z_0| = r \iff \left|\frac{\zeta}{a} - z_0\right| = r \iff |\zeta - az_0| = ar.$$

Thus above function transforms a circle with centre z_0 and radius r into a circle with centre az_0 and radius ar.

A straight line has the equation

$$L: \quad \bar{\alpha}z + \alpha\bar{z} + \gamma = 0. \tag{$**$}$$

Indeed, for $a, b, c \in \mathbb{R}$,

$$\begin{aligned} ax + by + c &= a\left(\frac{z + \bar{z}}{2}\right) + b\left(\frac{z - \bar{z}}{2i}\right) + c \\ &= \left(\frac{a - ib}{2}\right) z + \left(\frac{a + ib}{2}\right) \bar{z} + c. \end{aligned}$$

Now, under the map $z \mapsto \zeta := az$ with $a > 0$, we have

$$\bar{\alpha}z + \alpha\bar{z} + \gamma = 0 \iff \bar{\alpha}\left(\frac{\zeta}{a}\right) + \alpha\left(\frac{\bar{\zeta}}{a}\right) + \gamma = 0 \iff \bar{\alpha}\zeta + \alpha\bar{\zeta} + a\gamma = 0.$$

Thus, the function $z \mapsto az$ transforms a straight line into a a straight line.

Case (ii) $a \in \mathbb{C}$ and $|a| = 1$.

In this case a is of the form $a = \cos\theta_0 + i\sin\theta_0$ for some $\theta \neq 0$. Hence, by the map $z \mapsto az$, a point $z = r\cos_0\theta + i\sin\theta_0$ is rotated by an angle θ_0. Thus, circles and straight lines are mapped onto circles and straight lines, respectively.

Case (iii) $a \in \mathbb{C}$, $a \neq 0$.

Since

$$a = |a|\left(\frac{a}{|a|}\right),$$

this function is a composition of the functions considered in Case (i) and Case (ii). □

Exercise 2.2.1 Explain the last statement in Case(ii) analytically using the circle and straight line given in $(*)$ and $(**)$. ◁

EXAMPLE 2.2.4 Consider the function

$$z \mapsto z + b$$

for some nonzero $b \in \mathbb{C}$. In this case the circle in $(*)$ is mapped into the circle

$$\widetilde{C}: \quad |\zeta - (z_0 - b)| = r,$$

i.e., the original circle is translated by $-b$, and the straight line in $(**)$ is mapped into the straight line

$$\widetilde{L}: \quad \bar{\alpha}z + \alpha\bar{z} + (\bar{\beta} + \alpha\bar{b} + \gamma) = 0.$$

Note that for $\gamma \in \mathbb{R}$, $\bar{\beta} + \alpha\bar{b} + \gamma$ is also in $\mathbb{R}$. □

EXAMPLE 2.2.5 Consider the function

$$z \mapsto az + b$$

for some nonzero $a, b \in \mathbb{C}$. This case is combination of the function considered in the previous two examples, i..e, $f(z) = g(h(z))$, where $h(z) = az$ and $g(z) = z + b$. Therefore, circles and straight lines are mapped into circles and straight lines, respectively. □

Remark 2.2.2 We observe that under the under the map $z \mapsto az+b$ with $a \neq 0$, the tangent at a point z_0 on a curve Γ is mapped onto the tangent at the point $\zeta := f(z_0)$ on the curve $\widetilde{\Gamma} := f(\Gamma)$. Of course, we have not defined so far what we mean by a curve in $\mathbb{C}$. We shall do this now. ◊

2.2.3 Curves in the complex plane

Definition 2.2.7 A **curve** in the complex plane is a complex valued function γ defined on an interval I.

If $I = [a, b]$, then the point $z_1 := \gamma(a)$ is called the **initial point** of γ and $z_2 := \gamma(b)$ is called the **terminal point** of γ. ◊

If γ is a curve in $\mathbb{C}$, then we shall identify it with its image

$$\Gamma_\gamma := \{\gamma(t) : t \in I\}.$$

If Γ_γ lies in an open set Ω, then we say that the curve γ is in Ω.

The direction of a curve γ is along the direction in which the points on Γ vary as t increases on I.

EXAMPLE 2.2.6 Given $z_1, z_2 \in \mathbb{C}$, the line segment joining z_1 to z_2 is a curve given by

$$\gamma(t) := (1-t)z_1 + tz_2, \qquad t \in [0, 1].$$

This curve has the same image as the one given by

$$\gamma(t) := \left(\frac{b-t}{b-a}\right) z_1 + \left(\frac{t-a}{b-a}\right) z_2, \qquad t \in [a, b].$$

□

EXAMPLE 2.2.7 (i) The curve defined by

$$\gamma(t) := \cos t + i \sin t, \qquad 0 \leq t \leq 2\pi$$

traces the unit circle (with centre 0) once in anti-clockwise direction.

(ii) The curve defined by

$$\gamma(t) := \cos t - i \sin t, \qquad 0 \le t \le 2\pi$$

traces the unit circle (with centre 0) once in clockwise direction.

(iii) The curve defined by

$$\gamma(t) := \cos t + i \sin t, \qquad 0 \le t \le 2n\pi$$

traces the unit circle (with centre 0) n-times in anticlockwise direction. □

EXAMPLE 2.2.8 Using the definition of a line segment in Example 2.2.6, it can be seen that the following curve traces the boundary of the square with vertices at $0, 1, 1+i, i$ once in anticlockwise direction:

$$\gamma(t) := \begin{cases} t, & 0 \le t \le 1, \\ 1 + (t-1)i, & 1 < t \le 2, \\ 3 - t + i, & 2 < t \le 3, \\ (4-t)i, & 3 < t \le 4. \end{cases}$$

□

Definition 2.2.8 A curve $\gamma : I \to \mathbb{C}$ is said to be **differentiable** at a point $t_0 \in I$ if

$$\lim_{t \to t_0} \frac{\gamma(t) - \gamma(t_0)}{t - t_0}$$

exists, and in that case the above limit is called the derivative of γ at t_0, denoted by $\gamma'(t_0)$. ◊

If $\gamma'(t_0)$ exists and is nonzero, then it represents the direction of the *tangent vector* to the curve at the point $z_0 := \gamma(t_0)$. In this case, the direction of the curve γ at t_0 is specified by

$$\arg \gamma'(t_0) \quad \text{or by the unit vector} \quad \frac{\gamma'(t_0)}{|\gamma'(t_0)|}.$$

Definition 2.2.9 A curve $\gamma : I \to \mathbb{C}$ is said to be **regular** at a point $t_0 \in I$ if γ is differentiable at t_0 and $\gamma'(t_0) \ne 0$.

If $\gamma : I \to \mathbb{C}$ is regular at $t_0 \in I$, then abusing the terminology, we may say that γ is regular at $z_0 := \gamma(t_0)$. ◊

Definition 2.2.10 We say that curves γ_1 and γ_2 (or their images Γ_1 and Γ_2) **intersect** at a point $z_0 \in \mathbb{C}$ if there are points t_1, t_2 such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$. ◊

Remark 2.2.3 (i) In the above definition the curves γ_1 and γ_2 may be defined on different intervals I_1 and I_2.

(ii) As a particular case of the above definition, if we take $\gamma_1 = \gamma_2$, the we can say that γ is **self intersecting** at z_0. $\diamondsuit$

Definition 2.2.11 Suppose that curves γ_1 and γ_2 intersect at a point $z_0 \in \mathbb{C}$ and regular at z_0. Then the **angle** between γ_1 and γ_2 at z_0 is defined as

$$\Theta_{z_0}(\gamma_1, \gamma_2) := \arg {\gamma_2}'(t_2) - \arg {\gamma_1}'(t_1),$$

where t_1, t_2 are such that $\gamma_1(t_1) = \gamma_2(t_2) = z_0$. $\diamondsuit$

Note that

$$\Theta_{z_0}(\gamma_1, \gamma_2) = \arg {\gamma_2}'(t_2)\overline{{\gamma_1}'(t_1)}.$$

We observe that if $\gamma : I \to \mathbb{C}$ is a curve with its image as Γ, and if f is a continuous complex valued function on Γ, then $\tilde{\gamma} : I \to \mathbb{C}$ defined by

$$\tilde{\gamma}(t) = (f \circ \gamma)(t), \qquad t \in I$$

is also a curve in $\mathbb{C}$. Note that (*Exercise*) if f is defined on an open set containing Γ, γ is differentiable at $t_0 \in I$, and f is differentiable at $z_0 := \gamma(t_0)$, then $\tilde{\gamma}$ is also differentiable at t_0 and

$$\tilde{\gamma}\,'(t_0) = f'(z_0)\gamma\,'(t_0).$$

If, in addition, if γ is regular at t_0 and $f'(z_0) \neq 0$, then we obtain

$$\arg {\gamma_f}'(t_0) = \arg f'(z_0) + \arg \gamma\,'(t_0).$$

Definition 2.2.12 Let f be a continuous complex valued function defined on an open set Ω and $z_0 \in \Omega$. Let γ_1 and γ_2 be curves intersecting at z_0 and regular at z_0. Then f is said to **preserve angle** between γ_1 and γ_2 at z_0 if $\tilde{\gamma}_1 := f \circ \gamma_1$ and $\tilde{\gamma}_2 := f \circ \gamma_2$ are regular at $\tilde{z}_0 := f(z_0)$ and $\Theta_{\tilde{z}_0}(\tilde{\gamma}_1, \tilde{\gamma}_2) = \Theta_{z_0}(\gamma_1, \gamma_2)$. $\diamondsuit$

Thus we have the following theorem.

Theorem 2.2.1 *Let f be defined on an open set Ω and differentiable at a point $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Then f preserves angle between curves which are regular at z_0 and intersecting at z_0.*

In fact, with some additional conditions, we have the converse of the above theorem as well. We shall state it without proof. For a proof of this, see Sarason[1]

Theorem 2.2.2 *Let $f := u + iv$ be defined on an open set Ω such that u and v are differentiable at a point $z_0 \in \Omega$. If f preserves angle between curves which are regular at z_0 and intersecting at z_0, then f is differentiable at z_0 and $f'(z_0) \neq 0$.*

Definition 2.2.13 A function f defined on an open set Ω is said to be a **conformal map** at $z_0 \in \Omega$ if it preserves angles between any two curves in Ω intersecting at z_0 and regular at z_0. $\Diamond$

Thus, a holomorphic function is conformal at every point in its domain of definition.

Now, let f be defined on an open set Ω. Let

$$u = \text{Re} f, \qquad v = \text{Im} f.$$

Considering f as a function of two real variables and assuming the quantities involved are well-defined on Ω, we can write

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

Hence,

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right).$$

Thus, u and v satisfy CR-equations if and only if

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

Notation:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Thus, u and v satisfy CR-equations if and only if $\dfrac{\partial f}{\partial \bar{z}} = 0$.

[1] D. Sarason, *Notes on Complex Function Theory*, Hindustan Book Agency, New Delhi, 1994.

The following would justify the introduction of the above notations $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$: Recall that, for $z = x + iy$,

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

so that

$$\frac{\partial x}{\partial z} = \frac{1}{2}, \qquad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \qquad \frac{\partial y}{\partial z} = \frac{-i}{2}, \qquad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2}.$$

Thus,

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

We can also define higher partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Definition 2.2.14 Let f be defined on an open set $\Omega \subseteq \mathbb{C}$. Then

(i) f is said to be of **class** C^1 if first partial derivatives of f exist and are continuous.

(ii) f is said to be of **class** C^2 if second partial derivatives of f exist and are continuous.

(iii) f is said to be **harmonic** if f is of class C^2 and

$$\triangle f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

The above equation is called the **Laplace equation**, and the *operator* $\triangle$ is called the **Laplacian**. ◊

The following can be easily verified:

- f holomorphic on Ω and is of class of $C^2 \implies f$ is harmonic.
- f holomorphic on $\Omega \implies$ u and v are harmonic, i.e.,

$$\triangle u = 0 = \triangle v.$$

- f harmonic on $\Omega \iff \dfrac{\partial^2 f}{\partial \bar{z} \partial z} = 0$. In fact,

$$\frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

Exercise 2.2.2 Prove the above three statements. ◁

Definition 2.2.15 Let u and v be real valued functions of class C^2 defined on Ω (considered as a subset of $\mathbb{R}^2$). Then v is said to be a **harmonic conjugate** of u if $f := u + iv$ is holomorphic on Ω. ♢

We observe the following:

- If u and v are of class C^2 on an open set Ω, then then v is a harmonic conjugate of u if and only they and satisfy CR-equations, and in that case both u and v are harmonic on Ω.

Exercise 2.2.3 Prove that v is a harmonic conjugate of u if and only $-u$ is a harmonic conjugate of v. ◁

2.3 Fractional linear transformations

2.3.1 The map $z \mapsto 1/z$

Recall that, for nonzero complex numbers a and b, the functions

$$z \mapsto az, \qquad z \mapsto z + b$$

map circles onto circles and straight lines onto straight lines. Now let us look at another simple function

$$z \mapsto \frac{1}{z}, \qquad z \neq 0.$$

Let us see the images of circles and straight lines under this function:

Consider the image of the circle:

$$C: \quad |z - z_0| = r.$$

Since

$$|z - z_0|^2 = (\bar{z} - \bar{z}_0)(z - z_0) = |z|^2 - (\bar{z}_0 z + z_0 \bar{z}) + (|z_0|^2 - r^2),$$

the above equation can also be written as

$$|z|^2 - (\bar{z}_0 z + z_0 \bar{z}) + \rho = 0, \qquad \rho := |z_0|^2 - r^2. \qquad (*)$$

To find its image under $z \mapsto 1/z$, let us write $\zeta = 1/z$. Thus, the image is given by the equation

$$1 - (z_0\zeta + \bar{z}_0\bar{\zeta}) + \rho|\zeta|^2 = 0.$$

Thus, if $\rho = 0$, i.e., if $|z_0| = r$, i.e., if the circle passes through 0, then the image is given by

$$1 - (z_0\zeta + \bar{z}_0\bar{\zeta}) = 0$$

which is an equation of a straight line.

Now, assume that $\rho \neq 0$ i.e., $|z_0| \neq r$. Then the equation of the image takes the form

$$|\zeta|^2 - \left(\frac{z_0\zeta}{\rho} + \frac{\bar{z}_0\bar{\zeta}}{\rho}\right) + \frac{1}{\rho} = 0. \qquad (**)$$

Note that

$$\frac{1}{\rho} = \left|\frac{z_0}{\rho}\right|^2 - \left(\left|\frac{z_0}{\rho}\right|^2 - \frac{1}{\rho}\right) = \left|\frac{z_0}{\rho}\right|^2 - \left(\frac{r}{\rho}\right)^2.$$

Comparing $(**)$ with $(*)$, it follows that $(**)$ represents a circle with centre at $\bar{z}_0/\rho$ and radius r/ρ. Thus, under the function $z \mapsto 1/z$,

(i) circles passing trough 0 are mapped onto straight lines, and
(ii) circles not passing trough 0 are mapped onto circles.

Next, consider a straight line

$$L : \quad \bar{\alpha}z + \alpha\bar{z} + \gamma = 0.$$

This is mapped onto

$$\widetilde{L} : \quad \bar{\alpha}\left(\frac{1}{\zeta}\right) + \alpha\left(\frac{1}{\bar{\zeta}}\right) + \gamma = 0$$

$$\iff \bar{\alpha}\bar{\zeta} + \alpha\zeta + \gamma|\zeta|^2 = 0.$$

If $\gamma = 0$, i.e., if the line L passes through 0, then its image also a straight line passing through 0. If $\gamma \neq 0$, i.e., if L does not pass through 0, then

$$\widetilde{L} : \quad |\zeta|^2 + \left(\frac{\bar{\alpha}\bar{\zeta}}{\gamma} + \frac{\alpha\zeta}{\gamma}\right) = 0$$

which is an equation of a circle with centre at $-\bar{\alpha}/\gamma$ and radius $|\alpha|/\gamma$, which also pass through 0. Thus, under the function $z \mapsto 1/z$,

(i) straight lines passing through 0 are mapped onto straight lines trough 0, and

(ii) straight lines not passing through 0 are mapped onto circles passing through 0.

Thus, in general, under the function $z \mapsto 1/z$,

- circles and straight lines are are mapped onto either circles or straight lines.

2.3.2 Extended plane

Note that the function $z \mapsto 1/z$ is not defined at $z = 0$. However, we know that

$$\lim_{|z|\to 0} \frac{1}{|z|} = \infty,$$

i.e., for every $M > 0$, there exists $\delta > 0$ such that

$$|z| < \delta \Longrightarrow \frac{1}{|z|} > M.$$

We shall write this fact by

$$\lim_{z\to 0} \frac{1}{z} = \infty.$$

In the above, the ∞ is just a symbol which correspond to the north pole in *steriographic projection*. Let us extend the complex plane $\mathbb{C}$ by incorporating ∞, i.e., let us consider

$$\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$

Definition 2.3.1 The set $\widetilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called the **extended complex plane**. $\Diamond$

In view of the above definition, we say that $\lim_{z\to 0} \frac{1}{z} = \infty$ in the extended complex plane $\widetilde{\mathbb{C}}$.

Since, we have

$$\lim_{|z|\to \infty} \frac{1}{|z|} = 0,$$

i.e., for every $\varepsilon > 0$, there exists $M > 0$ such that

$$|z| > M \Longrightarrow \frac{1}{|z|} < \varepsilon,$$

we also write

$$\lim_{z\to\infty} \frac{1}{z} = 0$$

in the extended complex plane. Thus, for the functions

$$f(z) := az, \qquad g(z) = z + b, \qquad h(z) = \frac{1}{z}$$

for nonzero complex numbers a and b, we can write

$$\lim_{z\to\infty} f(z) = \infty, \qquad \lim_{z\to\infty} g(z) = \infty,$$

$$\lim_{z\to\infty} h(z) = 0, \qquad \lim_{z\to 0} h(z) = \infty.$$

Now, defining

$$f(\infty) = \infty, \qquad g(\infty) = \infty, \qquad h(\infty) = 0, \qquad h(0) = \infty,$$

we can say that f, g, h are defined on the extended complex plane $\widetilde{\mathbb{C}}$.

2.3.3 Fractional linear transformations

Now, for complex numbers a, b, c, d, consider the function

$$\varphi : z \mapsto \varphi(z) := \frac{az + b}{cz + d}.$$

Since,

$$\frac{az + b}{cz + d} = \frac{a}{c}\left(\frac{cz + bc/a}{cz + d}\right) = \frac{a}{c}\left(\frac{cz + d - [d - bc/a]}{cz + d}\right).$$

Thus,

$$\frac{az + b}{cz + d} = \frac{a}{c} - \left(\frac{1}{c}\right)\left(\frac{ad - bc}{cz + d}\right).$$

Thus, the function φ can be thought of as compositions of the functions f, g, h we can write the function. Note that if $ad - bc = 0$, then φ is a constant function. To avoid this case, we shall assume that $ad - bc \neq 0$.

Definition 2.3.2 For complex numbers a, b, c, d with $ad - bc \neq 0$, the function

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}},$$

is called a **fractional linear transformation** or a **linear fractional transformation** or a **Möbius transformation**. ◊

We observe that

$$\varphi(\infty) = \lim_{z\to\infty} \varphi(z) = \lim_{z\to\infty} \frac{a + b/z}{c + d/z} = \frac{a}{c}.$$

Thus,

$$\varphi(\infty) = \infty \iff c = 0.$$

Since φ is a composition of the functions $z \mapsto az$, $z \mapsto z + b$ and $z \mapsto 1/z$, we can infer that under φ, the family of circles and straight lines in $\mathbb{C}$ are mapped onto the family of circles and straight lines.

It can be easily seen that compositions of a finite number of fractional linear transformations is a fractional linear transformation.

Exercise 2.3.1 Suppose φ_1 and φ_2 are fractional linear transformations. Prove that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ are fractional linear transformations. ◁

Exercise 2.3.2 Consider a fractional linear transformation φ given by

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}}.$$

(i) Show that φ is one-one and onto and its inverse is given by

$$\varphi^{-1}(z) = \frac{-dz+b}{cz-a}, \qquad z \in \widetilde{\mathbb{C}}.$$

(ii) Show that φ is differentiable at every $z \in \mathbb{C}$ and its derivative is given by

$$\varphi'(z) = \frac{ad-bc}{(cz+d)^2}, \qquad z \in \mathbb{C}.$$

◁

Definition 2.3.3 A point $z_0 \in \widetilde{\mathbb{C}}$ is said to be a **fixed point** of a function $f : \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$ if $f(z_0) = z_0$. ◊

Consider a fractional linear transformation φ given by

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}}.$$

We have already observed that $\varphi(\infty) = \infty$ if and only if $c = 0$. Thus ∞ is a fixed point of φ if and only if $c = 0$. Suppose $c \neq 0$. Then we have

$$\varphi(z) = z \iff cz^2 + (d-a)z - b = 0.$$

Since the above equation has either two distinct roots or one repeated root, we can say that

Theorem 2.3.1 *A fractional linear transformation which is not an identity function has either one or two fixed points.*

In particular, the identity function is the only fractional linear transformation having more than two distinct fixed points.

