Calculus of Several Variables

Chapter 1

Functions of Several Variables

1.1 Introduction

Functions of more than one variables come naturally in applications. For example, in physics one comes across the relation

$$\frac{PV}{T} = c, \text{ constant},$$

where P, V, T represents the pressure, volume and temperature of an ideal gas. Since

$$P = \frac{cT}{V}, \quad V = \frac{cT}{P}, \quad T = \frac{PV}{c}$$

each of P, V, T can be thought of as a function of the remaining two variables.

In elementary geometry, we know that the area of a triangle of base length b and altitude h, area of a rectangle of sides a and b are given by

$$\frac{1}{2}bh, \quad ab \quad \text{and} \quad \pi r^2,$$

respectively, and they are functions of the variables (b, h), (a, b) and r, respectively. Also, distance of a point (x, y) in the plane from the origin $(0, 0)$ is given by

$$\sqrt{x^2 + y^2}.$$

Recall that the above follows from *Pythagoras theorem*.

We shall introduce some notations and basic definitions:

- $\mathbb{N}$: set of all natural numbers.
- $\mathbb{R}$: set of all real numbers.
- $\mathbb{R}^n$: set of all n-tuple of real numbers, i.e., the set of all $(x_1, \ldots, x_n)$ with $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$.
- For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote by $|u|$ (*absolute value* or *modulus* of u), the positive square root of $x_1^2 + x_2^2 + \ldots, x_k^2$, i.e.,

$$|x| := \sqrt{x_1^2 + x_2^2 + \ldots, x_k^2}.$$

Note that, if $u = (u_1, u_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2$, then

$$|x - u| = \sqrt{(x_1 - u_1)^2 + (x_2 - u_2)^2 + \ldots + (x_k - u_k)^2}.$$

Thus, for $u = (u_1, u_2) \in \mathbb{R}^2$, the set of all points $x = (x_1, x_2) \in \mathbb{R}^2$ such that $|x - u| = r$ represents the circle with centre u and radius r, that is, $\{x \in \mathbb{R}^2 : |x - u| = r\}$. The region inside this circle is $\{x \in \mathbb{R}^2 : |x - u| < r\}$ and the region inside the circle including the boundary is $\{x \in \mathbb{R}^2 : |x - u| \leq r\}$. Note that the above sets are same as

$$\{x \in \mathbb{R}^2 : (x_1 - u_1)^2 + (x_2 - u_2)^2 = r^2\},$$

$$\{x \in \mathbb{R}^2 : (x_1 - u_1)^2 + (x_2 - u_2)^2 < r^2\}$$

$$\{x \in \mathbb{R}^2 : (x_1 - u_1)^2 + (x_2 - u_2)^2 \leq r^2\},$$

respectively.

Definition 1.1 By a **function of several variables**, we mean a function f defined on a subset D of $\mathbb{R}^n$ for some $n \in \mathbb{N}$ with values in $\mathbb{R}$, and we write this fact as $f : D \to \mathbb{R}$.

The set D on which f is defined is called the **domain** of the function f, and the set of all values of f, i.e., the set $\{f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in D\}$, is called the **range** of f. $\Diamond$

A function $f : D \subseteq \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ is also written as

$$z = f(x_1, \ldots, x_n), \quad (x_1, \ldots, x_n) \in D\}$$

and in that case $x_1, \ldots, x_n$ are called the *independent variables* and z is called the *dependent variable.*

EXAMPLE 1.2 In the following, functions are specified by expressions of their values.

(1) $f(x, y) := \sqrt{x^2 + y^2}$ defines a function defined on any subset of $\mathbb{R}^3$ to $\mathbb{R}$.

(2) $f(x, y) := \frac{xy}{x^2+y^2}$ defines a function for all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \neq 0$. Thus, in this case domain of f is $D = \mathbb{R}^2 \setminus \{(0, 0)\} := \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$.

(3) $f(x, y) := \sqrt{1 - x^2 - y^2}$ is a function defined on $D := \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$. $\square$

Mostly we shall deal with functions of two or theree variables, i.e., $D \subseteq \mathbb{R}^2$ or $D \subseteq \mathbb{R}^3$. Most of the results that we study in these cases can be extended to the case of more variables.