By the above theorem we can identity a fractional linear transformation by requiring to map three distinct points z_1, z_2, z_3 to three distinct points w_1, w_2, w_3, respeoctively. This is done in the following theorem.

Theorem 2.3.2 *Given distinct points z_1, z_2, z_3 and distinct points w_1, w_2, w_3 in the plane $\widetilde{\mathbb{C}}$, there exists a unique fractional linear transformation φ such that $\varphi(z_j) = w_j$ for $j \in \{1,2,3\}$.*

Proof. First let us settle the uniqueness part. Suppose φ_1 and φ_2 are fractional linear transformations such that

$$\varphi(z_j) = w_j = \varphi_2(z_j), \qquad j \in \{1,2,3\}.$$

Then the fractional linear transformation $\psi := \varphi_2^{-1} \circ \varphi_1$ satisfies $\psi(z_j) = w_j$ for $j \in \{1,2,3\}$. Hence, ψ is the identity transformation, so that $\varphi - 1 = \varphi_2$.

Now, the question of existence. Let φ_1 and φ_2 be defined by

$$\varphi_1(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}, \qquad \varphi_2(z) = \frac{(z-w_1)(w_2-w_3)}{(z-w_3)(w_2-w_1)}.$$

If none of the points $z_1, z_2, z_3, w_1, w_2, w_3$ is ∞, then we see that

$$\varphi_1(z_1) = 0, \qquad \varphi_1(z_2) = 1, \qquad \varphi_1(z_3) = \infty,$$

$$\varphi_2(w_1) = 0, \qquad \varphi_2(w_2) = 1, \qquad \varphi_2(w_3) = \infty.$$

Hence, the transformation $\varphi := \varphi_2^{-1}\varphi_1$ satisfies the requirements. In case one of the points z_1, z_2, z_3 is ∞, then we define φ_1 as follows:

(i) If $z_1 = \infty$, then $\varphi_1(z) = \lim_{\zeta\to\infty} \frac{(z-\zeta)(z_2-z_3)}{(z-z_3)(z_2-\zeta)} = \frac{z_2-z_3}{z-z_3}$.

(ii) If $z_2 = \infty$, then $\varphi_1(z) = \lim_{\zeta\to\infty} \frac{(z-z_1)(\zeta-z_3)}{(z-z_3)(\zeta-z_1)} = \frac{z-z_1}{z-z_3}$.

(iii) If $z_3 = \infty$, then $\varphi_1(z) := \lim_{\zeta\to\infty} \frac{(z-z_1)(z_2-\zeta)}{(z-\zeta)(z_2-z_1)} = \frac{z-z_1}{z_2-z_1}$.

Similarly, if one of the points w_1, w_2, w_3 is ∞, then we define φ_2 as in the case of φ_1 by replacing z_j by w_j. Thus, in these cases also, the transformation $\varphi := \varphi_2^{-1}\varphi_1$ maps z_j onto w_j for $j \in \{1, 2, 3\}$. ∎

In the above theorem, if we write $w = \varphi(z)$, then it can be seen that

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

with the convention that if any of the points $z_1, z_2, z_3, w_1, w_2, w_3$ is ∞, the limit is taken for the corresponding expression.

2.3.4 Image of inverse points

Definition 2.3.4 Points p and q in the extended complex plane $\tilde{\mathbb{C}}$ are said to be inverse points with respect to a circle $\{z \in \mathbb{C} : |z - z_0| = r\}$ if

$$z_0 - p = \frac{r^2}{\bar{z}_0 - \bar{q}},$$

and in that case we say that p is inverse to q (respectively q is inverse to p). ◊

Thus, points p and q are inverse points with respect to a circle $\{z \in \mathbb{C} : |z - z_0| = r\}$ if and only if

(i) they line on a ray emanating from 0 and have same arguments, and

(ii) satisfies $|z_0 - p|\,|z_0 - q| = r^2$ whenever p and q are different from z_0, and in case one of them is z_0, then the other is ∞.

We are going to show that

Theorem 2.3.3 *Under a bilinear transformation, inverse points are mapped onto inverse points.*

First suppose that the bilinear transformation is linear, i..e, it is given by

$$\varphi(z) = az + b$$

for some nonzero $a, b \in \mathbb{C}$. Let p and q be inverse points with respect to a circle $\{z \in \mathbb{C} : |z - z_0| = r\}$. Then, taking $\alpha = ap + b$, $\beta = aq + b$, we have $p = (\alpha - b)/a$ and $q = (\beta - b)/a$ so that

$$z_0 - p = \frac{r^2}{\bar{z}_0 - \bar{q}} \iff z_0 - (\alpha - b)/a = \frac{r^2}{\bar{z}_0 - (\bar{\beta} - \bar{b})/\bar{a}}.$$

Thus, α and β are inverse points with respect to the circle with centre at $-b/a$ and radius r. Clearly, $p = \infty$ (resp. $q = \infty$) if and only if $\alpha = \infty$ (resp. $\beta = \infty$).

Next we consider the situation under the transformation

$$\varphi(z) = \frac{1}{z}.$$

Suppose p is inverse to q with respect to the circle, $|z - z_0| = r$.

Case (i): *Assume first that both p and q are different from z_0.*

Then we have

$$\begin{aligned} z_0 - p = \frac{r^2}{\bar{z}_0 - \bar{q}} &\iff (z_0 - p)(\bar{z}_0 - \bar{q}) = r^2 \\ &\iff (|z_0|^2 - r^2) - (\bar{z}_0 p + z_0 \bar{q}) + p\bar{q} = 0. \end{aligned}$$

Now, taking $\alpha = 1/p$, $\beta = 1/q$ and $\rho = |z_0|^2 - r^2$, we have

$$\begin{aligned} z_0 - p = \frac{r^2}{\bar{z}_0 - \bar{q}} &\iff (|z_0|^2 - r^2) - (\bar{z}_0 p + z_0 \bar{q}) + p\bar{q} = 0 \\ &\iff \rho - (\frac{\bar{z}_0}{\alpha} + \frac{z_0}{\bar{\beta}}) + \frac{1}{\alpha\bar{\beta}} = 0 \\ &\iff \left(\frac{\bar{z}_0}{\rho} - \alpha\right)\left(\frac{z_0}{\rho} - \bar{\beta}\right) = R^2, \end{aligned}$$

where

$$R^2 = \left|\frac{\bar{z}_0}{\rho}\right|^2 - \frac{1}{\rho} = \frac{r^2}{\rho^2}.$$

Thus $\alpha := 1/p$ is inverse to $\beta := 1/q$ with respect to the circle, $|z - w_0| = R$, where $w_0 = \bar{z}_0/\rho$ and $R = r/|\rho|$.

Case (ii): *Suppose $p = z_0$ so that $q = \infty$.*

Clearly, if $z_0 = 0$, then $\alpha := 1/p = \infty$ is inverse to $\beta := 1/q = 0$ with respect to the circle, $|z - w_0| = R$, where $w_0 = 0$ and $R = r/|\rho| = 1/r$.

Next assume that $z_0 \neq 0$. In this case, we have to show that $\alpha = 1/z_0$ and $\beta = 0$ are inverse points. Note that

$$\left(\frac{\bar{z}_0}{\rho} - \alpha\right)\left(\frac{z_0}{\rho} - \bar{\beta}\right) = \left(\frac{\bar{z}_0}{\rho} - \frac{1}{z_0}\right)\left(\frac{z_0}{\rho}\right) = \frac{|z_0|^2 - \rho}{\rho^2} = \frac{r^2}{\rho^2}.$$

Thus, the point $\alpha = 1/z_0$ is inverse to $\beta := 0$ with respect to the circle, $|z - w_0| = R$, where $w_0 = \bar{z}_0/\rho$ and $R = r/|\rho|$.

Case (iii): *Suppose $p = z_0$ so that $q = \infty$:* This case is similar to last case.

Thus, we have proved Theorem 2.3.3.

Theorem 2.3.3 helps in finding a general bilinear transformation that maps the open unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ onto itself.

Suppose the the required transformation is given by

$$\varphi(z) = \frac{az + b}{cz + d}, \qquad ad - bc \neq 0.$$

Suppose $z_0 \in D$ be such that $\varphi(z_0) = 0$, i.e.,

$$\frac{az_0 + b}{cz_0 + d} = 0.$$

Hence, $z_0 = -b/a$. Also we have

$$\varphi(z) = \left(\frac{a}{c}\right)\frac{z + b/a}{z + d/a} = \left(\frac{a}{c}\right)\frac{z - z_0}{z + d/a}.$$

Note that $z_1 := 1/\bar{z}_0 = -\bar{a}/\bar{b}$ is the inverse point of z_0 with respect to the circle $S := \{z \in \mathbb{C} : |z| = 1\}$. Since inverse points are mapped onto the respective inverse points, z_1 must be mapped onto ∞, as 0 and ∞ are inverse points. Thus, $z_1 = -d/a$, and hence, $d/a = -\bar{z}_0$ so that

$$\varphi(z) = \left(\frac{a\bar{z}_0}{c}\right)\frac{z - z_0}{z\bar{z}_0 - 1}.$$

Again, since

$$|\varphi(1)| = 1 \quad \text{and} \quad \left|\frac{1 - z_0}{1 - \bar{z}_0}\right| = 1,$$

we have $|a\bar{z}_0/c| = 1$. Thus, we obtain

$$\varphi(z) = \alpha\left(\frac{z - z_0}{z\bar{z}_0 - 1}\right) \quad \text{with} \quad |\alpha| = 1, \quad |z_0| < 1.$$

2.4 Problems

1. Suppose u is a real valued function defined on an open set $\Omega \subseteq \mathbb{R}^2$. Let $(x_0, y_0) \in \Omega$.

   (i) When do you say that u has partial derivatives u_x and u_y at (x_0, y_0)?

   (ii) When do you say that u is differentiable at (x_0, y_0)?

   (iii) What is gradient of u at (x_0, y_0)?

   (iv) What is the relation between gradient and derivative of u?

2. Show that a function f is differentiable at $z_0 \in \Omega$ if and only if its real part u and imaginary part v are differentiable at (x_0, y_0) and u_x, u_y, v_x, v_y satisfy the equations

   $$u_x(z_0) = v_y(z_0), \qquad u_y(z_0) = -v_x(z_0),$$

   and in that case

   $$f'(z_0) = u_x(z_0) + iv_x(z_0) = v_y(z_0) - iu_y(z_0).$$

3. Find points at which the following functions are differentiable:

   (i) $f(z) = x$, (ii) $f(z) = \bar{z} = x - iy$,

   (iii) $f(z) = |z|^2$, (iv) $f(z) = \bar{z}^2$.

4. Find points at which the functions in the last problem satisfy CR-equations.

5. Prove that the CR-equations in polar coordinates are q$ru_r = v_\theta$, $u_\theta = -rv_r$.

6. Suppose f is holomorphic on an open set Ω. Prove that if f satisfies any of the following conditions, then f is a constant function.

   (i) f' is constant on Ω,

   (ii) f is real valued on Ω,

   (iii) $|f|$ is constant on Ω,

   (iv) $\arg(f)$ is constant on Ω.

7. Suppose f is holomorphic on an open set Ω. Prove that the function $z \mapsto g(z) = \overline{f(\bar{z})}$ is holomorphic in $\Omega^* := \{\bar{z} : z \in G\}$.

8. (i) Show that the equation of a straight line is given by

$$\bar{\alpha}z + \alpha\bar{z} + \gamma = 0$$

for some α, β in $\mathbb{C}$ and $\gamma \in \mathbb{C}$.

(ii) Show that the above line passes through 0 if and only if $\gamma = 0$.

9. Show that the equation of a circle with centre at z_0 and radius $r > 0$ is given by

$$|z|^2 - (\bar{z}_0 z + z_0\bar{z}) + |z_0|^2 - r^2 = 0$$

10. Prove the following:

(i) For nonzero $a \in \mathbb{C}$, the function $z \mapsto az$ maps a straight line into a straight line and a circle into a circle.

(ii) For nonzero $b \in \mathbb{C}$, the function $z \mapsto z + b$ maps a straight line into a straight line and a circle into a circle.

11. For nonzero $a, b \in \mathbb{C}$, the function $z \mapsto az + b$ maps a straight line into a straight line and a circle into a circle - Why?

12. Given a curve $\gamma : [0,1] \to \mathbb{C}$, let $\tilde{\gamma} : [a,b] \to \mathbb{C}$ be defined by

$$\tilde{\gamma}(t) = \gamma(a + (b-a)t).$$

If Γ and $\widetilde{\Gamma}$ are the images of γ and $\tilde{\gamma}$ respectively, then show that Γ and $\widetilde{\Gamma}$ are homeomorphic.

13. Find the points at which the curve $\gamma : [0,4] \to \mathbb{C}$ defined in the following are not regular. Justify your answer:

$$\gamma(t) := \begin{cases} t, & 0 \le t \le 1, \\ 1 + (t-1)i, & 1 < t \le 2, \\ 3 - t + i, & 2 < t \le 3, \\ (4-t)i, & 3 < t \le 4. \end{cases}$$

14. Let f be defined on an open set Ω and differentiable at a point $z_0 \in \Omega$ and $f'(z_0) \neq 0$. Then prove that f preserves angle between curves which are regular at z_0 and intersecting at z_0.

15. Define $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and show that the real and imaginary parts of f satisfy the CR-equations if and only if

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

16. Define the operators $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ and show that the real and imaginary parts of f satisfy the CR-equations if and only if $\frac{\partial f}{\partial \bar{z}} = 0$.

17. Show that

$$\frac{\partial^2 f}{\partial \bar{z} \partial z} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

and deduce that f harmonic on $\Omega \iff \frac{\partial^2 f}{\partial \bar{z} \partial z} = 0$.

18. Prove that v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v.

19. Prove that v_1 and v_2 are harmonic conjugates of u if and only $v_1 - v_2$ is a constant.

20. Prove that if u is a real valued harmonic function on an open set Ω, then any two harmonic conjugates of u differ by a constant.

21. Prove that if u is a real valued on an open set Ω such that both u and u^2 are harmonic on Ω, then u is a constant function.

22. Prove that if u and v are harmonic functions on an open set Ω such that v is a harmonic conjugate of u, then uv and $u^2 - v^2$ are harmonic.

23. Prove that the Laplace equation $\triangle u = 0$ can be written in polar coordinates as $r^2 u_{rr} + r u_r + u_{\theta\tau} = 0$.

24. Prove that if u is a real valued harmonic function on an open set Ω, then $\frac{\partial u}{\partial z}$ is holomorphic on Ω.

25. Suppose φ_1 and φ_2 are fractional linear transformations. Prove that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ are fractional linear transformations.

26. Consider a fractional linear transformation φ given by

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}}.$$

(i) Show that φ is one-one and onto and its inverse is given by

$$\varphi^{-1}(z) = \frac{-dz+b}{cz-a}, \qquad z \in \widetilde{\mathbb{C}}.$$

(ii) Show that φ is differentiable at every $z \in \mathbb{C}$ and its derivative is given by

$$\varphi'(z) = \frac{ad-bc}{(cz+d)^2}, \qquad z \in \mathbb{C}.$$

27. Show that the set of all fractional linear transformations is in one-one correspondence with the set of all 2×2 nonsingular matrices with complex entries.

28. Let $\mathcal{F}$ be the set of all fractional linear transformations. Define a binary operation on $\mathcal{F}$ so that $\mathcal{F}$ becomes a group.

29. Show a fractional linear transformation maps every circle and straight line in $\mathbb{C}$ onto either a circle or a straight line.

30. Prove that the identity function is the only fractional linear transformation having more than two distinct fixed points.

31. Given distinct points z_1, z_2, z_3 and distinct points w_1, w_2, w_3 in the plane $\widetilde{\mathbb{C}}$, show that the fractional linear transformation $w = \varphi(z)$ defined by

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

map z_1, z_2, z_3 onto w_1, w_2, w_3, respectively.

32. Find the fractional linear transformation φ that maps -1, 0, i onto the points 0, 1, $-i$, respectively.

What is the image of the circle passing through -1, 0, i? A circle or a straight line? Why?

33. Find the fractional linear transformation φ that maps $0, 1, \infty$ onto the points $1, i, -1$, respectively.

    What is the image of the real axis under this transformation? Why?

34. Find the fractional linear transformation φ that maps $1, i, -1$ onto the points $i, 0, -i$, respectively.

    What is the image of the unit circle (with centre at 0) under this transformation? Why?

35. Suppose

$$\varphi(z) = \frac{az+b}{cz+d}, \qquad z \in \widetilde{\mathbb{C}}.$$

    is the fractional linear transformation that maps the real axis onto the unit circle (with centre 0). Show that

$$|a| = |c| \neq 0, \qquad |b| = |d| \neq 0.$$

36. If the fractional linear transformation in the last problem maps the upper half plane onto the open unit disk, then show that it is of the form

$$\varphi(z) = \alpha \, \frac{z - z_0}{z - \bar{z}_0}$$

    for some α, z_0 in $\mathbb{C}$ such that $|\alpha| = 1$ and $\mathrm{Im}(z_0) > 0$.

3

Elementary Functions

In this chapter we define the complex analogues of real elementary functions, namely, the functions such as polynomials, rational functions, exponential functions, and the functions obtained from these functions by applying the operations of addition, substraction, multiplications, divisions and compositions. We are already familiar with polynomials

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

and rational functions

$$f(z) = \frac{a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n}{b_0 z^m + b_1 z^{m-1} + \ldots + b_{m-1} z + b_m} .$$

Note that the above two types functions are holomorphic wherever they are defined. Now we shall consider exponential function and other elementary functions associated with it.

3.1 Exponential Function

Recall that the function $\exp(x)$, denoted by e^x for real x is defined by

$$e^x := \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

and we know that, it satisfies the relation

$$e^{x_1 + x_2} = e^{x_1} e^{x_2}, \qquad x_1, x_2 \in \mathbb{R}.$$

In view of this we may define

$$e^{iy} := \sum_{n=1}^{\infty} \frac{(iy)^n}{n!}, \qquad (*)$$

for real y. Note that the series in $(*)$ converge absolutely for every $y \in \mathbb{R}$ so that

$$
\begin{aligned}
e^{iy} &= \sum_{n=0}^{\infty} \frac{(iy)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(iy)^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} \\
&= \cos y + i \sin y.
\end{aligned}
$$

In view of the above we may define e^z for $z = x + iy$ with $x, y \in \mathbb{R}$ as

$$e^z := e^x(\cos y + i \sin y).$$

We note that the function $z \mapsto e^z$ satisfies

- $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for every $z_1, z_2 \in \mathbb{C}$,
- $e^{z+2\pi i} = e^z$ for every $z \in \mathbb{C}$, and
- it is holomorphic on the entire complex plane.

Further, we observe that for $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$,

- $|e^z| = e^x$ and $\arg(z) = y$.

The last property shows that the function $z \mapsto e^z$ maps straight lines parallel to y-axis onto concentric circles with centre 0, and straight lines parallel to y-axis onto rays emanating from 0. In fact,

- the strip $\{z \in \mathbb{C} : -\pi \leq \mathrm{Im}(z) < \pi\}$ is mapped onto the entire complex plane.

Exercise 3.1.1 Given θ_1 and $\theta_2 \in [0, 2\pi)$ with $\theta_1 < \theta_2$, , describe the image of the strip $\{z \in \mathbb{C} : \theta_1 \leq \mathrm{Im}(z) < \theta_2\}$ under the map $z \mapsto e^z$. ◁

Exercise 3.1.2 Given $r, R0$ with $0 < r < R$, describe the image of the strip $\{z \in \mathbb{C} : r \leq \mathrm{Re}(z) < R\}$ under the map $z \mapsto e^z$. ◁

Exercise 3.1.3 For the function $f(z) = e^z$, describe the curves

$$|f(z)| = \text{constant}, \qquad \arg(f(z)) = \text{constant}.$$

◁

Exercise 3.1.4 If f is is holomorphic on $\mathbb{C}$ satisfying $f'(z) = f(z)$, and g is defined by $g(z) = e^{-z}f(z)$, then show that $g'(z) = 0$, so that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$. ◁

Exercise 3.1.5 Find the most general form of a holomorphic f on $\mathbb{C}$ satisfying $f'(z) = cf(z)$ for some constant $c \in \mathbb{C}$. ◁

Having defined exponential function, we move on to define hyperbolic and trigonometric functions.