1.2 Geometric Representation of Functions

Geometrically a function of two variables represents a surface S in the 3-*dimensional space*, in such a way that the *projection* of S to the xy-plane is the domain D, and each line parallel to the z-axis and passing through a point in D cuts the surface at one and only one point. In fact, the surface S corresponding to the function f is the *graph* of f. Thus, given a function $f : D \to \mathbb{R}$, the corresponding surface is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ with $(x, y) \in D$, i.e.,

$$S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y),\ (x, y) \in D\}.$$

1.2.1 Basic concepts on regions

Definition 1.3 Let $u_0 = (x_0, y_0) \in \mathbb{R}^2$ and $r > 0$.

(1) The set of all points $u := (x, y) \in \mathbb{R}^2$ that satisfy $|u - u_0| < r$ is called the **open disc** with centre $u_0 := (x_0, y_0)$ and radius r.

(2) The set of all points $u := (x, y) \in \mathbb{R}^2$ that satisfy $|u - u_0| \leq r$ is called the **closed disc** with centre $u_0 := (x_0, y_0)$ and radius r. ◊

Definition 1.4 Let D be a region, i.e., a subset of $\mathbb{R}^2$.

(1) D is said to be a **bounded set** if it is contained in a disc (open or closed) for some radius $r > 0$. Sets which are not bounded are called **unbounded** sets.

(2) A point (x_0, y_0) is called an **interior point** of D if it is the centre of an open disc contained in D.

(3) A point (x_0, y_0) is called a **boundary point** of D every open disc containing this point contains some point from D as well as some point not in D.

(4) The set of all interior points of D is called the **interior** of D, and it is denoted by $int(D)$.

(5) The set of all boundary points of D is called the **boundary** of D, denoted by $bd(D)$.

(6) The set D is called an **open set** if its interior is itself.

(7) The set D is called a **closed set** if it contains all its boundary points. ◊

Thus, for $D \subseteq \mathbb{R}^2$,

- $u_0 \in bd(D) \iff \forall\, r > 0, \quad B(u_0, r) \cap D \neq \varnothing \quad \& \quad B(u_0, r) \cap \mathbb{R}^2 \setminus D \neq \varnothing;$
- D is open $\iff \exists\, r > 0, \quad B(u_0, r) \subseteq D,$

- D is closed $\iff bd(D) \subseteq D$.

EXAMPLE 1.5 An open disc is an open set (*verify*). If its centre is $u_0 = (x_0, y_0)$ and radius $r > 0$, then its boundary is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$, i.e., the circle with centre (x_0, y_0) and radius r. Also, the closed disc is a closed set as it contains all its boundary points. □

EXAMPLE 1.6 Let $D := \{(x, y) \in \mathbb{R}^2 : y \leq x^2\}$. This is the region in $\mathbb{R}^2$ below the parabola $y = x^2$, including it. It is a closed unbounded region. □

EXAMPLE 1.7 Let $D := \{(x, y) \in \mathbb{R}^2 : y \leq x^2 \,\&\, y - 1 \geq x^2\}$. This is the region in $\mathbb{R}^2$ below the parabola $y = x^2$ and above the parabola $y^2 = 1 + x^2$. It is closed and bounded. □

Exercise 1.8 A subset D of $\mathbb{R}^2$ is bounded $\iff \exists\, M > 0$ such that $x^2 + y^2 \leq M$ for some $M > 0$. ◄

More generally, we may define the following:

Definition 1.9 Given a particular point $u_0 := (u_1, \ldots, u_k) \in \mathbb{R}^k$ and $r > 0$, the set of all points $x = (x_1, \ldots, x_k) \in \mathbb{R}^n$ that satisfy $|x - u_0| < r$ is called an **open ball** with centre u and radius r, usually denoted denoted by $B(u_0, r)$. Thus,

$$B(u_0, r) := \{x \in \mathbb{R}^k : |x - u_0| < r\}.$$

Other concepts such as **interior, boundary, open set, closed set** also can be defined analogously using the notion of open balls instead of open discs. ◊

Definition 1.10 (a) An open set D is said to be **connected**[1] if any two points in D can be joined by a *line segment*[2] lying in D.

(b) An open connected set is called a **domain**[3].

(c) A set which consists of a domain together with some or all of its boundary points is called a **region**.

◊

In this course, we shall be mostly interested in those subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ which are domains or regions.

Definition 1.11 Let Let (u_n) be a sequence of points in $\mathbb{R}^k$. Then we say that (u_n) **converges** to $u \in \mathbb{R}^k$ if, corresponding to each open ball B centered at u, there is a positive integer N such that $u_n \in B$ for all $n \geq N$.