3.2 Hyperbolic and Trigonometric Functions

For $z \in \mathbb{C}$, we define hyperbolic and trigonometric functions.

Hyperbolic functions:

$$\sinh z = \frac{e^z + e^{-z}}{2}, \qquad \cosh z = \frac{e^z - e^{-z}}{2},$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z},$$
$$\mathrm{sech} z = \frac{1}{\cosh z}, \qquad \mathrm{cosech} z = \frac{1}{\sinh z}.$$

Trigonometric functions:

$$\sin z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \cos z = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$
$$\sec z = \frac{1}{\cos z}, \qquad \mathrm{cosec} z = \frac{1}{\sin z}.$$

One may observe that

$$\sin z = \frac{\sinh z}{i}, \qquad \cos z = \cosh iz.$$

Exercise 3.2.1 Derive the identities:

(i) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

(ii) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

(iii) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

(iv) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

(v) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

(vi) $|\sin z|^2 = \sin^2 x + \sinh^2 y$. ◁

Exercise 3.2.2 Show that

$$\cos(z + 2\pi) = \cos z, \qquad \sin(z + 2\pi) = \sin(z)$$

for all $z \in \mathbb{C}$. ◁

Exercise 3.2.3 Find zeros of $\sin z$ and $\cos z$. ◁

Exercise 3.2.4 Find all roots of the equation $\cos z = 1$. ◁

3.3 Logarithms

Definition 3.3.1 Logarithm of a complex number a is a complex number b such that $e^b = a$, and in that case we write $b = \log a$. ◊

Thus,

• a complex number b is a logarithm of a complex number a if and only if b is a zero of the function f defined by $f(z) = e^z - a$.

Clearly,

• if b is a logarithm of a, then $b + 2n\pi i$ is also a logarithm of a for every $n \in \mathbb{Z}$.

Observe that if $b = \log a$, then

$$|a| = |e^b| = e^{\mathrm{Re}(b)} \quad \text{and} \quad \arg(a) = \mathrm{Im}(b).$$

Hence,

$$\log a = \ln|a| + i\arg(a).$$

The value of $\log a$ corresponding to the principal value of $\arg a$ is denoted by $\mathrm{Log}\, a$, i.e.,

$$\mathrm{Log}\, a = \ln|a| + i\mathrm{Arg}\, a.$$

Exercise 3.3.1 Show that, for $n \in \mathbb{Z}$,

(i) $2n\pi i = \log 1$.

(ii) $(2n + 1)\pi i = \log(-1)$.

(iii) $\left(2n + \frac{1}{2}\right)\pi i = \log i$. ◁

Exercise 3.3.2 Find all values of

(i) $\cosh(\log 2)$,

(ii) $\log(\log i)$. ◁

Exercise 3.3.3 Does the relation $\log a_1 a_2 = \log a_1 + \log a_2$ hold for all nonzero a_1, a_2 in $\mathbb{C}$? ◁

3.4 Branches of $\arg(z)$ and $\log(z)$

Recall that

$$z \mapsto \arg(z) \quad \text{and} \quad z \mapsto \log(z)$$

are "multi-valued mappings". So, by the strict definition of the term, they are not functions. They can be considered as set-valued functions, that is, $\arg(z)$ and $\log(z)$ are subsets of $\mathbb{C}$ containing more than one elements rather than single numbers. Recall that

$$\log(z) = \ln|z| + i\arg(z). \tag{$*$}$$

Now, the question is the following:

> Given a multi-valued map $f(z)$ on certain open set Ω, can we identify certain values of $f(z)$ for each $z \in \Omega$, say $F(z) \in f(z)$ such that the $F : \Omega \to \mathbb{C}$ is a continuous function?

For instance,

$$z \mapsto \text{Arg}\,(z)$$

is a single valued function on $\mathbb{C} \setminus \{0\}$. But, this function is not continuous on the negative real axis:

$$z_n = -1 + i/n \to -1, \qquad z'_n = -1 - i/n \to -1,$$

but,

$$\text{Arg}\,(z_n) \to \pi, \qquad \text{Arg}\,(z'_n) \to -\pi.$$

However, if Ω_0 is the set obtained by deleting the whole of negative-real axis, including 0, from the complex plane, then the function

$$z \mapsto \text{Arg}\,(z)$$

is continuous on Ω_0. Similarly,

$$z \mapsto \text{Log}\,(z)$$

is continuous on Ω_0.

Definition 3.4.1 Suppose f is a multi-valued map defined on an open connected set $\Omega \subseteq \mathbb{C}$. Then by a **branch** of f we mean a continuous function $f_0 : \Omega_0 \to \mathbb{C}$ defined on an open set $\Omega_0 \subseteq \Omega$ such that for each $z \in \Omega_0$, $f_0(z)$ is one of the values of $f(z)$. $\Diamond$

Suppose f_a and f_ℓ are branches of $\arg(z)$ and $\log(z)$, respectively, on an open connected set Ω_0.

In view of $(*)$,

$$f_\ell(z) = \ln|z| + i f_a(z)$$

and hence

$$f_a(z) = \mathrm{Im} f_\ell(z).$$

It can be see that

- difference of any two branches of $\arg(z)$ is an integer multiple of 2π and

- difference of any two branches of $\log(z)$ is an integer multiple of $2\pi i$.

By the discussion preceding the definition, it is clear that

$$f_a(z) := \mathrm{Arg}\,(z) \quad \text{and} \quad f_\ell(z) := \mathrm{Log}\,(z)$$

are branches of $\arg(z)$ and $\log(z)$, respectively, on the set

$$\Omega := \{z = re^{i\theta} : r > 0,\ -\pi < \theta < \pi\}.$$

Also, $\arg(z)$ and $\log(z)$ have branches on any open disc which does not contain the point 0. If $z_0 = r_0 e^{i\theta_0}$ is the centre of the disc, then we can define

$$f_a(z) = \arg(z) \quad \text{with} \quad \theta_0 - \pi/2 < \arg(z) < \theta_0 + \pi/2.$$

The point 0 has special significance for the $\arg(z)$ and $\log(z)$ and hence has a special name for it, the *branch point*.

Definition 3.4.2 Let f be a multi-valued map defined on an open connected set Ω and f_0 be a branch of f on an open subset Ω_0 of Ω.

(i) A curve Γ in Ω is called a **branch cut** for f if f has a branch on $\Omega_0 := \Omega \setminus \Gamma$.

(ii) A point $z_0 \in \Omega$ is called a **branch point** for f if z_0 is the intersection of all branches of f.

$\Diamond$

Analyticity of branches of logarithm function

Let $f_a(z)$ be a branch of $arg(z)$ in an open connected set Ω. We know that Ω cannot contain 0, and $f_a(z)$ is a real valued continuous function. Let $f_\ell(z)$ be the corresponding branch of $\log(z)$, i.e.,

$$f_\ell(z) = \ln|z| + i f_a(z), \qquad z \in \Omega.$$

We show that f_ℓ is analytic in Ω.

Clearly,

$$f_\ell(z) = \frac{1}{2}\ln(r^2) + i\theta, \qquad z := re^{i\theta} \in \Omega.$$

Recall that if $f := u + iv$ is an analytic function on an open set, then the CR-equations in polar coordinates is given by

$$r u_r = v_\theta, \qquad u_\theta = -r v_r.$$

In the case of f_ℓ we have

$$u(r,\theta) = \frac{1}{2}\ln(r^2), \qquad v(r,\theta) = \theta.$$

Hence, we have

$$u_r = \frac{1}{r}, \quad u_\theta = 0, \quad v_r = 0, \quad v_\theta = 1.$$

Thus, the partial derivatives of u and v are continuous and they satisfy CR-equations. Hence, f_ℓ is analytic in Ω.

It can be also, seen that

$$f_\ell(z) = \frac{1}{2}\ln(x^2 + y^2) + i\tan^{-1}(y/x), \qquad z = x + iy \in \Omega.$$

Hence,

$$u_x = \frac{x}{x^2 + y^2}, \quad u_y = \frac{y}{x^2 + y^2}, \quad v_x = \frac{-y}{x^2 + y^2}, \quad v_y = \frac{x}{x^2 + y^2}.$$

Again we see that partial derivatives of u and v are continuous and they satisfy CR-equations. Further, we have

$$f'(z) = u_x + i v_x = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{|z|^2} = \frac{1}{z}.$$

3.5 Problems

1. For $y \in \mathbb{R}$, show that the series $e^{iy} := \sum_{n=1}^{\infty} \frac{(iy)^n}{n!}$ converges.

2. For $y \in \mathbb{R}$, $e^{iy} = \cos y + i \sin y$. Why?

3. Show that the function $z \mapsto e^z$ is holomorphic on the entire complex plane and satisfies the following:

   (i) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for every $z_1, z_2 \in \mathbb{C}$,

   (ii) $e^{z+2\pi i} = e^z$ for every $z \in \mathbb{C}$,

   (iii) if $x = \text{Re}(z)$ and $y = \text{Im}(z)$, then $|e^z| = e^x$ and $\arg(z) = y$, and

   (iv) the strip $\{z \in \mathbb{C} : -\pi \leq \text{Im}(z) < \pi\}$ is mapped onto the entire complex plane.

4. Given θ_1 and $\theta_2 \in [0, 2\pi)$ with $\theta_1 < \theta_2$, , describe the image of the strip $\{z \in \mathbb{C} : \theta_1 \leq \text{Im}(z) < \theta_2\}$ under the map $z \mapsto e^z$.

5. Given $r, R0$ with $0 < r < R$, describe the image of the strip $\{z \in \mathbb{C} : r \leq \text{Re}(z) < R\}$ under the map $z \mapsto e^z$.

6. For the function $f(z) = e^z$, describe the curves

$$|f(z)| = \text{constant}, \qquad \arg(f(z)) = \text{constant}.$$

7. If f is is holomorphic on $\mathbb{C}$ satisfying $f'(z) = f(z)$, and g is defined by $g(z) = e^{-z}f(z)$, then show that $g'(z) = 0$, so that $f(z) = ce^z$ for some constant $c \in \mathbb{C}$.

8. Find the most general form of a holomorphic f on $\mathbb{C}$ satisfying $f'(z) = cf(z)$ for some constant $c \in \mathbb{C}$.

9. Derive the identities:

   (i) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

   (ii) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$.

   (iii) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

   (iv) $\sin z = \sin x \cosh y + i \cos x \sinh y$.

(v) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

(vi) $|\sin z|^2 = \sin^2 x + \sinh^2 y$.

10. Show that

$$\cos(z + 2\pi) = \cos z, \qquad \sin(z + 2\pi) = \sin(z)$$

for all $z \in \mathbb{C}$.

11. Find zeros of $\sin z$ and $\cos z$.

12. Find all roots of the equation $\cos z = 1$.

13. Show that if b is a logarithm of a, then $b + 2n\pi i$ is also a logarithm of a for every $n \in \mathbb{Z}$.

14. Show that if $b = \log a$, $\log a = \ln|a| + i \arg(a)$.

15. Show that, for $n \in \mathbb{Z}$,

(i) $2n\pi i = \log 1$.

(ii) $(2n + 1)\pi i = \log(-1)$.

(iii) $\left(2n + \frac{1}{2}\right) \pi i = \log i$.

16. Find all values of

(i) $\cosh(\log 2)$,

(ii) $\log(\log i)$.

17. Does the relation $\log a_1 a_2 = \log a_1 + \log a_2$ hold for all nonzero a_1, a_2 in $\mathbb{C}$?

4

Power Series

4.1 Convergence

In this chapter we study convergence and other properties of series of the form $a_0 + \sum_{n=1}^{\infty} a_n(z - z_0)^n$ which we write as

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{$*$}$$

whenever sequence (a_n) in $\mathbb{C}$ and $z_0 \in \mathbb{C}$ are given.

Clearly, the series $(*)$ is a special case of the series

$$\sum_{n=0}^{\infty} f_n(z), \tag{$**$}$$

where (f_n) is a sequence of complex valued functions defined on some subset $\Omega \subseteq \mathbb{C}$.

Definition 4.1.1 The series in $(**)$ is said to **converge** at a point $z \in \Omega$ if the sequence of its partial sums converge at z, i.e., if the sequence $(g_n(z))$ of complex numbers, where

$$g_n(z) = \sum_{j=0}^{n} f_j(z), \qquad z \in \Omega, \tag{$+$}$$

converges. $\Diamond$

Definition 4.1.2 The series in $(**)$ is said to **converge**

(i) **point-wise** to a function g on Ω if it converges to $g(z)$ at every point $z \in \Omega$;

(ii) **absolutely** on Ω if the series $\sum_{n=0}^{\infty} |f_n(z)|$ converges at every point $z \in \Omega$;

(iii) **uniformly** to a function g on Ω if the sequence (g_n) defined as in (+) converge uniformly to g on Ω.

$\Diamond$

Remark 4.1.1 It can be seen that if $(**)$ converges at a point z, then

$$f_n(z) \to 0 \quad \text{as} \quad n \to \infty.$$

This follows from the face that

$$|f_n(z)| = |g_n(z) - g_{n-1}(z)| \to 0 \quad \text{as} \quad n \to \infty.$$

It should be noted that a series of the form $\sum_{n=1}^{\infty} a_n$, where (a_n) is a sequence of complex numbers, is a particular case of $(**)$ with $f_n(z) = a_n$ for all $z \in \mathbb{C}$. $\Diamond$

EXAMPLE 4.1.1 Consider the series $\sum_{n=0}^{\infty} z^n$. Note that

$$g_n(z) = \sum_{j=0}^{n} z^j = \frac{1 - z^{n+1}}{1 - z}, \qquad z \neq 1,$$

so that

$$\frac{1}{1-z} - g_n(z) = \frac{z^{n+1}}{1-z}.$$

Thus, if $|z| < 1$, then

$$g_n(z) \to \frac{1}{1-z} \quad \text{as} \quad n \to \infty.$$

Also, for $z| < 1$, we have

$$\sum_{j=0}^{n} |z^j| = \frac{1 - |z|^{n+1}}{1 - |z|} \to \frac{1}{1 - |z|} \quad \text{as} \quad n \to \infty.$$

Thus, the series $\sum_{n=0}^{\infty} z^n$ converges absolutely for $|z| < 1$. Further, if $0 < r < 1$, then for $|z| \leq r$, we have

$$\left| \frac{1}{1-z} - g_n(z) \right| = \frac{|z|^{n+1}}{|1-z|} \leq \frac{r^{n+1}}{1-r}.$$

Since $r^n \to 0$, it follows that $\sum_{n=0}^{\infty} z^n$ converges uniformly on the set $\{z \in \mathbb{C} : |z| < r\}$. $\square$

As in the case of real valued functions of real variable, we have the following result.

Theorem 4.1.1 (M-test) *Suppose (f_n) is a sequence of complex valued functions defined on some subset $\Omega \subseteq \mathbb{C}$. Suppose there exists a sequence (M_n) of positive real numbers such that*

(i) $|f_n(z)| \leq M_n$ *for all* $n \in \mathbb{N}$ *and for all* $z \in \Omega$, *and*

(ii) $\sum_{n=1}^{\infty} M_n$ *converges.*

Then the series $\sum_{n=1}^{\infty} f_n$ *converges uniformly on* Ω.

Proof. Note that for $n > m$,

$$|g_n(z) - g_m(z)| = \left|\sum_{j=m+1}^{n} f_j(z)\right| \leq \sum_{j=m+1}^{n} |f_j(z)| \leq \sum_{j=m+1}^{n} M_j. \quad (\star)$$

Now, let $\varepsilon > 0$, and let $N \in \mathbb{N}$, Since $\sum_{n=1}^{\infty} M_n$ converges, from the above it follows from $(\star)$ that $(g_n(z))$ is a Cauchy sequence, and hence it converges. Let

$$g(z) = \lim_{n\to\infty} g_n(z), \qquad z \in \Omega.$$

Again from $(\star)$,

$$|g(z) - g_m(z)| = \lim_{n\to\infty} |g_n(z) - g_m(z)| \leq \sum_{j=m+1}^{\infty} M_j.$$

Now, let $\varepsilon > 0$ be given and $N \in \mathbb{N}$ be such that $\sum_{j=m+1}^{n} M_j < \varepsilon$ for all $m \geq N$. Then we have

$$|g(z) - g_m(z)| < \varepsilon \quad \forall\, n \geq N, \forall\, z \in \Omega.$$

Thus, (g_n) converges uniformly to g. ∎

Now, we prove an important theorem due to Abel[1] in the theory of power series.

[1] Niels Henrik Abel (5 August 1802 – 6 April 1829) was a noted Norwegian mathematician who proved the impossibility of solving the quintic equation in radicals - Curtsey Wikepedia.

Theorem 4.1.2 *Consider a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. If this series converges at a point z_1, then it converges at every point z such that $|z-z_0| < |z_1-z_0|$.*

An immediate corollary:

Corollary 4.1.3 *If the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ diverges at a point z_2, then it diverges at every point z such that $|z-z_0| > |z_2-z_0|$.*

Proof of Theorem 4.1.4. Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges at z_1. Let z be such that $|z-z_0| < |z_1-z_0|$. Note that for every $n \in \mathbb{N}$,

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n.$$

Since $|a_n(z_1-z_0)^n| \to 0$, there exists $M > 0$ such that $|a_n(z_1-z_0)^n| \leq M$ for all $n \in \mathbb{N}$, and since $|z-z_0| < |z_1-z_0|$, $\sum_{n=1}^{\infty} \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n$ converges. Hence, by comparison test, the series $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ converges. ∎

Theorem 4.1.4 *Suppose a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z with $|z-z_0| < r$ for some $r > 0$. Then, for any ρ with $0 < \rho < r$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on the set $\{z : |z-z_0| \leq \rho\}$.*

In particular, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on every compact subset of the disc $\{z : |z-z_0| < r\}$.

Proof. Let $\rho < r_1 < r$ and let z_1 be such that $|z_1-z_0| = r_1$. Then we have

$$|a_n(z-z_0)^n| = |a_n(z_1-z_0)^n| \left(\frac{|z-z_0|}{r_1}\right)^n \leq |a_n(z_1-z_0)^n| \left(\frac{\rho}{r_1}\right)^n.$$

Let $M > 0$ be such that $|a_n(z_1-z_0)^n| \leq M$ for all $n \in \mathbb{N}$. Thus,

$$|a_n(z-z_0)^n| \leq M \left(\frac{\rho}{r_1}\right)^n \qquad \forall\, n \in \mathbb{N}.$$

Hence, by M-test, $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on the set $\{z : |z-z_0| \leq \rho\}$. ∎

Suppose

$$R := \sup\{|z-z_0| : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges at } z\}.$$

By Theorem 4.1.4 and Corollary 4.1.3, it follows that

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges for } |z-z_0| < R$$

and

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ diverges for } |z-z_0| > R.$$

Definition 4.1.3 For the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, the number

$$R := \sup\{|z-z_0| : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges at } z\}$$

is called the **radius of convergence** of the series. The disc

$$\{z : |z-z_0| < R\}$$

is its **disc of convergence** and the set

$$\{z \in \mathbb{C} : \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ converges at } z\}$$

is called its **region of convergence**. $\Diamond$

Clearly, if Ω is the region of convergence, then

$$\{z : |z-z_0| < R\} \subseteq \Omega \subseteq \{z : |z-z_0| \le R\}.$$

Thus, region of convergence may include some of the boundary points of the disc of convergence. See the following examples.

EXAMPLE 4.1.2 (i) Consider the series $\sum_{n=0}^{\infty} z^n$. Its radius of convergence is 1. Its disc of convergence and region of convergence are the same, the disc:

$$\{z : |z-z_0| < 1\}.$$

(ii) Consider the series $\sum_{n=0}^{\infty} \frac{z^n}{n^2}$. Since $|z^n/n^2| \le 1/n^2$ for all z with $|z| \le 1$, the series converges absolutely on

$$\Omega := \{z \in \mathbb{C} : |z| \le 1\}.$$

If $z \notin \Omega$, then $|z|^n/n^2 \not\to 0$ (see Exercise 4.1.1). Thus, the series does not converge outside Ω. Hence, the region of convergence is Ω, and the disc of convergence is interior of Ω.

(iii) Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n}$. Since $|z^n/n| \leq |z|^n$ for all n , the series converges absolutely for $|z| < 1$. Also, the series does not converge at $z = 1$. Hence, the radius of convergence is 1.

Now, let z be such that $|z| = 1$ and $z \neq 1$. Since

$$\Big|\sum_{j=0}^{n} z^j\Big| = \Big|\frac{1 - z^{n+1}}{1 - z}\Big| \leq \frac{2}{|1 - z|} \qquad \forall\, n \in \mathbb{N}.$$

Now, recall (see Theorem 3.42 in Rudin[2]) that if $\left(\sum_{j=1}^{n} a_j\right)$ is bounded and (b_n) is a decreasing sequence of non-negative real numbers which converges to 0, then $\sum_{n=1}^{\infty} a_n b_n$ converges. In the present example, we have $a_n = z^n$ with $|z| = 1$, $z \neq 1$ and $b_n = 1/n$. Thus, $\sum_{n=0}^{\infty} z^n/n$ converge. Thus, the region of convergence of $\sum_{n=0}^{\infty} z^n/n$ is

$$\{z : |z - z_0| \leq 1\} \setminus \{1\}.$$

(iv) Consider the series $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$. Since $|z^n/n!| \leq 1/n!$ for all n and for all $z \in \mathbb{C}$, the series converges absolutely in the entire plane. Hence, the region of convergence is the entire $\mathbb{C}$.

(v) Consider the series $\displaystyle\sum_{n=0}^{\infty} n!z^n$. Writing $a_n = n!z^n$, we have for $z \neq 0$

$$\Big|\frac{a_{n+1}}{a_n}\Big| = (n + 1)|z| \to \infty \quad \text{as} \quad n \to \infty.$$

Hence, for $z \neq 0$, $a_n \not\to 0$ (see Exercise 4.1.1). Thus, region of convergence of the series is the singleton set $\{0\}$. □

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[2] W. Rudin, *Principles of Mathematical Analysis*, 1976: Let $A_n := \sum_{j=1}^{n} a_n$ and $M > 0$ is such that $|A_n| \leq M$ for all $n \in \mathbb{N}$. For $q > p$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p-1}^{q-1} A_n(b_n - b_{n+1}) + A_p b_p + A_q b_q.$$

Hence, $\Big|\displaystyle\sum_{n=p}^{q} a_n b_n\Big| \leq M \sum_{n=p-1}^{q-1} (b_n - b_{n+1}) + M b_p + M b_q = M(b_{p-1} + b_q) \to 0$ as $p, q \to \infty$.

Exercise 4.1.1 Suppose $a_n > 0$ for all $n \in \mathbb{N}$ and $\dfrac{a_{n+1}}{a_n} \to \alpha$. Prove that

(i) if $\alpha \geq 1$, then $a_n \not\to 0$, and

(ii) if $\alpha > 1$, then $a_n \to \infty$. ◁

For the next theorem, we recall the following from real analysis:

Theorem 4.1.5 *Let* (a_n) *be a sequence of positive real numbers and* $b = \limsup_n a_n$. *Then the following hold:*

(i) *If* $b < \ell$, *then there exists* $k \in \mathbb{N}$ *such that* $a_n < \ell$ *for all* $n \geq k$.

(ii) *If* $b > \ell$, *then* $a_n > \ell$ *for infinitely many* $n \in \mathbb{N}$.

Proof. Let $b_n := \sup\{a_j : j \geq n\}$. Then, we see that (b_n) is a decreasing sequence, and hence

$$b := \limsup_n a_n := \lim_{n\to\infty} b_n.$$

(i) Suppose $b < \ell$. Then there exists $k \in \mathbb{N}$ such that $b_k < \ell$. Hence, $a_n \leq b_k < \ell$ for all $n \geq k$.

(ii) Suppose $b > \ell$. Then $b_n > \ell$ for all $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that $k_n > n$ and $a_{k_n} > \ell$. ∎

Using the above theorem we have the following result on convergence of a sequence of complex numbers.

Theorem 4.1.6 *Suppose* (a_n) *is a sequence in* $\mathbb{C}$, *and*

$$b := \limsup_n |a_n|^{1/n}.$$

Then the series $\sum_{n=1}^{\infty} a_n$

(i) *converges absolutely if* $b < 1$, *and*

(ii) *diverges if* $b > 1$.

Proof. (i) Suppose $b < 1$, and let ℓ be such that $b < \ell < 1$. Then by Theorem 4.1.5 (i), there exists $k \in \mathbb{N}$ such that $|a_n|^{1/n} < \ell$ for all $n \geq k$. Thus,

$$|a_n| < \ell^n \qquad \forall\, n \geq k.$$

Since $0 \le \ell < 1$, the series $\sum_{n=1}^{\infty} \ell^n$, and hence the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(ii) Suppose $b > 1$, and let ℓ be such that $1 < \ell < b$. Then by Theorem 4.1.5 (ii), there exists a sequence (k_n) in $\mathbb{N}$ such that $|a_{k_n}|^{1/k_n} > \ell$ for all $n \in \mathbb{N}$. Thus,

$$|a_{k_n}| > \ell^{k_n} > 1 \qquad \forall\, n \in \mathbb{N}.$$

Hence, $a_n \not\to 0$. Consequently, the series $\sum_{n=1}^{\infty} a_n$ diverges. ∎

Theorem 4.1.7 (Cauchy[3]–Hadamard[4] theorem) *For a sequence (a_n) in $\mathbb{C}$, let*

$$\beta := \limsup_n |a_n|^{1/n}.$$

Then $R := 1/\beta$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Proof. By Theorem 4.1.6, the series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

(i) converges if $|z - z_0| \limsup_n |a_n|^{1/n} < 1$, and

(ii) diverges if $|z - z_0| \limsup_n |a_n|^{1/n} > 1$.

From the above, the conclusion follows. ∎

Using similar arguments as in the proof of Theorem 4.1.7, we obtain the following.

Theorem 4.1.8 (Ratio test) *For a sequence (a_n) of nonzero complex numbers, let*

$$\gamma := \limsup_n \left| \frac{a_{n+1}}{a_n} \right|.$$

Then $R := 1/\gamma$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n (z - z_0)^n$.

Exercise 4.1.2 Find the radius of convergence for each of the following series[5]:

$$\text{(i)} \sum_{n=0}^{\infty} n^2 z^n, \quad \text{(ii)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n, \quad \text{(iv)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n. \ \triangleleft$$

---

[3] Augustine Luis Cauchy (21 August 1789 – 23 May 1857)
[4] Jaques Hadamard (8 December 1865 - 17 October 1963)
[5] W.Rudin, Chapter 3, Exercise 9.

Theorem 4.1.9 *Suppose $R > 0$ be the radius of convergence of the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, and*

$$f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n \quad \textit{for} \quad |z-z_0| < R.$$

Then f is differentiable and the series $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ converges for $|z| < R$ and

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1} \quad \textit{for} \quad |z-z_0| < R.$$

Proof. Since $\limsup_n |na_n|^{1/n} = \limsup_n |a_n|^{1/n}$ the radii of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ are the same. Thus, $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ converges for $|z-z_0| < R$. Let

$$g(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}, \qquad |z-z_0| < R.$$

For the sake simplicity of presentation, without loss of generality, we assume that $z_0 = 0$. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} na_n z^{n-1} \quad \text{for} \quad |z-z_0| < R.$$

Now, let $0 < \rho < R$ and let $z \neq z_1$ with $|z| \leq \rho$, $|z_1| \leq \rho$. Note that

$$\begin{aligned} f(z) - f(z_1) &= \sum_{n=0}^{\infty} a_n(z^n - z_1^n) \\ &= \sum_{n=0}^{\infty} a_n(z-z_1) \sum_{k=0}^{n-1} z^k z_1^{n-k-1} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{f(z)-f(z_1)}{z-z_1} - g(z_1) &= \sum_{n=1}^{\infty} a_n \Big[\sum_{k=0}^{n-1} z^k z_1^{n-k-1} - n z_1^{n-1} \Big] \\ &= \sum_{n=2}^{\infty} a_n \Big[\sum_{k=0}^{n-1} z_1^{n-k-1} (z^k - z_1^k) \Big] \end{aligned}$$

Now,

$$z_1^{n-k-1}(z^k - z_1^k) = z_1^{n-k-1}(z - z_1)\sum_{j=0}^{k-1} z^j z_1^{k-j-1}$$

so that

$$|z_1^{n-k-1}(z^k - z_1^k) \leq |z - z_1|\rho^{n-k-1}\sum_{j=0}^{k-1} \rho^j \rho^{k-j-1} \leq |z - z_1|k\rho^{n-2}.$$

Thus,

$$\begin{aligned}\Big|\frac{f(z) - f(z_1)}{z - z_1} - g(z_1)\Big| &\leq \sum_{n=2}^{\infty} |a_n| \Big[\sum_{k=0}^{n-1} |z - z_1|k\rho^{n-2} \\ &\leq |z - z_1|\sum_{n=2}^{\infty} |a_n|\frac{n(n-1)}{2}\rho^{n-2}.\end{aligned}$$

Since $\sum_{n=2}^{\infty} |a_n|\frac{n(n-1)}{2}\rho^{n-2}$ converges (Why?), we have

$$\frac{f(z) - f(z_1)}{z - z_1} \to g(z_1) \quad \text{as} \quad z \to z_1.$$

Thus, the proof is complete. ∎

By the above theorem, a power series can be differentiated term by term within its disc of convergence. Further, if f represents a power series on its disc of convergence, i.e.,

$$f(z) := \sum_{n=0}^{\infty} a_n(z - z_0)^n \quad \text{for} \quad |z - z_0| < R.$$

then f is infinitely times differentiable, and for any $k \in \mathbb{N}$,

$$f^{(k)}(z) := \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k} \quad \text{for} \quad |z - z_0| < R.$$

Hence, we have

$$a_k = \frac{f^{(k)}(z_0)}{k!}, \qquad k \in \mathbb{N}.$$

The above discussion urges us to ask the following question:

> Suppose f is holomorphic in a disc centered at z_0. Then, does f have the representation
>
> $$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$
>
> in that disc?

We answer this question affirmatively.

Suppose f is holomorphic in a neighborhood of z_0. Let Ω be the largest open disc centered at z_0 in which f is holomorphic, i.e., if

$$\rho := \sup\{|z-z_0| : f \text{ is holomorphic at } z\},$$

then $\Omega = \{z \in \mathbb{C} : |z-z_0| < \rho\}$. We shall show that there exists (a_n) in $\mathbb{C}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad \forall\, z \in \Omega,$$

and hence

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \forall\, z \in \Omega.$$

This is one of the biggest differences between

(i) a real valued differentiable function of a real variable on an open set in $\mathbb{R}$ and

(ii) a complex valued differentiable function of a complex variable on an open set in $\mathbb{C}$.

4.2 Problems

Note: Problems from 1-3 and 9-12 are discussed in class, either by proving them, or by way of indicating their proofs.

1. Suppose a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all z with $|z-z_0| < r$ for some $r > 0$. Then, prove that for any ρ with $0 < \rho < r$, the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on the set $\{z : |z-z_0| \le \rho\}$.

2. Let R be the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Prove the following:

   (i) If the series converges at z_1, then $R \geq |z_1 - z_0|$.

   (ii) If the series diverges at z_2, then $R \leq |z_2 - z_0|$.

   (iii) If $\sum_{n=0}^{\infty} |a_n(z-z_0)^n|$ diverges at z_2, then $R \leq |z_2 - z_0|$.

3. If (a_n) and (b_n) are sequences of complex numbers such that $|a_n| \leq M|b_n|$ for all $n \in \mathbb{N}$, and if R_1 nd R_2 are the radius of convergence of $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} b_n(z-z_0)^n$ respectively, then prove that $R_1 \leq R_2$.

4. Using Problem 3, show that radius of convergence of $\sum_{n=1}^{\infty} n^{-n} z^n$ is ∞.

5. Prove that radius of convergence of $\sum_{n=1}^{\infty} n^n z^n$ is 0.

6. Find a power series in a neighborhood of $z_0 = 1$ which represents the function $f(z) := 1/z$.

7. Find the radius of convergence for each of the following series:

   $$\text{(i)} \sum_{n=0}^{\infty} n^2 z^n, \quad \text{(ii)} \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n, \quad \text{(iii)} \sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n, \quad \text{(iv)} \sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n.$$

   $$\text{(v)} \sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^{3n}, \quad \text{(vi)} \sum_{n=1}^{\infty} \frac{z^{n!}}{n}, \quad \text{(vii)} \sum_{n=0}^{\infty} n^n z^{n^2}, \quad \text{(viii)} \sum_{n=0}^{\infty} \frac{n+1}{n!} z^{n^3}.$$

8. Give one example each of a power series which

   (a) converges only on the interior of the disc of convergence,

   (b) converges diverges on a proper subset of the boundary of the disc of convergence,

   (c) converges on the closure of the disc of convergence.

9. Show that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$ have the same radius of convergence.

10. If (a_n) and (b_n) are sequences of complex numbers such that $\limsup_n |b_n|^{1/n}$, then show that the series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=1}^{\infty} na_nb_n(z-z_0)^{n-1}$ have the same radius of convergence.

11. For a sequence (a_n) of nonzero complex numbers, let $\gamma := \limsup_n \left|\frac{a_{n+1}}{a_n}\right|$. Then show that $R := 1/\gamma$ is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-z_0)^n$.

12. If f represents a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ on its disc of convergence, then $a_k = \frac{f^{(k)}(z_0)}{k!}$ for every $k \in \mathbb{N}$. Justify.

13. Let f be a holomorphic function in an open set Ω such that $f' = f$ and $f(0) = 1$. Then show that $f(z) = e^z$. Deduce that, for all $z \in \mathbb{C}$,

    (i) $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

    (ii) $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

    (iii) $$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}.$$

14. Prove that, for $|z| < 1$,

    (i) $$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

    (ii) $$\operatorname{Log} \frac{1}{1-z} = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

15. Find the function represented by the series $\sum_{n=1}^{\infty} n^2 z^n$.

5

Integration

5.1 Integrals along Piecewise Smooth Curves

Recall that a *curve* in $\mathbb{C}$ is a continuous function from a closed (non-degenerate) interval to C. Thus, a continuous function

$$\gamma : [a, b] \to \mathbb{C}$$

is a curve in $\mathbb{C}$. If the range of γ is contained in a set $\Omega \subseteq \mathbb{C}$, then we say that γ is a curve in Ω. The point $z_1 := \gamma(a)$ is called the *initial point* of γ and the point $z_2 := \gamma(b)$ is called the *final point* or *terminal point* of γ.

Definition 5.1.1 Given a curve $\gamma : [a, b] \to \mathbb{C}$, the corresponding **reversed curve** $\tilde{\gamma} : [a, b] \to \mathbb{C}$ is defined by

$$\tilde{\gamma}(t) = \gamma(a + b - t), \qquad t \in [a, b].$$

$\Diamond$

Note that, $\tilde{\gamma}$ has the same range as that of γ, but its orientation as t varies from a to b is reversed.

Definition 5.1.2 We shall call the range of a curve $\gamma : [a, b] \to \mathbb{C}$ with orientation as t varies from a to b as **oriented range** of γ, and denote it by Γ_γ or, simply, Γ.

If Γ is the oriented range of $\gamma : [a, b] \to \mathbb{C}$, then we say that γ is a **parametrization** of Γ. $\Diamond$

Thus, oriented range of $\gamma : [a, b] \to \mathbb{C}$ is

$$\Gamma_\gamma := \{\gamma(t) : a \leq t \leq b\}.$$

with orientation as t varies from a to b.

Similarly, the oriented range of $\tilde{\gamma}$ will be denoted by $\tilde{\Gamma}$.

Convention: Hereafter, if we say Γ is a curve, then we mean that Γ is the oriented range of a curve $\gamma : [a, b] \to \mathbb{C}$.

Definition 5.1.3 Curves $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$ are said to be *equivalent*, if there exists continuous increasing bijection $\varphi : [a_1, b_1] \to [a_2, b_2]$ such that $\gamma_1 = \gamma_2 \circ \varphi$, and in that case we write $\gamma_1 \sim \gamma_2$.

If $\gamma_1 \sim \gamma_2$, then for γ_2 is called a **reparameterization** of γ_1. $\Diamond$

Exercise 5.1.1 (i) Show that equivalence of curves defines an equivalence relation.

(ii) If $\gamma_1 \sim \gamma_2$, then show that $\Gamma_{\gamma_1} = \Gamma_{\gamma_2}$.

(iii) Given a curve $\gamma : [a, b] \to \mathbb{C}$ and a closed interval $[\alpha, \beta]$, find a curve $\eta : [\alpha, \beta] \to \mathbb{C}$ such $\gamma \sim \eta$. $\lhd$

Exercise 5.1.2 Given a curve $\gamma : [a, b] \to \mathbb{C}$, show that the curve $\eta : [-b, -a] \to \mathbb{C}$ defined by

$$\eta(t) = \gamma(-t), \qquad -b \le t \le -a,$$

is equivalent to the reverse of γ, i.e., $\eta \sim \tilde{\gamma}$. $\lhd$

Given curves $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_1 : [c, d] \to \mathbb{C}$, define $\gamma_3 : [a, b + d - c] \to \mathbb{C}$ by

$$\gamma_3(t) = \begin{cases} \gamma_1(t), & a \le t \le b, \\ \gamma_2(t + c - b), & b \le t \le b + d - c. \end{cases}$$

Then, $\gamma_3 : [a, b + d - c] \to \mathbb{C}$ is a curve such that terminal point of γ_1 is the initial point of γ_2. This curve γ_3 is called the sum of the curves γ_1 and g_2 and is denoted by $\gamma_1 + \gamma_2$.

In the sequel, we shall be dealing with *piecewise smooth* curves.

Definition 5.1.4 A curve $\gamma : [a, b] \to \mathbb{C}$ is said to be **piecewise smooth** if

(i) γ is differentiable except possibly at a finite number of points in $[a, b]$, and

(ii) right and left derivative of γ exists at every point in $[a, b]$. $\Diamond$

Definition 5.1.5 If $\gamma : [a, b] \to \mathbb{C}$ is a piecewise smooth curve, then its length is defined by

$$\ell(\Gamma) := \int_a^b |\gamma'(t)| dt,$$

where Γ is the oriented range of γ. $\Diamond$

In order to define integrals of complex valued functions of a complex variable along piecewise smooth curves, first we define integral of complex valued functions on bounded intervals.

Suppose $\varphi : [a, b] \to \mathbb{C}$ be a continuous function. Then the integral

$$\int_a^b \varphi(t)\, dt$$

is defined in a natural way:

Definition 5.1.6 If $\varphi = \varphi_1 + i\varphi_2$, where φ_1 and φ_2 are real and imaginary parts of φ, respectively, then

$$\int_a^b \varphi(t)\, dt := \int_a^b \varphi_1(t)\, dt + i \int_a^b \varphi_2(t)\, dt.$$

◊

The following properties can be easily verified.

- $\displaystyle\int_a^b [\varphi(t) + \psi(t)]\, dt = \int_a^b \varphi(t)\, dt + \int_a^b \psi(t)\, dt,$
- $\displaystyle\int_a^b [\alpha\varphi(t)]\, dt = \alpha \int_a^b \varphi(t)\, dt$ for all $\alpha \in \mathbb{R}$.

Exercise 5.1.3 Verify the above properties. ◁

Further, we have the following.

Proposition 5.1.1

$$\Big| \int_a^b \varphi(t)\, dt \Big| \leq \int_a^b |\varphi(t)|\, dt. \qquad (*)$$

Proof. If $\int_a^b \varphi(t)\, dt = 0$, then clearly $(*)$ holds. So, assume that $\int_a^b \varphi(t)\, dt$ is nonzero and $\lambda := \Big| \int_a^b \varphi(t)\, dt \Big| / \int_a^b \varphi(t)\, dt$. Then we have

$$\begin{aligned} \Big| \int_a^b \varphi(t)\, dt \Big| &= \lambda \int_a^b \varphi(t)\, dt = \int_a^b \lambda\varphi(t)\, dt \\ &= \int_a^b \mathrm{Re}\varphi(t)\, dt \leq \int_a^b |\lambda\varphi(t)|\, dt = \int_a^b |\varphi(t)|\, dt \end{aligned}$$

This completes the proof. ∎

Exercise 5.1.4 For $a > 0$ and $\alpha > 0$, show that
$\lim_{\alpha\to\infty} \int_0^a e^{-(\alpha+it)^2} dt = 0$. ◁

We have the following analogue of the *fundamental theorem of integration.*

Proposition 5.1.2 *If $\varphi : [a,b] \to \mathbb{C}$ is continuously differentiable, then*

$$\int_a^b \varphi'(t)\,dt = \varphi(b) - \varphi(a). \qquad (**)$$

Now we define the integral for a continuous complex valued function of a complex variable along a piecewise smooth curve.