If (u_n) converges to u, then we write, $u_n \to u$ or $\lim\limits_{n\to\infty} u_n = u$. ◊

[1]We define here the notion of connectedness only for an open set. It can be done for a more general set also; but we are not doing that here.

[2]If $u, v \in \mathbb{R}^k$, then the line segment $L_{u,v}$ joining u and v is the set of all points of the form $\gamma(t) := (1 - t)u + tv$ for $0 \leq t \leq 1$. The point u is the initial point of $L_{u,v}$ and v is the terminal point of $L_{u,v}$. Note that $\gamma(0) = u$ and $\gamma(1) = v$.

[3]Note that, this is different from saying that D is the domain of a function.

1.2.2 Level curves and level surfaces

Definition 1.12 Let f be a function defined on $D \subseteq \mathbb{R}^2$. Then the subsets of D of the form

$$\{(x, y) \in D : f(x, y) = c\}$$

are called **level curves** of f, where $c \in \mathbb{R}$.

If f is a function of three variables, defined on $D \subseteq \mathbb{R}^3$, then the subsets of D of the form

$$\{(x, y, z) \in D : f(x, y, z) = c\}$$

are called **level surfaces** of f, where $c \in \mathbb{R}$. $\Diamond$

Recall that if f is a function on $D \subset \mathbb{R}^2$, then the graph of f is the *surface*,

$$S_f := \{(x, y, z) : z = f(x, y),\ (x, y) \in D\}.$$

- Thus a level curve of f corresponds to the intersection of S_f with the plane $z = c$. In fact, the level curve is the projection of the this intersection onto the domain of f. Similarly, we can interpret level surfaces geometrically.

EXAMPLE 1.13 Let $f(x, y) = x^2 + y^2$ for $(x, y) \in \mathbb{R}^2$. Then, for $c > 0$, the level curve of f corresponding to c is the circle with centre 0 and radius $\sqrt{c}$. The level curve of f corresponding to $c = 0$ is the point $(0, 0)$, and the level curve of f corresponding to $c < 0$ is the empty set. $\Box$

EXAMPLE 1.14 Let $f(x, y, z) = x^2 + y^2 + z^2$ for $(x, y, z) \in \mathbb{R}^3$. Then, for $c > 0$, the level curve of f corresponding to c is the sphere with centre 0 and radius $\sqrt{c}$. The level curve of f corresponding to $c = 0$ is the point $(0, 0, 0)$, and the level curve of f corresponding to $c < 0$ is the empty set. $\Box$

Remark 1.15 In he case of Example 1.13, we could visualize how level curves are obtained as the graph of f is a surface in $\mathbb{R}^3$, whereas it was not possible for Example 1.14, as the graph of f lies in $\mathbb{R}^4$. $\blacklozenge$

1.3 Limit and Continuity

We shall discuss limit, continuity and differentiability of functions of several variables. One of the primary concepts required to do these is that of a *neighbourhood of a point* in $\mathbb{R}^2$.

Definition 1.16 By a **neighbourhood** of a point $u_0 = (x_0, y_0) \in \mathbb{R}^2$ we mean an open disc with centre u_0 and radius δ for some $\delta > 0$, and such a neighbourhood is also a called a **δ-neighbourhood** of $u_0 = (x_0, y_0)$. ◊

A δ-neighbourhood of a point u_0 is usually denoted by $B_\delta(u_0)$.

Definition 1.17 A neighbourhood with its centre u_0 deleted is called a **deleted neighbourhood** of u_0. ◊

Thus, $\{u \in \mathbb{R}^2 : 0 < |u - u_0| < \delta\}$ is a deleted neighbourhood of u_0.

Definition 1.18 A point $u_0 \in \mathbb{R}^2$ is called a **limit point** of $D \subseteq \mathbb{R}^2$, if every deleted neighbourhood of u_0 contains atleast one point from D. ◊

Exercise 1.19 Show that, if D is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$, then every point in D is a limit point of D. ◄

Exercise 1.20 Show that a point $u_0 \in \mathbb{R}^2$ is a limit point of $D \subseteq \mathbb{R}^2$ if and only if there exists a sequence (u_n) of distinct points in D which converges to u_0, that is, $|u_n - u_0| \to 0$ as $n \to \infty$. ◄

EXAMPLE 1.21 Let $D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Then $u_0 := (1, 0)$ is a limit point of D, as the sequence[4] (u_n) in D with $u_n := \left(\frac{n}{n+1}, 0\right)$ converges to u_0. Also, the sequence (v_n) in D with $v_n := \left(\frac{n}{n+1}, \frac{1}{n+1}\right)$ converges to u_0. In fact every point in the *closed disc* $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ is a limit point of D. □