Definition 5.1.7 Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve with oriented range Γ and $f : \Gamma \to \mathbb{C}$ be a continuous function. Then we define

$$\int_\Gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

We shall also denote the above integral by $\int_\gamma f(z)dz$. ♢

Proposition 5.1.3 *The following hold.*

1. $\int_\Gamma [f(z) + g(z)]\,dz = \int_\Gamma f(z)\,dz + \int_\Gamma g(z)\,dz,$
2. $\int_\Gamma [\alpha f(z)]\,dz = \alpha \int_\Gamma f(z)\,dz$ *for all* $\alpha \in \mathbb{C}$.
3. $\int_{\tilde{\Gamma}} f(z)\,dz = -\int_\Gamma f(z)\,dz.$
4. $\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$

Proof. Exercise. ▮

Remark 5.1.1 It is to be observe that the integral $\int_{\Gamma_\gamma} f(z)dz$ depends essentially on the way the point $\gamma(t)$ moves along Γ_γ as t varies on $[a,b]$. It can happen that two different curves γ and η can have same range Γ, but $\int_\gamma f(z)dz \neq \int_\eta f(z)dz$. For example, consider the curves

$$\gamma(t) = t^2, \qquad \eta(t) = t \quad \text{for} \quad 0 \le t \le 1.$$

Clearly, the range of γ and η coincide and it is the line segment $[0,1]$. However, if we take $f(z) = z$, then

$$\int_\gamma f(z)dz = \int_0^1 t^2(2t)dt = \frac{1}{2}, \qquad \int_\eta f(z)dz = \int_0^1 t(t)dt = \frac{1}{3}.$$

◊

Definition 5.1.8 (i) A curve $\gamma : [a,b] \to \mathbb{C}$ is called a **closed curve** if its initial and terminal points are the same.

(ii) A curve $\gamma : [a,b] \to \mathbb{C}$ is said to **intersect** at a point z_0 if there exists distinct $t_1, t_2 \in [a,b]$ such that $\gamma(t_1) = z_0 = \gamma(t_2)$.

(iii) A closed curve $\gamma : [a,b] \to \mathbb{C}$ is said to be a **simple closed curve** if for distinct points t_1, t_2 in $[a,b]$, $\gamma(t_1) = \gamma(t_2)$ implies $\{t_1, t_2\} = \{a, b\}$. ◊

Proposition 5.1.4 $\left|\int_\Gamma f(z)\,dz\right| \le M\ell_\Gamma$, *where* $M = \max_{z\in\Gamma} |f(z)|$.

Proof. By $(*)$,

$$\left|\int_\Gamma f(z)\,dz\right| \le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt \le M\ell_\Gamma.$$

Thus, the proof is complete. ∎

Proposition 5.1.5 *Let f be continuously differentiable in an open set containing a piecewise smooth curve with initial point z_1 and terminal point z_2. Then*

$$\int_\Gamma f'(z)\,dz = f(z_2) - f(z_1).$$

In particular, if Γ is closed, then $\displaystyle\int_\Gamma f'(z)\,dz = 0.$

Proof. By $(**)$, we have

$$\begin{aligned} \int_\Gamma f'(z)\,dz &= \int_a^b f'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b (f\circ\gamma)'(t)\,dt = (f\circ\gamma)(b) - (f\circ\gamma)(a) \\ &= f(z_2) - f(z_1). \end{aligned}$$

The particular case follows, since in this case $z_2 = z_1$. ∎

Corollary 5.1.6 *Let Γ be a closed curve and f has a primitive on an open set Ω containing Γ, i.e., there exists a continuously differentiable function g such that $g' = f$ on Ω. Then*

$$\int_\Gamma f(z)\,dz = 0.$$

Proof. Follows from Theorem 5.1.4. ∎

Remark 5.1.2 We shall prove in the next section that if f is holomorphic on a *simply connected domain* Ω, then integral of f along every closed curve in Ω is 0. This result is known as *Cauchy's theorem.* ◊

EXAMPLE 5.1.1 Let Γ be the circle with center at z_0 and radius r given by

$$\Gamma := \{z_0 + re^{it},\ 0 \le t \le 2\pi\}.$$

Then, we can take $\gamma(t) := z_0 + re^{it}$, $0 \le t \le 2\pi$ so that

$$\int_\Gamma \frac{dz}{z - z_0}\,dz = \int_0^{2\pi} \frac{\gamma'(t)dt}{\gamma(t) - z_0} = \int_a^b \frac{rie^{it}dt}{re^{it}} = 2\pi i.$$

Note that the value if the above integral does not depend on the centre and the radius. □

Exercise 5.1.5 Let Γ_n be the circle with center at z_0 and radius r traced n times, i.e., $\Gamma_n := \{z_0 + re^{it},\ 0 \le t \le 2n\pi\}$. Then, show that

(i) $\displaystyle\int_{\Gamma_n} \frac{dz}{z - z_0} = 2n\pi i$, and if $p(z)$ is a polynomial, then

(ii) $\displaystyle\int_{\Gamma_n} p(z)\,dz = 0.$ ◁

EXAMPLE 5.1.2 Let Ω be the disc of convergence of a power series $\sum_{n=0}^\infty a_n(z - z_0)^n$ and let $f(z)$ represent this series in Ω. Let $g(z) = \sum_{n=0}^\infty \frac{a_n}{n+1}(z - z_0)^{n+1}$. Then we have $g'(z) = f(z)$. Hence, if Γ is a closed piecewise smooth curve in the disc of convergence, then

$$\int_\Gamma f(z)\,dz = 0.$$

Thus, integral of a function which represents a power series, over any closed piecewise curve in its disc of convergence, is 0. □

Exercise 5.1.6 For $a > 0$ and $\alpha > 0$, let Γ_α be the line segment joining $z_0 = \alpha$ to $z_1 = \alpha + ia$. Show that $\lim_{\alpha\to\infty} \int_{\Gamma_a} e^{-z^2} dz = 0$. ◁

Exercise 5.1.7 Suppose $f_n \to f$ uniformly on Γ. Then show that $\int_\Gamma f_n(z)dz \to \int_\Gamma f(z)\, dz$. ◁

Exercise 5.1.8 If f is continuous on $\{z : |z - z_0| \le 1\}$ and if $\gamma(t) = z_0 + re^{it}$, $0 \le t \le 2\pi\}$ and Γ is the oriented range of γ for $0 < r \le 1$, then show that $\lim_{r\to 0} \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz = f(z_0)$. ◁

5.2 Cauchy's Theorem

Now, we prove one of the most important theorems in complex analysis, the so called *Cauchy's theorem.* For its statement we shall use the following definitions.

Definition 5.2.1 An open connected subset of the complex plane is called a **domain**. ♢

It is easy to conceive the statement in the following theorem, though its proof is much involved and beyond the scope of this course:

Theorem 5.2.1 (Riemann's theorem) *If Γ is a simple closed curve, then it is the boundary of two disjoint domains one of which is bounded and the other is unbounded.*

Definition 5.2.2 If Γ is a simple closed curve, then the bounded domain as in Definition 5.2.1 is called the **domain enclosed by** Γ. ♢

Definition 5.2.3 A domain Ω is said to be **simply connected** if for every simple closed curve Γ in Ω, the domain enclosed by Γ is contained in Ω. ♢

In the following, we shall call a simple closed piecewise smooth curve as *simple closed contour.*

Definition 5.2.4 A simple closed contour γ is said to be **positively oriented** if the domain enclosed by it is on the left while traversing along γ.

More precisely, for each $t \in [a, b]$, the normal vector $\gamma'(t)e^{i\pi/2}$ at the point $\gamma(t)$ must direct towards the domain enclosed by γ, i.e., for each $t \in [a,b]$, there exists $\varepsilon_t > 0$ such that

$$\gamma(t) + \varepsilon\gamma'(t)e^{i\pi/2} \in \Omega_\gamma \quad \text{whenever} \quad 0 < \varepsilon < \varepsilon_t,$$

where Ω_γ is the domain enclosed by γ. ◊

Theorem 5.2.2 (Cauchy's theorem - using Green's theorem) *Let Γ be a positively oriented simple closed contour and let Ω be the domain enclosed by Γ. Let f be holomorphic and its derivative continuous on $\Omega \cup \Gamma$. Then*

$$\int_\Gamma f(z)dz = 0.$$

Proof. Let u and v be the real and imaginary parts of f. Let Γ have the parametrization γ on $[a,b]$. Let $\gamma(t) = x(t)+iy(t)$, $t \in [a,b]$. Then

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))][x'(t) + iy'(t)]\,dt \\
&= \int_a^b [ux' - vy']dt + i\int_a^b [uy' - vx']dt \\
&= \int_\Gamma [udx - vdy] + i\int_\Gamma [udy + vdx].
\end{aligned}$$

Now, by Green's theorem[1] and using the CR-equations, we have

$$\int_\Gamma [udx - vdy] = \iint\limits_\Omega \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = 0,$$

$$\int_\Gamma [vdx + udy] = \iint\limits_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0.$$

This completes the proof. ∎

Exercise 5.2.1 Prove Cauchy's theorem if Γ is any piecewise closed curve which intersects itself only at a finite number of points. ◁

[1]Green's theorem, named after the British mathematician and physicist George Green (14 July 1793 31 May 1841) : $\int_\Gamma Pdx+Qdy = \iint\limits_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$

We shall see that the condition that the real and imaginary parts of f have continuous partial derivatives is redundant. In fact, we shall prove the following general version of *Cauchy's theorem.*

Theorem 5.2.3 (Cauchy's theorem) *Let Ω be a simply connected domain and f be a holomorphic function on Ω. Then for every piecewise smooth closed curve Γ in Ω,*

$$\int_{\Gamma} f(z) = 0.$$

Our proof involves the following steps:

1. The theorem holds with γ as a triangle.

2. The theorem holds if γ is a rectangle.

3. Every holomorphic function on a simply connected domain has a holomorphic primitive.

4. Use Corollary 5.1.6.

First we require the following simple property of a domain. For simplicity of expression, we shall call a curve a *rook-path*[2] if it is a polygonal curve consisting of line segments parallel to coordinate axes.

Lemma 5.2.4 *Any two points in an open connected set can be joined by a rook-path.*

Proof. Let Ω be an open connected set. If Ω is empty, then the lemma holds vacuously. Hence assume that Ω is nonempty and $z_0 \in \Omega$. Now, consider the set Ω_0 of all those points in Ω which can be joined with z_0 by rook-paths. We have to show that $\Omega_0 = \Omega$. Since Ω is connected and $z_0 \in \Omega_0$, it is enough to show that both Ω_0 and its compliment $\Omega_1 := \Omega \setminus \Omega_0$ are open.

Let $z \in \Omega_0$, and let $r > 0$ be such that $B(z, r) \subseteq \Omega$. Since every point in $B(z, r)$ can be joined to z by a rook-path, we have $B(z, r) \subseteq \Omega_0$. Thus, Ω_0 is an open set.

---

[2] borrowed from Persian *rokh*, in Sanskrit *rath* meaning "chariot", is a piece in the strategy board game of chess.

Next, let $\zeta \in \Omega_1$, and let $\rho > 0$ be such that $B(\zeta, \rho) \subseteq \Omega$. Since every point in $B(\zeta, \rho)$ can be joined to ζ by a rook-path, and since $\zeta \notin \Omega_0$, we have $B(\zeta, \rho) \subseteq \Omega_1$. Thus, Ω_1 is also an open set. This completes the proof. ∎

Now, we proceed to prove (1)-(3).

Theorem 5.2.5 (Goursat's lemma)[3] *Suppose* Γ *is a positively oriented triangle and* f *is holomorphic on and inside* Γ. *Then*

$$\int_\Gamma f(z)dz = 0.$$

Proof. By joining midpoints of the sides of Γ construct four positively oriented triangles, say $\Gamma_{0,1}, \Gamma_{0,2}, \Gamma_{0,3}, \Gamma_{0,4}$. Then we obtain

$$I := \int_\Gamma f(z)dz = \sum_{j=1}^{4} \int_{\Gamma_{0,j}} f(z)dz.$$

Since $|I| \leq \sum_{j=1}^{4} \Big| \int_{\Gamma_{0,j}} f(z)dz \Big|$, it follows that there exists $j_0 \in \{1, 2, 3, 4\}$ such that

$$\Big| \int_{\Gamma_{0,j_0}} f(z)dz \Big| \geq \frac{1}{4} \Big| \int_\Gamma f(z)dz \Big|.$$

Denote this Γ_{0,j_0} by Γ_1.

Now, joining midpoints of the sides of Γ_1 construct four positively oriented triangles, say $\Gamma_{1,1}, \Gamma_{1,2}, \Gamma_{1,3}, \Gamma_{1,4}$. Following the same argument as above with Γ_1 in place of Γ, there exists $j_1 \in \{1, 2, 3, 4\}$ such that

$$\Big| \int_{\Gamma_{1,j_1}} f(z)dz \Big| \geq \frac{1}{4} \Big| \int_{\Gamma_1} f(z)dz \Big| \geq \frac{1}{4^2} \Big| \int_\Gamma f(z)dz \Big|. \qquad (1)$$

Denote Γ_{1,j_1} by Γ_2 and continue the above procedure. Then we obtain a sequence $\{\Gamma_n\}$ of positively oriented triangles such that

$$\Big| \int_{\Gamma_n} f(z)dz \Big| \geq \frac{1}{4^n} \Big| \int_\Gamma f(z)dz \Big|.$$

[3] Edouard Goursat (1858-1936, a French mathematician, was the first recognized, in 1800, that continuity of the derivative is not required for proving Cauchy's theorem.

Let Ω_n be the closure of the domain enclosed by Γ_n. Then we see that $\Omega_n \supseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$ and

$$\text{diam}(\Omega_n) \le \ell(\Gamma_n) = \frac{1}{2^n}\ell(\Gamma). \qquad (2)$$

Hence, by *nested compact sets theorem* in real analysis, $\bigcap_{n=1}^{\infty} \Omega_n$ is a singleton set, say $\bigcap_{n=1}^{\infty} \Omega_n = \{z_0\}$.

Now, let $\varepsilon > 0$ be given. Since $z_0 \in \Gamma \cup \Omega$ and f is holomorphic at z_0, there exists $\delta > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \varepsilon|z - z_0| \quad \text{whenever} \quad |z - z_0| < \delta.$$

Let $N \in \mathbb{N}$ be such that $\Omega_n \subseteq B(z_0, \delta)$ for all $n \ge N$. Then, we have

$$\Big|\int_{\Gamma_n} (f(z) - f(z_0) - f'(z_0)(z - z_0)dz\Big| \le \varepsilon[\ell(\Gamma_n)]^2. \qquad (3)$$

Note that, since the function $z \mapsto f(z_0) + f'(z_0)(z - z_0)$ has a primitive, by Corollary 5.1.6,

$$\int_{\Gamma_n} [f(z_0) - f'(z_0)(z - z_0)]dz = 0 \qquad \forall\, n \in \mathbb{N}. \qquad (4)$$

Hence, by (1)-(4),

$$\Big|\int_{\Gamma} f(z)dz\Big| \le 4^n \Big|\int_{\Gamma_n} f(z)dz\Big| \le \varepsilon 4^n [\ell(\Gamma_n)]^2 = \varepsilon\ell(\Gamma).$$

This is true for every $\varepsilon > 0$. Hence, $\displaystyle\int_{\Gamma} f(z)dz = 0$. ∎

Corollary 5.2.6 *Conclusion in Theorem 5.2.5 holds if the positive oriented triangle* Γ *is replaced by a rectangle.*

Theorem 5.2.7 *Every holomorphic function on a simply connected domain has a holomorphic primitive.*

Proof. Let Ω be a simply connected domain and f be a holomorphic function on Ω. We show that there exists a holomorphic function g on Ω such that $g'(z) = f(z)$ for every $z \in \Omega$.

Let $z_0 \in \Omega$, and z be any arbitrary point in Ω. By Lemma 5.2.4 there exists a rook-path $\Gamma_{z_0,z}$ joining z_0 to z. Define

$$g(z) = \int_{\Gamma_{z_0,z}} f(\zeta)d\zeta.$$

In view of Corollary 5.2.6, the integral along any two rook-paths joining z_0 to z will give the same value. Hence, g is well defined on Ω as long as we restrict the integration along rook-paths joining z_0 to z. Further, if $\delta > 0$ and if $B(z,\delta) \subseteq \Omega$, then for any $h \in \mathbb{C}$ with $z+h \in B(z,\delta)$, we have

$$g(z+h) = \int_{\Gamma_{z_0,z+h}} f(\zeta)d\zeta.$$

Hence, by Corollary 5.2.6, we obtain

$$\frac{g(z+h)-g(z)}{h} - f(z) = \frac{1}{h}\int_{\Gamma_{z,z+h}} [f(\zeta)-f(z)]d\zeta.$$

Here, we also used the fact that $\displaystyle\int_{\Gamma_{z,z+h}} d\zeta = h$, by Proposition 5.1.5. Also, by Theorem 5.2.5,

$$\frac{1}{h}\int_{\Gamma_{z,z+h}} [f(\zeta)-f(z)]d\zeta = \frac{1}{h}\int_{C_h} [f(\zeta)-f(z)]d\zeta,$$

where C_h is the straight line segment joining z to $z+h$. Hence, by Proposition 5.1.4,

$$\Big|\frac{g(z+h)-g(z)}{h} - f(z)\Big| = \frac{\ell(C_h)}{|h|}\max_{\zeta\in C_h}|f(\zeta)-f(z)|.$$

Now, let $\varepsilon > 0$ be give. Since f is uniformly continuous on $\mathrm{cl}\,(\Omega)$, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$z+h \in \Omega, \quad |h| < \delta \Longrightarrow \max_{\zeta\in C_h}|f(\zeta)-f(z)| < \varepsilon.$$

Thus,

$$\Big|\frac{g(z+h)-g(z)}{h} - f(z)\Big| < \delta \quad \text{whenever} \quad |h| < \delta.$$

Consequently, g is differentiable at z and $g'(z) = f(z)$. This completes the proof. ∎

Proof of Theorem 5.2.3. The proof follows from Theorem 5.2.7 and Corollary 5.1.6. ∎

If we observe the proof of Theorem 5.2.7, it is apparent that we have actually proved the following.

Theorem 5.2.8 *Suppose f is a continuous function on a simply connected domain Ω and*

$$\int_\Gamma f(z)dz = 0$$

for every positively oriented triangle Γ in Ω. Then f has a holomorphic primitive on Ω.

5.3 Cauchy's Integral Formulas

Theorem 5.3.1 (Cauchy's integral formula) *Suppose f is analytic on and inside a simple closed contour Γ, and let Ω_Γ be the domain enclosed by Γ. Then for every $z \in \Omega_\Gamma$,*

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Proof. Let $z \in \Omega_\Gamma$ and let $C := C_r$ be a positively oriented circle with centre at z and radius r such that $C_r \subseteq \Omega_\Gamma$. Take two points on C_r and join with Γ by a line segments, say L_1 and L_2. More precisely, take two points $z_1, z_2 \in C_r$ and join with $\zeta_1, \zeta_2 \in \Gamma$ such that

$$|z_1 - \zeta_1| = \text{dist}\,(z_1, \Gamma), \qquad |z_2 - \zeta_2| = \text{dist}\,(z_2, \Gamma).$$

Cut Γ into into two pieces Γ_1, Γ_2 at the points ζ_1, ζ_2, and cut C into into two pieces C_1, C_2 at the points z_1, z_2 retaining the original orientations. Note that the function $\zeta \mapsto f(\zeta)/(\zeta - z)$ is analytic in the region between Γ and C, Hence by Cauchy's theorem, integral of $f(\zeta)/(\zeta - z)$ over the curves $\Gamma_1 + L_1 + \tilde{C}_1 + L_2$ and $\Gamma_2 + \tilde{L}_2 + \tilde{C}_2 + \tilde{L}_1$ are zeros. Therefore, we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Since $\frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z} = 1$, we have

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta. \qquad (*)$$

Since

$$\left|\frac{f(\zeta) - f(z)}{\zeta - z}\right| \to |f'(z)| \quad \text{as} \quad \zeta \to z,$$

there exists $\delta > 0$ such that

$$\left|\frac{f(\zeta) - f(z)}{\zeta - z}\right| \to |f'(z)| + 1 \quad \text{whenever} \quad |\zeta - z| < \delta.$$

Now, taking $r < \delta$, we have

$$\left|\int_C \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta\right| \leq (|f'(z)| + 1)2\pi r. \tag{$**$}$$

From ($*$) and ($**$), we have

$$\left|\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - f(z)\right| \leq (|f'(z)| + 1)r.$$

This is true for all r such that $0 < r < \text{dist}\,(z, \Gamma)$. Hence,

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = f(z),$$

completing the proof. ∎

Corollary 5.3.2 *Suppose f is holomorphic on $\{z \in \mathbb{C} : |z - z_0| \leq r\}$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Our next attempt is to show that if f is analytic at a point z_0, then in neigbourhood of z_0, f can be expressed as a power series.