Definition 1.22 Suppose f is defined on a set $D \subseteq \mathbb{R}^2$, and $u_0 = (x_0, y_0) \in \mathbb{R}^2$ is a limit point of D. We say that $f(x, y)$ approaches the **limit** ℓ as $u = (x, y) \in D$ approaches u_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(u) - \ell| < \varepsilon \quad \text{whenever} \quad u \in D,\ 0 < |u - u_0| < \delta.$$

We write the above fact by

$$\lim_{u \to u_0} f(u) = \ell \quad \text{or} \quad \lim_{(x,y) \to (x_0, y_0)} f(x, y) = \ell$$

or as $f(x, y) \to \ell$ as $(x, y) \to (x_0, y_0)$. ◊

[4]We may recall that a sequence $(a_1, a_2, \ldots)$ of numbers is said to converge to a number a if $|a_n - a|$ can be made *arbitrarily close to* 0 by taking n *sufficiently large*. More precisely, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \ge N$.

Thus, for a function f defined on a set D, if u_0 is a limit point of D, then

> $\lim_{u\to u_0} f(u) = \ell$ iff for every $\varepsilon > 0$, there exists a deleted neighbourhood D_0 of u_0 such that $f(u) \in (\ell - \varepsilon, \ell + \varepsilon)$ for every $u \in D_0 \cap D$.

Exercise 1.23 Let u_0 be a limit point of D and $f : D \to \mathbb{R}$. Then $\lim_{u\to u_0} f(u) = \ell$ if and only if corresponding to every open interval I_ℓ containing ℓ, there is a deleted neighbourhood D_0 of u_0 such that $f(u) \in I_\ell$ whenever $u \in D_0$. ◀

An important fact is the the following:

Theorem 1.24 *Let* $\lim_{u\to u_0} f(u)$ *exists. Then the limit is unique.*

Proof. Suppose there are ℓ_1 and ℓ_2 such that $\lim_{u\to u_0} f(u) = \ell_1$ and $\lim_{u\to u_0} f(u) = \ell_2$. Let $\varepsilon > 0$ be given. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(u) - \ell_1| < \varepsilon \quad \text{whenever} \quad u \in D, \quad 0 < |u - u_0| < \delta_1, \tag{1}$$

$$|f(u) - \ell_2| < \varepsilon \quad \text{whenever} \quad u \in D, \quad 0 < |u - u_0| < \delta_2. \tag{2}$$

Then both (1) and (2) hold for all $u \in D$ with $0 < |u - u_0| < \delta := \min\{\delta_1, \delta_2\}$. So, let $u \in D$ with $0 < |u - u_0| < \delta$. Then we have

$$|\ell_1 - \ell_2| = |(\ell_1 - f(u) + (f(u) - \ell_2)| \leq |f(u) - \ell_1| + |f(u) - \ell_2| < 2\varepsilon.$$

That is $|\ell_1 - \ell_2| < 2\varepsilon$ for all $\varepsilon > 0$. This is possible only if $\ell_1 = \ell_2$. □

Proof. [**Alternatively:**] Let $0 < \varepsilon < |\ell_1 - \ell_2|/2$. Let $\delta_1 > 0$ and $\delta > 0$ be such that

$$f(u) \in (\ell_1 - \varepsilon, \ell_1 + \varepsilon) \quad \text{whenever} \quad u \in D, \quad 0 < |u - u_0| < \delta_1, \tag{1}$$

$$f(u) \in (\ell_2 - \varepsilon, \ell_2 + \varepsilon) \quad \text{whenever} \quad u \in D, \quad 0 < |u - u_0| < \delta_2, \tag{1}$$

This is impossible, since $(\ell_1 - \varepsilon, \ell_1 + \varepsilon)$ and $(\ell_2 - \varepsilon, \ell_2 + \varepsilon)$ are disjoint. □

EXAMPLE 1.25 Let

$$f(x, y) = \sqrt{1 - x^2 - y^2}, \quad D := \{(x, y) : x^2 + y^2 \leq 1\}.$$

Then

$$\lim_{(x,y)\to(0,0)} f(x, y) = 1.$$

Let us show this: Let $\varepsilon > 0$ be given. If $\varepsilon \geq 1$, then $|f(u) - 1| = 1 - f(u) < \varepsilon$ for all $u \in D$. So, let $0 < \varepsilon < 1$. Then we have

$$\begin{aligned} |f(u) - 1| < \varepsilon \quad &\iff \quad 1 - \varepsilon < f(u) \\ &\iff \quad (1 - \varepsilon)^2 < 1 - |u|^2 \\ &\iff \quad |u|^2 < 1 - (1 - \varepsilon)^2. \end{aligned}$$

Thus, taking $\delta := \sqrt{1-(1-\varepsilon)^2} = \sqrt{2\varepsilon - \varepsilon^2} > 0$, we have $|f(u)-1| < \varepsilon$ for $|u| < \delta$.
$\square$

Exercise 1.26 Show that the function f in Example 1.25 is continuous at all points in $D := \{(x,y) : x^2 + y^2 \le 1\}$. ◀

EXAMPLE 1.27 Let

$$f(x,y) := xy\frac{x^2-y^2}{x^2+y^2}, \quad D := \{(x,y) : x^2+y^2 \neq 0\}.$$

Then taking $x = r\cos\theta$ and $y = r\sin\theta$ we have $r^2 = x^2 + y^2$, and

$$|f(x,y)| = \left|\frac{r^2}{4}\sin 4\theta\right| \le \frac{r^2}{4} \to 0 \quad \text{as} \quad r^2 \to 0.$$

Thus, for $\varepsilon > 0$,

$$|f(x,y)| \le \frac{|u|^2}{4} < \varepsilon \quad \text{whenever} \quad |u| < 2\sqrt{\varepsilon}.$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Note that f is not defined at $(0,0)$. $\square$

EXAMPLE 1.28 Let

$$f(x,y) := \frac{xy}{x^2+y^2}, \quad D := \{(x,y) : x^2+y^2 \neq 0\}.$$

Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. To see this, for each $m \in \mathbb{R}$, consider the straight line $L_m := \{(x,y) : y = mx\}$, i.e., the straight line passing through the origin with slope m. Then we see that for $(x,y) \in L_m$, $f(x,y) = \dfrac{m}{1+m^2}$. In particular, in every deleted neighbourhood of $u_0 = (0,0)$, the function takes the values 1/2 and 2/5, by taking $m = 1$ and $m = 2$, respectively. Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. $\square$

EXAMPLE 1.29 Let $z = f(x,y) := \dfrac{x^2y}{x^4+y^2}$, $D := \{(x,y) : x^2+y^2 \neq 0\}$. Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. To see this, for each $m \in \mathbb{R}$, consider the set $A_m := \{(x,y) : y = mx^2\}$. Then we see that for $(x,y) \in A_m$, $f(x,y) = \dfrac{m}{1+m^2}$. Again, by the same argument as in last example, the function does not have limit at $(0,0)$. $\square$

Remark 1.30 It is to be observed that the limit defined above is quite different from the limits:

$$\lim_{x\to x_0}\lim_{y\to y_0} f(x,y), \qquad \lim_{y\to y_0}\lim_{x\to x_0} f(x,y).$$

It is possible that

- the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist but one or both of the limits $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$, $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$ exist, and
- any one or both of $\lim_{x\to x_0}\lim_{y\to y_0} f(x,y)$ and $\lim_{y\to y_0}\lim_{x\to x_0} f(x,y)$ may not exist, but $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists. ♦

To illustrate the statements in the above remark we consider a two examples.

EXAMPLE 1.31 Let

$$f(x,y) := \frac{(y-x)(1+x)}{(y+x)(1+y)}, \qquad D := \{(x,y) : x+y \neq 0,\ y \neq -1\}.$$

Then we see that

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{y}{y} = 1, \qquad \lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}(-1)(1+x) = -1,$$

and

$$\lim_{y=mx,x\to 0} f(x,y) = \frac{m-1}{m+1}.$$

Thus, separate limit exit, but the limit does not exists at $(0,0)$. Note that the above function is not defined in a deleted neighbourhood of $(0,0)$. □

EXAMPLE 1.32 Let

$$f(x,y) := x\sin\frac{1}{y} + y\sin\frac{1}{x}, \qquad D := \{(x,y) : xy \neq 0\}.$$

Then we see that separate limits do not exist at $(0,0)$, but

$$|f(x,y)| \leq |x| + |y| \quad \forall\,(x,y)\in D$$

so that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. □

Theorem 1.33 *Let f and g be defined in D and u_0 be a limit point of D. Suppose $\lim_{u\to u_0} f(u) = \ell_1$ and $\lim_{u\to u_0} g(u) = \ell_2$. Then*