Proposition 5.3.3 *Let C be a circle with centre z_0 and Ω_C be the domain enclosed by C. Let g be continuous on C and let*

$$\varphi(z) = \int_C \frac{g(\zeta)}{\zeta - z} d\zeta, \qquad z \in \Omega_C.$$

Then for every $z \in \Omega_C$,

$$\varphi(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n := \int_C \frac{g(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Proof. Let $\zeta \in \mathbb{C}$ and z lies inside C. Since $|z - z_0| < |\zeta - z_0|$, we have

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{(\zeta - z_0)[1 - \frac{z - z_0}{\zeta - z_0}]} \\ &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n.\end{aligned}$$

Since the above series converges uniformly on C, we have

$$\int_C \frac{g(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n := \int_C \frac{g(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

This completes the proof. ∎

Theorem 5.3.4 *Suppose f is analytic at a point z_0. Then f is infinitely differentiable in a neigbourhood D_0 of z_0 and we have the following.*

(i) **(Talor series expansion)**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \qquad z \in D_0 \qquad (*)$$

in that neigbourhood.

(ii) **(Cauchy's integral formula of higher orders)**

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

where Γ is a simple closed curve enclosing z_0 such that f is analytic on and inside Γ.

Proof. Using Proposition 5.3.3 taking $g = f$ and applying Cauchy's integral formula (Theorem 5.3.1), we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n := \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

for z in a neigbourhood of z_0, where C is a circle with centre at z_0 lying inside that neigbourhood. Since the function

$$\zeta \mapsto \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}$$

is analytic on $\overline{\Omega}_\Gamma \cap \Omega_C^c$, by Cauchy's theorem (Theorem 5.2.3), we obtain (*How?*)

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Now, from the discussion at the end of Chapter 4, we know that

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

This completes the proof. ∎

Definition 5.3.1 The series expansion $(*)$ of f is called the **Taylor series expansion** of f around z_0. ◊

By the above theorem derivative of a holomorphic function is holomorphic. Hence, in view of Theorem 5.2.8, we have a converse to the Cauchy's theorem.

Theorem 5.3.5 (Morera's theorem) *Suppose f is a continuous function on a simply connected domain Ω and*

$$\int_\Gamma f(z)dz = 0$$

for every positively oriented triangle Γ in Ω. Then f is holomorphic on Ω.

Remark 5.3.1 In fact, the conventional Morera's theorem is slightly weaker form of the above theorem, namely, *if f is a continuous function on a simply connected domain Ω and $\int_\Gamma f(z)dz = 0$ for every closed contour Γ in Ω, then f is holomorphic on Ω.* ◊

Exercise 5.3.1 Suppose f is an entire function, $M > 0$, $R > 0$ and $n \in \mathbb{N}$ such that

$$|f(z)| \leq M|z^n| \qquad \forall\, z \quad \text{with} \quad |z| \geq R.$$

Show that f is a polynomial of degree atmost n. ◁

Exercise 5.3.2 Suppose f is holomorphic for $|z| < 1$ and

$$|z| < 1 \Longrightarrow |f(z)| < 1.$$

Show that

$$|f'(z)| \leq \frac{1}{1 - |z|}.$$

◁

Theorem 5.3.6 (Liouville' therem) *Suppose f is an entire function, i.e., f is holomorphic on the entire $\mathbb{C}$. If f is bounded, then f is a constant function.*

Proof. Let $z \in \mathbb{C}$. Then, by Theorem 5.3.4,

$$f'(z) = \frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

where Γ_R is the circle with centre z_0 and radius R. Hence, we have

$$|f'(z)| = \frac{M\ell(\Gamma_R)}{2\pi R^2} = \frac{M}{R}.$$

Now, letting $R \to \infty$, we have $f'(z) = 0$. This is true for all $z \in \mathbb{C}$. Hence, f is a constant function. ∎

Theorem 5.3.7 (Fundamental theorem of algebra) *Suppose $p(z)$ is a nonconstant polynomial with complex coefficients. Then there exists $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.*

Proof. For $a_0 \neq 0$, let $p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$. Suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$. Then, for $z \neq 0$,

$$p(z) = a_0 z^n \left(1 + \frac{a_1}{a_0 z} + \ldots + \frac{a_{n-1}}{a_0 z^{n-1}} + \frac{a_n}{a_0 z^n}\right).$$

Hence,

$$|p(z)| \geq |a_0 z^n| \left(1 - \frac{|a_1|}{|a_0 z|} + \ldots + \frac{|a_{n-1}|}{|a_0 z^{n-1}|} + \frac{|a_n|}{|a_0 z^n|}\right).$$

Let $R > 0$ be such that

$$|z| \geq R \Longrightarrow \frac{|a_1|}{|a_0 z|} + \ldots + \frac{|a_{n-1}|}{|a_0 z^{n-1}|} + \frac{|a_n|}{|a_0 z^n|} \leq \frac{1}{2}.$$

Then, for $|z| \geq R$, we have

$$\frac{1}{|p(z)|} \leq \frac{2}{|a_0 z^n|} \leq \frac{2}{|a_0| R^n}.$$

Since $1/p(z)$ is bounded for $|z| \leq R$, it then follows that $1/p(z)$ is a bounded entire function. Hence, by Liouville's theorem, $1/p(z)$ is a constant function, so that $p(z)$ is a constant polynomial, which is a contradiction to our assumption on $p(z)$. ∎

5.3.1 Appendix

We state again the Cauchy's integral formula for the derivatives of analytic functions, and give another proof for the same.

Theorem 5.3.8 *Suppose f is analytic on an inside a positively oriented simple closed curve Γ and z lies inside Γ. Then, for every $n \in \mathbb{N} \cup \{0\}$, $f^{(n)}(z)$ exists and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Proof. We know that the above result is true for $n = 0$. Assume it for $n-1$. We shall prove it for n. Let $\rho > 0$ be such that the circle with centre z and radius ρ lies inside Γ. Then for $h \in \mathbb{C}$ with $|h| < \rho$,

$$\frac{f^{(n-1)}(z+h) - f^{(n)}(z)}{h} - \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

takes the form

$$\frac{(n-1)!}{2\pi i} \int_\Gamma \frac{f(z)}{h} \left[\frac{(\zeta - z)^n - (\zeta - z - h)^n}{(\zeta - z - h)^n (\zeta - z)^n} - \frac{n}{(\zeta - z)^{n+1}} \right] d\zeta.$$

Now, writing $a = \zeta - z$ and $b = \zeta - z - h$), we have

$$\begin{aligned} \frac{(\zeta - z)^n - (\zeta - z - h)^n}{h(\zeta - z - h)^n (\zeta - z)^n} - \frac{n}{(\zeta - z)^{n+1}} &= \frac{a^n - b^n}{h a^n b^n} - \frac{n}{a^{n+1}} \\ &= \frac{a(a^n - b^n) - nhb^n}{h a^{n+1} b^n} \end{aligned}$$

Since $a(a^n - b^n) = a(a-b) \sum_{j=0}^{n-1} a^{n-1-j} b^j = h \sum_{j=0}^{n-1} a^{n-j} b^j$, we have

$$\begin{aligned} a(a^n - b^n) - nhb^n &= \sum_{j=0}^{n-1} h b^{n-1-j} (a^{j+1} - b^{j+1}) \\ &= \sum_{j=0}^{n-1} h^2 b^{n-1-j} [a^j + a^{j-1} b + \ldots + a b^{j-1} + b^j] \end{aligned}$$

Since $|a| = \rho$ and $\rho - |h| \leq |b| \leq \rho + |h|$, taking $\alpha = \rho + |h|$ we obtain,

$$|a(a^n - b^n) - nhb^n| \leq |h|^2 \sum_{j=0}^{n-1} \alpha^{n-1-j} [\rho^j + r^{j-1} \alpha + \ldots + \rho \alpha^{j-1} + \alpha^j].$$

Thus, absolute value of

$$\frac{f(z)}{h}\left[\frac{(\zeta-z)^n-(\zeta-z-h)^n}{(\zeta-z-h)^n(\zeta-z)^n}-\frac{n}{(\zeta-z)^{n+1}}\right]$$

is less than or equal to

$$\varepsilon_h := \frac{M|h|\sum_{j=0}^{n-1}\alpha^{n-1-j}[\rho^j+r^{j-1}\alpha+\ldots+\rho\alpha^{j-1}+\alpha^j]}{\rho^{n+1}(\rho-|h|)^n}.$$

Hence,

$$\left|\frac{f^{(n-1)}(z+h)-f^{(n)}(z)}{h}-\frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta\right| \le \varepsilon_h\rho.$$

Since $\varepsilon_h \to 0$ as $|h| \to 0$, it follows that $f^{(n-1)}$ is differentiable at z and

$$f^{(n)}(z) = \lim_{|h|\to 0}\frac{f^{(n-1)}(z+h)-f^{(n)}(z)}{h} = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta.$$

This completes the proof. ∎

5.4 Zeros of analytic functions

Suppose f is analytic in a domain Ω and z_0 is a zero of f, i.e., $f(z_0) = 0$. Then, using the Taylor series expansion of f around z_0, it follows that

$$f(z) = (z-z_0)g(z)$$

in a neigbourhood of z_0, where g is analytic in a neigbourhood of z_0. In fact,

$$g(z) = f'(z_0) + \sum_{n=2}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-1}$$

in a neigbourhood of z_0. Note that, if $f'(z_0) = 0$, then we can write

$$f(z) = (z-z_0)^2 g_1(z),$$

where

$$g(z) = \frac{f^{(2)}(z_0)}{2} + \sum_{n=3}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-2}$$

in a neigbourhood of z_0.

Definition 5.4.1 (i) A point $z_0 \in \Omega$ is said to be a **zero of f of order** m if $f^{(k)}(z_0) = 0$ for $k = 0, 1, \ldots, m-1$ and $f^{(m)}(z_0) \neq 0$.

(ii) A point $z_0 \in \Omega$ is said to be a **zero of f of finite order** if it is a zero of f of order m for some $m \in \mathbb{N}$.

(iii) A zero of f which is not of finite order is called a **zero of f of infinite order**.

♢

Definition 5.4.2 A zero $z_0 \in \Omega$ of an analytic function f is said to be an **isolated zero** if there exists a $r > 0$ such that $B(z_0, r) \subseteq \Omega$ and $f(z) \neq 0$ for every $z \in B(z_0, r) \setminus \{z_0\}$. ♢

Remark 5.4.1 We observe the following:

(a) If $z_0 \in \Omega$ is a zero of f of order m, then

$$f(z) = (z - z_0)^m g(z)$$

in a neigbourhood of z_0, where g is analytic in a neigbourhood of z_0 and $g(z_0) \neq 0$.

(b) If $z_0 \in \Omega$ is a zero of f of infinite order, then $f^{(k)}(z_0) = 0$ for all $k \in \mathbb{N}$; consequently, $f = 0$ in a neigbourhood of z_0.

♢

Exercise 5.4.1 Prove the statements in the above remark. ◁

Theorem 5.4.1 *Every zero of finite order of an analytic function is isolated.*

Proof. Follows from Remark 5.4.1(a). ∎

Theorem 5.4.2 *Suppose f is analytic in an open connected set Ω. If Ω contains a zero of f of infinite order, then $f = 0$ on Ω.*

Proof. Follows from Remark 5.4.1(b) using the fact that Ω is open and connected. ∎

5.4.1 Identity theorem

Theorem 5.4.3 (Identity theorem-I) *Suppose f is analytic in an open connected set Ω. If Ω contains a point which is the limit point of a set of zeros of f, then $f = 0$ on Ω.*

Proof. Follows from Theorem 5.4.2. ∎

Theorem 5.4.4 (Identity theorem-II) *Suppose f and g are analytic in an open connected set Ω. If $f = g$ on a set having a limit point in Ω, then $f = g$ on Ω.*

Proof. Follows from Theorem 5.4.3. ∎

EXAMPLE 5.4.1 Let f be analytic in $\{z \in \mathbb{C} : |z| < 1\}$ and

$$f\Big(\frac{1}{n+1}\Big) = \frac{1}{n+1} \qquad \forall\, n \in \mathbb{N},$$

and

$$g\Big(\frac{1}{n+1}\Big) = 0 \qquad \forall\, n \in \mathbb{N}.$$

Then, by Theorem 5.4.4, $f(z) = z$ and $g(z) = 0$ for all z. □

EXAMPLE 5.4.2 We show that there is no analytic function f on $\Omega := \{z \in \mathbb{C} : |z| < 1\}$ satisfying

$$f\Big(\frac{1}{n}\Big) = \frac{(-1)^n}{n^2} \qquad \forall\, n \in \mathbb{N}.$$

Suppose there is an analytic function f satisfying the above requirements. Then we have

$$f\Big(\frac{1}{2n}\Big) = \frac{1}{(2n)^2} \qquad \forall\, n \in \mathbb{N},$$

$$f\Big(\frac{1}{2n-1}\Big) = -\frac{1}{(2n-1)^2} \qquad \forall\, n \in \mathbb{N}.$$

Then, by Theorem 5.4.4, we have $f(z) = z^2$ and $f(z) = -z^2$ for all $z \in \Omega$, which is not possible. □

EXAMPLE 5.4.3 Suppose Ω is a connected (nonempty) open set which is symmetric with respect to the real axis, i.e., $z \in \Omega \iff \bar{z} \in \Omega$. Suppose f is holomorphic on Ω such that it is real on $\Omega \cap \mathbb{R}$. We show that $f(\bar{z}) = \overline{f(z)}$.

It can be shown that g defined by $g(z) := \overline{f(\bar{z})}$ is analytic on Ω (*verify*). For $z \in \Omega \cap \mathbb{R}$, we have $g(z) = \overline{f(z)}$. Sine f is real on $\Omega \cap \mathbb{R}$, we have $g(z) = f(z)$ for all $z \in \Omega \cap \mathbb{R}$. Hence, by Theorem 5.4.4, $f = g$ on Ω so that $f(\bar{z}) = \overline{f(z)}$ on Ω. □

5.4.2 Maximum modulus principle

Theorem 5.4.5 (Maximum modulus principle) *Suppose f is holomorphic in a connected open set Ω. If f is not a constant function, then $|f|$ cannot attain maximum at a point in Ω.*

In other words, if there exists $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$, then f is a constant function.

Proof. Suppose there exists $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. Let $r > 0$ such that $\{z \in \mathbb{C} : |z - z_0| \leq r\} \subseteq \Omega$. Recall that

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it}) dt.$$

If $f(z_0) = 0$, then we have $f(z) = 0$ for all $z \in \Omega$. Hence, assume that $f(z_0) \neq 0$, and let $\lambda := |f(z_0)|/f(z_0)$. Then we have

$$|f(z_0)| = \lambda f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \lambda f(z_0+re^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}[\lambda f(z_0+re^{it})] dt.$$

Hence,

$$\frac{1}{2\pi} \int_0^{2\pi} [|f(z_0)| - \mathrm{Re}(\lambda f(z_0 + re^{it}))] dt = 0.$$

Since $|\mathrm{Re}(\lambda f(z_0 + re^{it})| \leq |f(z_0)|$, we have

$$|f(z_0)| = \mathrm{Re}(\lambda f(z_0 + re^{it})) \qquad \forall\, t \in [0, 2\pi],$$

i.e.,

$$|f(z_0)| = \mathrm{Re}[\lambda f(z)] \qquad \forall\, z \in C_r := \{\zeta : |\zeta - z_0| = r\}.$$

Again, since $|\lambda f(z)| \leq |f(z)|$ for all $z \in C_r$, we have

$$|f(z_0)| = \lambda f(z) \qquad \forall\, z \in C_r.$$

Hence,

$$f(z) = f(z_0) \qquad \forall\, z \in C_r.$$

By, identity theorem, $f(z) = f(z_0)$ for all $z \in \Omega$. ∎

Exercise 5.4.2 Suppose f is holomorphic in a connected open set Ω. If f is not a constant function and $|f|$ attains minimum at $z_0 \in \Omega$, then $f(z_0) = 0$. ◁

5.4.3 Schwarz's lemma

Theorem 5.4.6 (Schwarz's lemma) *Suppose f is holomorphic in the open unit disc $D := \{z \in \mathbb{C} : |z| < 1\}$ such that*

$$f(0) = 0 \quad \text{and} \quad f(z) \in D \quad \forall\, z \in D.$$

Then $|f(z)| \le |z|$ for all $z \in D$. Strict inequality follows unless f is of the form $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$.

Proof. By the assumptions on f, we have

$$f(z) = zg(z), \qquad z \in D,$$

where g is holomorphic on D and $g(0) = f'(0)$. Since $f(z)| \le 1$, we have

$$|g(z)| \le \frac{1}{|z|} = \frac{1}{r} \quad \text{whenever} \quad |z| = r < 1.$$

By maximum modulus principle,

$$|g(z)| \le \frac{1}{r} \quad \text{whenever} \quad |z| \le r < 1.$$

Now, let $z \in D$, and $0 < r < 1$ such that $|z| \le r$. By the above arguments,

$$|g(z)| \le \frac{1}{r}.$$

Letting $r \to 1$, we obtain $|g(z)| \le 1$. Thus, $|f(z)| \le |z|$ for all $z \in D$. ∎

5.4.4 On harmonic functions

Theorem 5.4.7 *Suppose u is real harmonic in a simply connected domain Ω. Then u has a harmonic conjugate which is unique up to addition of an imaginary constant.*

Theorem 5.4.8 *Suppose u is real harmonic in a simply connected domain Ω. Then $u \in C^\infty(\Omega)$.*

Theorem 5.4.9 *Suppose u is real harmonic in an open set Ω. Then, for $z_0 \in \Omega$,*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{it})dt, \qquad 0 < r < \text{dist}\,(z_0, \mathbb{C} \setminus \Omega).$$

Theorem 5.4.10 *Suppose u is real harmonic in a connected open set Ω. If u is not a constant function, then $|u|$ cannot attain maximum at a point in Ω.*

In other words, if there exists $z_0 \in \Omega$ such that $|u(z)| \leq |u(z_0)|$ for all $z \in \Omega$, then u is a constant function.

Theorem 5.4.11 *Suppose u is real harmonic in an open connected set Ω. If $f = 0$ on an open set $\Omega_0 \subseteq \Omega$, then $f = 0$ on Ω.*

5.5 Problems

1. A curve $\eta : [\alpha, \beta] \to \mathbb{C}$ is called a *reparameterization* of the curve $\gamma : [a, b] \to \mathbb{C}$ if there exists continuous increasing bijection $\varphi : [\alpha, \beta] \to [a, b]$ such that $\eta = \gamma \circ \varphi$, and in that case we may say that η is equivalent to γ and we write $\eta \sim \gamma$.

   (a) Given a curve $\gamma : [a, b] \to \mathbb{C}$ and a closed interval $[\alpha, \beta]$, find a curve $\eta : [\alpha, \beta] \to \mathbb{C}$ such $\gamma \sim \eta$.

   (b) If $\tilde{\gamma}$ is the reverse of $\gamma : [a, b] \to \mathbb{C}$ and $\eta : [-b, -a] \to \mathbb{C}$ is defined by $\eta(t) = \gamma(-t)$ for $-b \leq t \leq -a$, then show that $\eta \sim \tilde{\gamma}$.

   (c) If γ and η are piecewise smooth curves such that $\eta \sim \gamma$, then show that $\ell(\Gamma_\eta) = \ell(\Gamma_\gamma)$.

2. Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ and a partition $\Pi_n := a = t_0, < t_1 < \ldots < t_n = b$ of $[a, b]$, let $S_n := \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})|$. If $\max_{1 \leq j \leq n} |t_j - t_{j-1}| \to 0$, then show that $S_n \to \int_a^b |\gamma'(t)| dt$.

3. If η is a differentiable reparamterization of a piecewise smooth curve γ, and if f is continuous on Γ_γ, then show that $\int_\gamma f(z) dz = \int_\eta f(z) dz$.

4. Prove: $\displaystyle\int_{\gamma_1 + \gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz.$

5. Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, a partition $\Pi_n := a = t_0, < t_1 < \ldots < t_n = b$ of $[a, b]$, and a continuous function f on Γ, let $S_n(f) := \sum_{j=1}^{n} f(\gamma(t_j))[\gamma(t_j) - \gamma(t_{j-1})]$. If $\max_{1 \leq j \leq n} |t_j - t_{j-1}| \to 0$, then show that $S_n(f) \to \int_\Gamma f(z) dz$.

6. If $\varphi : [a, b] \to \mathbb{C}$ is continuously differentiable, then show that
$$\int_a^b \varphi'(t)\, dt = \varphi(b) - \varphi(a).$$

7. Let Γ be a closed curve and f has a primitive on an open set Ω containing Γ, i.e., there exists a continuously differentiable function g such that $g' = f$ on Ω. Then $\displaystyle\int_\Gamma f(z)\, dz = 0$. Justify.

8. Let Γ_n be the circle with center at z_0 and radius r traced n times, i.e., $\Gamma_n := \{z_0 + re^{it}, 0 \le t \le 2n\pi\}$. Then, show that $\displaystyle\int_{\Gamma_n} \frac{dz}{z - z_0} = 2n\pi i,.$

9. If $p(z)$ is a polynomial, then prove that $\displaystyle\int_{\Gamma_n} p(z)\, dz = 0.$

10. If Γ is the circle with centre z_0 and radius r, then show that $\displaystyle\int_\Gamma \frac{dz}{(z - z_0)^n} = 0$ for every $n \in \mathbb{Z} \setminus \{-1\}$.

11. Given distinct points α and β in $\mathbb{C}$, evaluate the integrals $\int_{[\alpha,\beta]} z^n dz$ and $\int_{[\alpha,\beta]} \bar{z}^n dz$, where $[\alpha, \beta]$ denotes the line segment joining α to β.

12. If f is a real valued function defined on the interval $[a, b]$ and if $\gamma(t) = t$, $a \le t \le b$, then show that $\int_{[a,b]} f(z)dz = \int_a^b f(t)dt$.

13. Let Γ be a closed piecewise smooth curve in the disc of convergence of a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ and let $f(z)$ represent this series in that disc. Then $\displaystyle\int_\Gamma f(z)\, dz = 0$. Justify.

14. Evaluate the integrals $\int_\gamma f(z)dz$ in following, where

    (a) γ is the curve joining $z_1 = -1 - i$ to $z_2 = 1 + i$ consisting of the line segment from $-1 - i$ to 0 and the portion of the curve $y = x^2$ from 0 to $1 + i$ and $f(z) = \begin{cases} 1, & y < 0, \\ 4y, & y > 0 \end{cases}$.
    Answer: $-\frac{5}{8} + \frac{5}{2}i$

    (b) γ is the curve consisting of the line segments joining the points 0 to 1 and 1 to $1 + 2i$ and $f(z) = 3x^2 - y + ix^3$.
    Answer: $2 + 3i$

(c) $f(z) = \bar{z}$ and γ is the curve joining 1 to $1+i$ along the parabola $y = x^2$. Answer: $1 + \frac{1}{3}i$

(d) γ is the positive oriented circle $z - 1| = 4$ traced once and $f(z) = \frac{1}{z-1} + \frac{6}{(z-1)^2}$ Answer: $-6\pi i$

15. Show that $\left| \int_\gamma \frac{z}{z^2+1} dz \right| \leq \frac{1}{2}$, where γ be the line segment joining 2 to $2 + i$.

16. For piecewise smooth curve $\gamma :: [a, b] \to \mathbb{C}$ and continuous function $f : \Gamma \to \mathbb{C}$, define $\displaystyle\int_\gamma f(z)|dz| := \int_a^b f(g(t))|\gamma'(t)|dt$. Show that
$$\left| \int_\gamma f(z)dz \right| \leq \int_\gamma |f(z)|\,|dz|.$$

17. If Γ_r is the circle $\gamma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ and if f is continuous on and inside Γ_r, then prove that
$$\lim_{r\to 0} \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{z - z_0} dz = f(z_0).$$

18. For every closed piecewise smooth curve Γ, $\displaystyle\int_\Gamma e^{-z^2} dz = 0$. Why?

19. For positive real numbers, let I_1, I_2, I_3, I_4 be the integrals of e^{-z^2} over the line segments
$$[-a, a], \quad [a, a + ib], \quad [a + ib, -a + ib], \quad [-a + ib, -a],$$
respectively. Prove that

(i) $I_1 = \displaystyle\int_{-a}^{a} e^{-x^2} dx$,

(ii) $|I_2| \leq be^{-a^2+b^2}$.

(iii) $I_3 = -e^{b^2} \displaystyle\int_{-a}^{a} e^{-t^2} \cos(2bt)dt$,

(iv) $|I_4| \leq be^{-a^2+b^2}$.

(v) $e^{b^2} \displaystyle\int_{-\infty}^{\infty} e^{-t^2} \cos(2bt)dt = \int_{-\infty}^{\infty} e^{-x^2} dx$.

20. Suppose $f_n \to f$ uniformly on Γ. Then show that
$$\int_\Gamma f_n(z)dz \to \int_\Gamma f(z)\,dz.$$

21. Prove that $\displaystyle\int_0^\infty e^{-t^2}\cos t^2 dt = \frac{1}{4}\sqrt{\pi}\sqrt{1+\sqrt{2}}$, by integrating e^{-z^2} over the positive oriented triangle with vertices at $0, R, Re^{i\pi/8}$ for $R > 0$ and letting $R \to \infty$.

22. Evaluate the integrals $\displaystyle\int_0^\infty \cos t^2 dt$, $\displaystyle\int_0^\infty \cos t^2 dt$ by integrating e^{-z^2} over the positive oriented triangle with vertices at $0, R, Re^{i\pi/4}$ for $R > 0$ and letting $R \to \infty$.

23. Let (Ω_n) be a sequence of nonempty compact sets in $\mathbb{C}$ such that $\Omega_n \supseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$ and $\text{diam}(\Omega_n) \to 0$ as $n \to \infty$. Let $z_0 \in \cap_{n=1}^\infty \Omega_n$. If f is a continuous function defined on Ω_1, show that
$$\max_{z\in\Omega_n} |f(z) - f(z_0)| \to 0 \quad \text{as} \quad n \to \infty.$$

24. Suppose Ω be a simply connected domain and f be holomorphic on Ω. Suppose integral of f over every positively oriented triangle is zero. Prove that if Γ_1 and Γ_2 are two polygonal lines joining any two points z_0 and ζ_0 in Ω, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$.

25. Let f be continuous in a neigbourhood of z_0 and $\Gamma_r := \{z \in \mathbb{C} : |z - z_0| = r\}$. Show that
$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z - z_0}dz \to f(z_0) \quad \text{as} \quad r \to 0.$$

26. Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}$ (using complex integrals).

27. Let f be an entire function such that for some $n \in \mathbb{N}$ and $R > 0$, $\left|\frac{f(z)}{z^n}\right|$ is bounded for $|z| > R$. Prove tht f is a polynomial of degree atmost n.

28. Let f be holomorphic and map $\Omega := \{z \in \mathbb{C} : |z| < 1\}$ into itself. Prove that $|f'z)| \leq 1/(1 - |z|)$ for all $z \in \Omega$.

29. Prove that there is no analytic function f on $\Omega := \{z \in \mathbb{C} : |z| < 1\}$ such that

    (i) $f(1/n) = (1/2^n$ for $n \in \mathbb{N} \setminus \{1\}$.

    (ii) $f(1/n) = (-1)^n/n^2$ for $n \in \mathbb{N} \setminus \{1\}$.

30. Let f be a nonconstant holomorphic function in a connected opens set Ω. If $z_0 \in \Omega$ is such that $|f(z_0)| \leq |f(z)|$ for all $z \in \Omega$, then prove that $f(z_0) = 0$.

31. Let u be a (real valued) harmonic function in a connected opens set Ω. Let $g := u_x - iu_y$ on Ω. Justify the following:

    (i) g is holomorphic on Ω.

    (ii) There exists a holomorphic function f on Ω such that $\text{Re} f = u$.

    (iii) u is infinitely differentiable.

6

Laurent Series and Isolated Singularities

6.1 Laurent Series

In this chapter we consider series expansions of functions which are analytic in an annulus. Such series are known as **Laurent series**.

We know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

whenever $|z| < 1$. We also have

$$\frac{1}{1-z} = \frac{-1}{z(1-1/z)} = (-1/z)\sum_{n=0}^{\infty}(1/z^n)$$

whenever $|z| > 1$. Note also that

$$\begin{aligned}\frac{1}{(z-1)(z-2)} &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \sum_{n=1}^{\infty}\frac{(-1)}{z^n} + \sum_{n=0}^{\infty}\frac{(-1)}{2^{n+1}}z^n\end{aligned}$$

whenever $1 < |z| < 2$.

In view of the above examples, for a function f which is analytic in an annulus $\{D := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$, we may look for an expansion of f of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

Definition 6.1.1 A series of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ for a given $z_0 \in \mathbb{C}$ and sequences (a_n) and $a_{-n})$ in $\mathbb{C}$ is said to converge at a point $z \neq z_0$ if both the series $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ and $\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converge at z. If the series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges for all z in a subset Ω of $\mathbb{C}$, then we say that it converges in Ω. $\Diamond$

Note that

- the series $\displaystyle\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converge for all z such that

$$|z-z_0| > R_1 := \limsup_{n\in\mathbb{N}} |a_{-n}|^{1/n}$$

  and

- the series $\displaystyle\sum_{n=1}^{\infty} a_n(z-z_0)^n$ converge for all z such that

$$|z-z_0| < R_2 := \frac{1}{\limsup_{n\in\mathbb{N}} |a_n|^{1/n}}.$$

Here, R_1 can be 0 and R_2 can be ∞. Thus, the series $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges in the *annulus* z such that $R_1 < |z-z_0| < R_2$.

Questions: Consider an annulus

$$D := \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}.$$

- Suppose a series $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges in D. Then, is the function represented by this series is holomorphic in D? In such case, are the coefficients uniquely determined?
- If f is holomorphic in D, does it have a series expansion of the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ in D?

We answer both the above questions affirmatively.

Theorem 6.1.1 *Suppose* $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ *converges in an annulus* $D := \{z \in \mathbb{C} : R_1 < |z-z_0| < R_2\}$ *and let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \qquad z \in D.$$

Then f is holomorphic in D and

$$a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta}{(\zeta - z_0)^{n+1}}\,d\zeta \qquad \forall\, n \in \mathbb{Z},$$

where Γ is any simple closed contour in D enclosing z_0.

Proof. Let $\Gamma_r := \{z : |z - z_0| = r\}$, $R_1 < r < R_2$. Since f is uniformly continuous on Γ_r,

$$\frac{1}{2\pi i}\int_{\Gamma_r} f(z)dz = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi i}\int_{\Gamma_r} (z - z_0)^n dz.$$

Hence,

$$a_{-1} = \frac{1}{2\pi i}\int_{\Gamma_r} f(z)dz.$$

Also, for $k \in \mathbb{N}$, we have

$$\frac{f(z)}{z - z_0)^{k+1}} = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^{n-k-1}, \qquad z \in D.$$

Hence,

$$\frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z - z_0)^{k+1}}dz = \sum_{n=-\infty}^{\infty} a_n \frac{1}{2\pi i}\int_{\Gamma_r} (z - z_0)^{n-k-1}dz$$

so that (since $n - k - 1 = -1 \iff n = k$)

$$a_k = \frac{1}{2\pi i}\int_{\Gamma_r} \frac{f(z)}{z - z_0)^{k+1}}dz.$$

By Cauchy's theorem, Γ_r can be replaced by any cure Γ as in the theorem. ∎

By the above theorem the coefficients of the convergent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ is uniquely determined by the function which it represents. In view of this fact, we have the following definition.

Definition 6.1.2 Suppose the series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ converges in an annulus $D := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ and let f be the function represented by the series in D. Then the series is said to be the **Laurent series** of f in D. ♢

Proposition 6.1.2 *Suppose φ is continuous on piecewise smooth curve Γ and*

$$\psi(z) := \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta \qquad \forall\, z \notin \Gamma.$$

Then ψ is holomorphic in $\mathbb{C} \setminus \Gamma$ and ψ has the series expansions

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{whenever} \quad |z - z_0| < \inf_{\zeta\in\Gamma} |\zeta - z_0|,$$

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \quad \textit{whenever} \quad |z - z_0| > \sup_{\zeta\in\Gamma} |\zeta - z_0|,$$

where

$$a_n = \int_\Gamma \frac{f(\zeta}{(\zeta - z_0)^{n+1}}\, d\zeta \qquad \forall\, n \in \mathbb{Z},$$

with $a_{-n} = b_n$ for $n \in \mathbb{N}$.

Proof. We note that for $\zeta \neq z$,

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{-1}{(z - z_0) - (\zeta - z_0)} \\ &= \frac{-1}{(z - z_0)\left[1 - \frac{\zeta - z_0}{z - z_0}\right]}. \end{aligned}$$

Hence, for $|z - z_0| > \max_{\zeta\in\Gamma} |\zeta - z_0|$,

$$\frac{1}{\zeta - z} = \frac{-1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^n = \sum_{n=1}^{\infty} (-1) \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n}$$

Thus,

$$\frac{\varphi(z)}{\zeta - z} = \sum_{n=1}^{\infty} (-1)\varphi(z) \frac{(\zeta - z_0)^{n-1}}{(z - z_0)^n}$$

so that

$$\int_\Gamma \frac{\varphi(z)}{\zeta - z} d\zeta = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where

$$b_n = -\int_\Gamma \varphi(z)(\zeta - z_0)^{n-1} d\zeta, \qquad n \in \mathbb{N}.$$

Also, for $\zeta \neq z$,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}.$$

Hence, for $|z - z_0| < \inf_{\zeta \in \Gamma} |\zeta - z_0|$,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.$$

Thus,

$$\int_\Gamma \frac{\varphi(z)}{\zeta - z} d\zeta = \sum_{n=1}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \int_\Gamma \frac{\varphi(z)}{(\zeta - z_0)^{n+1}} d\zeta, \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

This completes the proof. ∎

Theorem 6.1.3 *Suppose f is holomorphic annulus $D := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$. Then f has a series expansion*

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad on \quad D,$$

where

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \qquad \forall\, n \in \mathbb{Z}.$$

Proof. Let $R_1 < r < R < R_2$ and let $C := \{\zeta : |\zeta - z_0| = r\}$ and $\Gamma := \{\zeta : |\zeta - z_0| = R\}$. Then it can be seen that (verify!) for every z with $r < |z - z_0| < R$,

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = f_1(z) + f_2(z).$$

By Proposition 6.1.2, we have

$$f_1(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

and

$$f_2(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \sum_{n=1}^{\infty} a_n (z - z_0)^n,$$

where,

$$b_n = \int_\Gamma f(z)(\zeta - z_0)^{n-1} d\zeta, \qquad n \in \mathbb{N},$$

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}, \qquad n \in \mathbb{N}_0.$$

Thus,

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{on} \quad D,$$

where

$$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta \qquad \forall\, n \in \mathbb{Z}.$$

This completes the proof. ∎

6.2 Isolated Singularities

Definition 6.2.1 Let f be analytic in an open set Ω. A point $z_0 \notin \Omega$ is said to be a singularity of f if z_0 is a limit point of Ω and f cannot be extended to an open set $\tilde{\Omega}$ which contains Ω and z_0. ♢

- z_0 is a singularity of an analytic function $f : \Omega \to \mathbb{C}$ if and only if z_0 is a limit point of Ω and for every $r > 0$, f cannot be extended analytically to $\Omega \cup B(z_0, r)$, i.e., there is not analytic function $g : \Omega \cup B(z_0, r)\tau\mathbb{C}$ such that $g(z) = f(z)$ for all $z \in \Omega$.

Definition 6.2.2 Let Ω be an open set in which a holomorphic function f is defined and $z_0 \in \mathbb{C}\backslash\Omega$. Then z_0 is said to be an **isolated singularity** of f if a deleted neigbourhood of z_0 is contained in Ω.

Let z_0 be an isolated singularity of a holomorphic function f, and let

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

be its Laurent series expansion in a deleted neigbourhood D_0 of f. Then z_0 is said to be

- a **removable singularity** of f if $a_{-n} = 0$ for all $n \in \mathbb{N}$;
- a **pole order** m of f if $a_{-m} \neq 0$ and $a_n = 0$ for all $n < -m$;
- an **essential singularity** of f if $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$.

If z_0 is a pole of order 1, then z_0 is called a **simple pole**. $\diamondsuit$

EXAMPLE 6.2.1 (i) For the function $f(z) := 1/z^3$, $z_0 = 0$ is a pole of order 3.

(ii) For the function $f(z) := \sin z/z$, $z_0 = 0$ is a removable singularity.

(iii) For the function $f(z) := e^{1/z^2}$, $z_0 = 0$ is an essential singularity. $\square$

Theorem 6.2.1 *Let z_0 be an isolated singularity of a holomorphic function f. Then the following are equivalent.*

(i) *z_0 is a removable singularity of f.*

(ii) *f can be extended holomorphically to a neigbourhood of z_0.*

(iii) *f is bounded in a deleted neigbourhood of z_0.*

(iv) $\lim_{z\to z_0}(z - z_0)f(z) = 0$.

Proof. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be the Laurent series expansion f in a deleted neigbourhood D_0 of z_0.

(i) $\iff$ (ii): Suppose z_0 is a removable singularity of f. Defining $\tilde{f}(z_0) = \begin{cases} f(z), & z \neq z_0, \\ a_0, & z \neq z_0, \end{cases}$, we see that $\tilde{f}$ is a holomorphic extension of f to a neigbourhood of z_0.

Conversely, suppose $\tilde{f}$ is a holomorphic extension of f to neigbourhood D of z_0. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be the Taylor series expansion of $\tilde{f}$ in D. Then, $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is the Laurent series expansion of f in the deleted neigbourhood $D \setminus \{0\}$.

(i) $\iff$ (iii): Suppose z_0 is a removable singularity of f. Since $\lim_{z\to z_0} f(z) = a_0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Longrightarrow |f(z) - a_0| \leq 1.$$

Hence,

$$0 < |z - z_0| < \delta \Longrightarrow |f(z)| \leq |a_0| + 1.$$

Conversely, suppose that f is bounded in a deleted neigbourhood D_0 of z_0, say $|f(z)| \leq M$ for all $z \in D_0$. Let $r > 0$ be such that $\Gamma_r := \{z \in \mathbb{C} : |z - z_0| = r\} \subseteq D_0$. Then, for each $n \in \mathbb{N}$,

$$|a_{-n}| = \left|\frac{1}{2\pi i}\int_{\Gamma_r} (z - z_0)^{n-1} f(z)dz\right| \leq Mr^n.$$

Letting $r \to 0$, we obtain $a_{-n} = 0$.

(i) $\iff$ (iv): Suppose z_0 is a removable singularity of f. Since $\lim_{z\to z_0} f(z) = a_0$, we have $\lim_{z\to z_0}(z - z_0)f(z) = 0$.

Conversely, suppose $\lim_{z\to z_0}(z - z_0)f(z) = 0$. Let $\varepsilon > 0$ and let $\delta > 0$ be such that

$$0 < |z - z_0| < \delta \Longrightarrow |(z - z_0)f(z)| < \varepsilon.$$

Let $0 < r < \min\{\delta, 1\}$. Then, for each $n \in \mathbb{N}$,

$$\begin{aligned} |a_{-n}| &= \Big|\frac{1}{2\pi i}\int_{\Gamma_r}(z - z_0)^{n-1}f(z)dz\Big| \\ &= \Big|\frac{1}{2\pi i}\int_{\Gamma_r}(z - z_0)^{n-2}(z - z_0)f(z)dz\Big| \\ &\leq r^{n-1}\varepsilon \leq \varepsilon. \end{aligned}$$

Hence, $a_{-n} = 0$ for all $n \in \mathbb{N}$, and hence z_0 is a removable singularity of f. ∎

Theorem 6.2.2 *Let z_0 be an isolated singularity of a holomorphic function f. Then the following are equivalent.*

(i) *z_0 is a pole of f.*

(ii) *There exists $m \in \mathbb{N}$ and a holomorphic function φ in a neigbourhood D of z_0 such that $f(z) = (z - z_0)^{-m}\varphi(z)$ for all $z \in D \setminus \{0\}$ with $\varphi(z_0) \neq 0$.*

(iii) *$|f(z)| \to \infty$ as $z \to z_0$.*

(iv) *There exists $m \in \mathbb{N}$ such that z_0 is a removable singularity of $\varphi(z) := (z - z_0)^m f(z)$ with $\lim_{z\to z_0}\varphi(z) \neq 0$.*

(v) *The function $1/f$ defined in a deleted neigbourhood of z_0 can be extended holomorphically to a neigbourhood of z_0 and z_0 is a zero of order m of the extended function.*

Proof. Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be the Laurent series expansion f in a deleted neigbourhood D_0 of z_0.

(i) $\iff$ (ii): This is obvious from the definition of the pole.

(i) $\Longleftrightarrow$ (iii): Suppose z_0 is a pole of order m of f. By (ii), There exists $m \in \mathbb{N}$ and a holomorphic function φ in a neigbourhood D of z_0 such that

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\}$$

with $\varphi(z_0) \neq 0$. Since $\varphi(z) \to \varphi(z_0) \neq 0$, we obtain

$$|f(z)| = |(z - z_0)^{-m}\varphi(z)| \to \infty \quad \text{as} \quad z \to z_0.$$

Conversely, suppose $|f(z)| \to \infty$ as $z \to z_0$. Then f is nonzero in a deleted neigbourhood D_0 of z_0, and hence, the function defined by $g(z) = 1/f(z)$ for $z \in D_0$, satisfies

$$|g(z)| \to 0 \quad \text{as} \quad z \to z_0.$$

Define $g(z_0) = 0$. Then g is holomorphic in $D := D_0 \cup \{0\}$. Hence, there exists $m \in \mathbb{N}$ and a holomorphic function φ in D such that

$$g(z) = (z - z_0)^{m}\varphi(z) \qquad \forall\, z \in D$$

$\varphi(z_0) \neq 0$. Thus, for $z \in D_0$,

$$f(z) = \frac{1}{g(z)} = (z - z_0)^{-m}\psi(z), \qquad \psi(z) := \frac{1}{\varphi(z)}$$

and $\psi(z_0) \neq 0$. Hence, z_0 is a pole of order m of f.

(i) $\Longleftrightarrow$ (iv): Suppose z_0 is a pole of f. By (ii), there exists $m \in \mathbb{N}$ and a holomorphic function φ in a neigbourhood D of z_0 such that

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\}$$

with $\varphi(z_0) \neq 0$. Then, $\varphi(z) = (z - z_0)^m f(z)$ in D_0. Clearly, $\lim_{z \to z_0} \varphi(z) = \varphi(z_0)$. Hence, z_0 is a removable singularity of φ.

Conversely, suppose there exists $m \in \mathbb{N}$ such that z_0 is a removable singularity of $\varphi(z) := (z - z_0)^m f(z)$ with $\lim_{z \to z_0} \varphi(z) \neq 0$. Let $\tilde{\varphi}$ be the holomorphic extension of φ to a neigbourhood D of z_0. Then we have

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\}$$

and $\tilde{\varphi}(z) \neq 0$. Hence z_0 is a pole of order m of f.

(i) $\Longleftrightarrow$ (v): Suppose z_0 is a pole of f. By (ii), there exists $m \in \mathbb{N}$ and a holomorphic function φ in a neigbourhood D of z_0 such that

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\}$$

with $\varphi(z_0) \neq 0$. Then,

$$\frac{1}{f(z)} = (z - z_0)^{m}\psi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\},$$

where ψ, defined by $\psi(z) := 1/\varphi(z)$, is analytic on D. Thus, the function $1/f$ can be extended holomorphically to D by defining 0 at z_0, and z_0 is a zero of order m of the extended function.

Conversely, if $g := 1/f$ can be extended holomorphically to a neigbourhood D of z_0 and z_0 is a zero of order m of the extended function $\tilde{g}$, then there exists a an analytic function ψ in D such that

$$\tilde{g}(z) = (z - z_0)^{m}\psi(z) \qquad \forall\, z \in D.$$

Thus, we have

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{0\}$$

where $\varphi(z) = 1/\psi(z)$ on D and $\varphi(z_0) \neq 0$ so that z_0 is a pole of f order m. ∎

The following theorem shows that if z_0 is an essential singularity of a holomorphic function f, then in a neigbourhood of z_0, there are values of f which are arbitrarily close to any complex number.

Theorem 6.2.3 (Casorati-Weierstrass theroem) *Suppose z_0 is an essential singularity of a holomorphic function f. Then for every $w \in \mathbb{C}$, there exists a sequence (z_n) in the domain of analyticity of f such that $f(z_n) \to w$ as $n \to \infty$.*

Proof. Suppose for a moment that the conclusion in the theorem does not hold. Then there exists $w \in \mathbb{C}$ such that for any sequence (z_n) with $z_n \to z_0$, $f(z_n) \not\to w$. Then the function $g(z) := \frac{1}{f(z)-w}$ is analytic in a deleted neigbourhood of z_0. Further, $|g|$ is bounded in a deleted neigbourhood of z_0. (If $|g|$ is not bounded in any deleted neigbourhood of z_0, then there exists a sequence (z_n) such that $z_n \to z_0$ and $|g(z_n)| \to \infty$.) Hence, by Theorem 6.2.1, z_0 is a removable singularity of g. Suppose $g(z) \to \alpha_0$ as $z \to z_0$. We have the following two cases:

Case (i): $\alpha_0 \neq 0$: In this case, $f(z) = w + 1/g(z) \to w + 1/\alpha_0$ as $z \to z_0$, so that by Theorem 6.2.1, z_0 is a removable singularity of f as well.

Case (ii): $\alpha_0 = 0$: In this case, we have $|f(z)| = |w+1/g(z)| \to \infty$ as $z \to z_0$, and hence by Theorem 6.2.2, z_0 is a pole of f.

Since z_0 is an essential singularity, cases (i) and (ii) can not occur. Thus, our assumption that the conclusion in the theorem does not hold is not true. ∎

6.3 Problems

1. For $0 < a < 1$, find the annulus of convergence of the series $\sum_{n=-\infty}^{\infty} a^{n^2} z^n$.

2. Locate and classify the isolated singularities of the following functions:

$$\text{(i)}\ \frac{z^5}{1 = z + z^2 = z^3 + z^4}, \quad \text{(ii)}\ \frac{1}{\sin^2 z}, \quad \text{(iii)}\ \sin(1/z).$$

   Also, check whether $z_0 = \infty$ is an isolated singularity (i.e., $w_0 = 0$ is an isolated singularity of $f(1/z)$) in each case.

3. If f and g are holomorphic functions having z_0 a pole of the same order for both, then prove that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

7

Residues and Real Integrals

7.1 Residue theorem

Definition 7.1.1 Suppose f is holomorphic in a deleted neigbourhood D_0 of $z_0 \in \mathbb{C}$ and $\Gamma_r := \{z \in \mathbb{C} : |z - z_0| = r\} \subseteq D_0$. Then **residue** of f at z_0 is defined by

$$Res(f, z_0) = \frac{1}{2\pi i} \int_{\Gamma_r} f(z) dz.$$

$\diamondsuit$

Recall that if f is holomorphic in a deleted neigbourhood D_0 of $z_0 \in \mathbb{C}$, then f has Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in D_0,$$

and we know that

$$a_n = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{f(z)}{(z - z_0)^{n+1}} dz, \qquad n \in \mathbb{Z}.$$

Thus,

$$Res(f, z_0) = a_{-1}.$$

The following theorem, known as **residue theorem** follows from Cauchy's theorem.

Theorem 7.1.1 (Residue theorem) *Suppose Γ is a simple closed contour and $z_1, \ldots, z_k$ are points in Γ_Γ which are the only singular points of f in $\Gamma \cup \Omega_\Gamma$. Then*

$$\int_\Gamma f(z) dz = 2\pi i \sum_{j=1}^{k} Res(f, z_j).$$

7.2 Calculation of Residues

Suppose z_0 is a pole of order m of a holomorphic function f. Then we know that there exists a holomorphic function φ in a neigbourhood D of z_0 such that

$$f(z) = (z - z_0)^{-m}\varphi(z) \qquad \forall\, z \in D_0 := D \setminus \{z_0\}.$$

Let

$$\varphi(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n, \qquad z \in D.$$

Then we have

$$\alpha_n = \frac{\varphi^{(n)}(z_0)}{n!}, \qquad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

so that

$$f(z) = (z - z_0)^{-m}\varphi(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^{n-m} = \sum_{n=-m}^{\infty} a_n (z - z_0)^n.$$

Hence,

$$Res(f, z_0) = a_{-1} = \alpha_{m-1} = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

Thus, we have proved the following theorem.

Theorem 7.2.1 *Suppose z_0 is a pole of order m of a holomorphic function f. Then the function*

$$z \mapsto \varphi(z) := (z - z_0)^m f(z)$$

defined in a deleted neigbourhood of z_0 has a holomorphic extension to a neigbourhood D of z_0, again denoted by φ, and

$$Res(f, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

In particular, if z_0 is a simple pole of f, then

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Corollary 7.2.2 *Suppose g and h are holomorphic in a neigbourhood D of z_0 and z_0 is a zero of h of order m. If $h(z) = (z - z_0)^m h_0(z)$ with $h_0(z_0) \neq 0$, then*

$$Res(\frac{g}{h}, z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!},$$

where $\varphi(z) = g(z)/h_0(z)$. In particular, if $m = 1$, then

$$Res(\frac{g}{h}, z_0) = \frac{g(z_0)}{h_0(z_0)} = \frac{g(z_0)}{h'(z_0)}.$$

EXAMPLE 7.2.1 Let us find $\int_\Gamma f(z)dz$, where $f(z) = \dfrac{1}{z(z-1)}$ and Γ is the positively oriented circle with centre 0 and radius 2.

By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)} = 2\pi i[Res(f, z_1) + Res(f, z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ and $z_2 = 1$ are simple poles of the function

$$Res(f, z_1) = \lim_{z \to 0} z f(z) = -1,$$

$$Res(f, z_2) = \lim_{z \to 1} (z-1) f(z) = 1.$$

Hence, $\displaystyle\int_\Gamma \frac{dz}{z(z-1)} = 0.$ □

EXAMPLE 7.2.2 Let us find $\int_\Gamma f(z)dz$, where $f(z) = \dfrac{1}{z(z-1)^2}$ and Γ is the positively oriented circle with centre 0 and radius 2. By residue theorem,

$$\int_\Gamma \frac{dz}{z(z-1)^2} = 2\pi i[Res(f, z_1) + Res(f, z_2)], \qquad z_1 = 0,\ z_2 = 1.$$

Since $z_1 = 0$ is a simple pole and $z_2 = 1$ is a pole of order 2,

$$Res(f, z_1) = \lim_{z \to 0} z f(z) = 1, \qquad Res(f, z_2) = \varphi'(1)$$

where $\varphi(z) = \frac{1}{z}$ so that $\varphi'(1) = -1$. Thus, $\displaystyle\int_\Gamma \frac{dz}{z(z-1)^2} = 0.$ □

Exercise 7.2.1 1. Find $Res(f, z_0)$, where

(a) $f(z) = ze^{1/z}, \quad z_0 = 0.$

(b) $f(z) = \dfrac{z+2}{z(z+1)}, \quad \text{(i) } z_0 = 0, \quad \text{(ii) } z_0 = -1.$

2. Evaluate $\displaystyle\int_\Gamma f(z)dz$, where

(a) $\dfrac{3z+1}{z(z-1)^3}$ and $\Gamma = \{z : |z| = 2\}$.

(b) $\dfrac{z+1}{2z^3 - 3z^2 - 2z}$ and $\Gamma = \{z : |z| = 1\}$.

(c) $\dfrac{z + 1/z}{z(2z - 1/(2z)}$ and $\Gamma = \{z : |z| = 1\}$.

(d) $\dfrac{\log(z+2)}{2z+1}$ and $\Gamma = \{z : |z| = 1\}$.

(e) $\dfrac{\cosh(1/z)}{z}$ and $\Gamma = \{z : |z| = 1\}$.

$\triangleleft$

7.3 Evaluation of Improper Integrals

In this section we shall evaluate integrals of the form

$$\int_0^\infty f(x)dx \quad \text{and} \quad \int_{-\infty}^\infty f(x)dx,$$

where f is a continuous function.

EXAMPLE 7.3.1 Let us evaluate $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$. For this consider the function $f(z) = \dfrac{dx}{1+z^2}$ for $z \neq 0$. Note that $z = i$ is the only singularity of f in the upper half plane and it is a simple pole. Consider the positively oriented curve Γ_R consisting of the semicircle with centre 0 and radius R, i.e., $S_R := \{z : |z| = R, \text{Im}(z) > 0\}$ and the line segment $L_R := [-R, R]$. Then, by Cauchy's theorem,

$$\int_{\Gamma_R} f(z)dz = \int_{C_r} f(z)dz,$$

where $C_r = \{z : |z - i| = r\}$ with $0 < r < R$. But,

$$\int_{C_r} f(z)dz = 2\pi i \, Res(f, i) = 2\pi i \lim_{z \to i}(z - i)f(z) = \pi.$$

Thus,

$$\int_{\Gamma_R} f(z)dz = \pi.$$

Also, we have

$$\int_{\Gamma_R} f(z)dz = \int_{S_R} f(z)dz + \int_{L_R} f(z)dz = \int_{S_R} f(z)dz + \int_{-R}^{R} f(x)dx.$$

But, for $z \in S_R$,

$$|f(z)| \le \frac{1}{R^2 - 1}.$$

Hence,

$$\Big| \int_{S_R} f(z)dz \Big| \le \frac{\ell(S_R)}{R^2 - 1} = \frac{\pi R}{R^2 - 1}.$$

Hence,

$$\begin{aligned} \lim_{R \to \infty} \int_{\Gamma_R} f(z)dz &= \lim_{R \to \infty} \int_{S_R} f(z)dz + \lim_{R \to \infty} \int_{L_R} f(z)dz \\ &= 0 + \int_{-\infty}^{\infty} f(x)dx. \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} \int_{\Gamma_R} f(z)dz = \pi.$$

□

EXAMPLE 7.3.2 Let us evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx$. Since

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx = \mathrm{Re} \left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} dx \right)$$

we consider the function

$$f(z) = \frac{e^{iz}}{1 + z^2}, \qquad z \notin \{i, -i\}.$$

Following the arguments as in the previous example, one arrive at

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} dx = \frac{\pi}{e}.$$

But,

$$\text{Im}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}dx\right) = \int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}dx = 0.$$

Hence,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}.$$

□

EXAMPLE 7.3.3 Let us evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x}dx$. Since

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \text{Re}\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{x}dx\right)$$

we consider the function

$$f(z) = \frac{e^{iz}}{z}, \qquad z \neq 0.$$

For $r > 0$, let $S_r := \{z : |z| = r\}$ with positive orientation. Then, taking $0 < \varepsilon < R$ and Γ as the curve consisting of S_R, $[-R, -\varepsilon]$, $\widetilde{S}_\varepsilon$, $[\varepsilon, R]$, using Cauchy's theorem,

$$\begin{aligned} 0 &= \int_\Gamma f(z)dz \\ &= \int_{S_R} f(z)dz + \int_{-R}^{-\varepsilon} f(x)dx + \int_{\widetilde{S}_\varepsilon} f(z)dz + \int_\varepsilon^R f(x)dx \\ &= \int_{S_R} f(z)dz + \int_{-R}^{-\varepsilon} f(x)dx - \int_{S_\varepsilon} f(z)dz + \int_\varepsilon^R f(x)dx \end{aligned}$$

But,

$$\int_{-R}^{-\varepsilon} \frac{\cos x}{x}dx + \int_\varepsilon^R \frac{\cos x}{x}dx = 0$$

and

$$\int_{-R}^{-\varepsilon} \frac{\sin x}{x}dx + \int_\varepsilon^R \frac{\sin x}{x}dx = 2\int_\varepsilon^R \frac{\sin x}{x}dx.$$

Hence,

$$\int_\varepsilon^R f(x)dx + \int_{-R}^{-\varepsilon} f(x)dx = 2i\int_\varepsilon^R \frac{\sin x}{x}dx.$$

Thus,

$$2i\int_\varepsilon^R \frac{\sin x}{x}dx = \int_{S_\varepsilon} f(z)dz - \int_{S_R} f(z)dz$$

Now, we note that with the parametrization $\gamma(t) = Re^{it}$, $0 \le t \le \pi$ of S_R,

$$\int_{S_R} f(z)dz = \int_0^\pi \frac{e^{iR(\cos t + i\sin t)}}{Re^{iRt}} iRe^{iRt}dt = i\int_0^\pi e^{iR(\cos t+i\sin t)}dt.$$

Hence,

$$\Big|\int_{S_R} f(z)dz\Big| \le \int_0^\pi e^{-R\sin t}dt = 2\int_0^{\pi/2} e^{-R\sin t}dt.$$

Since $\dfrac{\sin t}{t}$ is decreasing in $[0, \pi/2]$, we have $\dfrac{\sin t}{t} \ge \dfrac{\sin \pi/2}{\pi/2}$ so that $\sin t \ge 2t/\pi$.Thus,

$$\Big|\int_{S_R} f(z)dz\Big| \le 2\int_0^{\pi/2} e^{-2Rt/\pi}dt = \frac{\pi}{2R}(1-e^{-2R}) \to 0 \quad \text{as} \quad R \to \infty.$$

Next, we observe that

$$\frac{e^{iz}}{z} = \frac{1}{z} + \varphi(z)$$

where φ is an entire function. Hence, there exists $M > 0$ such that $|\varphi(z)| \le M$ for all z with $|z| \le 1$. Thus,

$$\int_{S_\varepsilon} f(z)dz = \int_{S_\varepsilon} \frac{dz}{z} + \int_{S_\varepsilon} \varphi(z)dz,$$

where

$$\Big|\int_{S_\varepsilon} \varphi(z)dz\Big| \le M\pi\varepsilon, \qquad 0 < \varepsilon \le 1.$$

Hence,

$$\begin{aligned} 2i\int_\varepsilon^R \frac{\sin x}{x}dx &= \int_{S_\varepsilon} \frac{dz}{z} + \int_{S_\varepsilon} \varphi(z)dz - \int_{S_R} f(z)dz \\ &= \pi i + \int_{S_\varepsilon} \varphi(z)dz - \int_{S_R} f(z)dz, \end{aligned}$$

where

$$\int_{S_\varepsilon} \varphi(z)dz \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{and} \quad \int_{S_R} f(z)dz \to 0 \quad \text{as} \quad R \to \infty.$$

Thus,

$$\int_\varepsilon^R \frac{\sin x}{x}dx = \frac{\pi}{2}.$$

□

7.4 Problems

1. Find the residues of the following functions:

   (i) $\dfrac{z^3}{z-1}$ (ii) $\dfrac{z^3}{(z-1)^2}$

2. If f and g are holomorphic in a neigbourhood of z_0, and z_0 is a simple a pole of g, then prove that $Res(f/g, z_0) = f(z_0)/g'(z_0)$.

3. Determine the residues of each of the following functions at each of their singularities:

$$\text{(i) } \frac{z^3}{1-z^4}, \quad \text{(ii) } \frac{z^5}{(z^2-1)^2}, \quad \text{(iii) } \frac{\cos z}{1+z+z^2}.$$

4. If f is holomorphic in a neigbourhood of z_0, and z_0 is a zero of f order m, then prove that $Res(f'/f, z_0) = m$.

5. Evaluate the following using complex integrals:

$$\text{(i) } \int_0^\infty \frac{e^{ix}}{x} dx, \quad \text{(ii) } \int_0^\infty \frac{dx}{1+x^2},$$

$$\text{(iii) } \int_0^\infty \frac{\sin^2 x}{x} dx, \quad \text{(iii) } \int_0^\infty \frac{\cos ax}{x^2+b^2} dx, \quad a \geq 0,\ b > 0.$$

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