(i) $\lim_{u\to u_0}[f(u)+g(u)] = \ell_1 + \ell_2$;

(ii) $\lim_{u\to u_0} f(u)g(u) = \ell_1\ell_2$;

(iii) *If $\ell_2 \neq 0$, then there exists a deleted neighbourhood D_0 of u_0 such that $g(u) \neq 0$ for all $u \in D \cap D_0$;*

(iv) $\displaystyle\lim_{u \to u_0} \frac{f(u)}{g(u)} = \frac{\ell_1}{\ell_2}$, *whenever $\ell_2 \neq 0$.*

Definition 1.34 A function f defined on a set $D \subseteq \mathbb{R}^2$ is said to be **continuous** at a point $u_0 \in D$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u) - f(u_0)| < \varepsilon$ whenever $|u - u_0| < \delta$. ♢

Remark 1.35 We note that in order to define limit of a function f at a point $u_0 = (x_0, y_0)$, *it is not necessary* that the function is defined at u_0, whereas to define continuity of f at u_0 *it is necessary* that u_0 belongs to the domain of f. ♦

> Throughout this course, whenever we talk about continuity of a function f at a point u_0, we assume that u_0 is a limit point of the domain of definition of D.

Thus, for us,

- f is continuous at u_0 if and only if $\displaystyle\lim_{u \to u_0} f(u)$ exists and it is equal to $f(u_0)$.

Remark 1.36 Suppose a function f is not defined at a point u_0, but $\displaystyle\lim_{u \to u_0} f(u)$ exists. Then we may extend the function f to a new function $\tilde{f}$ defined on $\tilde{D} := D \cup \{u_0\}$, where D is the domain of f, by

$$\tilde{f}(u) := \begin{cases} f(u), & u \in D, \\ \displaystyle\lim_{u \to u_0} f(u), & u = u_0. \end{cases}$$

Then we have $\lim_{u \to u_0} \tilde{f}(u) = \tilde{f}(u_0)$. Thus, $\tilde{f}$ is continuous at u_0.

♦

Exercise 1.37 Give an example of $f : D \to \mathbb{R}$ such that f is continuous at some point $u_0 = (x_0, y_0) \in D$, but $\displaystyle\lim_{u \to u_0} f(u)$ does not exist. ◀

Exercise 1.38 Let $D = \{(x, y) : x^2 + y^2 < 1 \text{ or } (x, y) = (2, 0)\}$. Let $f(x, y) = 1$ for $x^2 + y^2 < 1$ and $f(2, 0) = 2$. Show that f is continuous at every point in D. ◀

EXAMPLE 1.39 In Example 1.25, the function f is continuous at every point in its domain of definition. □

EXAMPLE 1.40 In Example 1.27, the function is not defined at the point $u_0 = (0,0)$. But, the extended function

$$\tilde{f}(x,y) := \begin{cases} f(x,y), & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0), \end{cases}$$

is continuous at every point in $\mathbb{R}^2$. □

EXAMPLE 1.41 In Examples 1.28 and 1.29, the functions cannot be redefined at $u_0 = (0,0)$ so as to make them continuous at u_0. □

Theorem 1.42 *Suppose that f and g are continuous at a point $u_0 \in D$, where D is the domain of both f and g. Then the functions $f+g$ and fg defined by*

$$(f+g)(u) = f(u) + g(u), \quad (fg)(u) = f(u)g(u) \quad \textit{for} \quad u \in D$$

are continuous at u_0. Also, if $g(u) \neq 0$ for u in a neighbourhood of u_0, then the function $u \mapsto f(u)/g(u)$ is defined in that neighbourhood and it is continuous at u_0.

EXAMPLE 1.43 Let $f(x,y) = \frac{2xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$. This function is continuous at every point in $D := \mathbb{R}^2 \setminus \{(0,0)\}$. Note that $(0,0)$ is a limit point of D and $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Hence, f does not have a continuous extension to all of $\mathbb{R}^2$. □

1.4 Partial Derivatives

Let f be a function of two variables defined on a domain $D \subseteq \mathbb{R}^2$ and let $(x_0, y_0) \in D$. Let

$$D_0 := \{(x, y_0) : (x, y_0) \in D\},$$

which may be called the *y_0-section* of D. Note that D_0 is a subset of D and hence f is defined on D_0 and its values $f(x, y_0)$ vary according as x varies. Thus, we can talk about the continuity and differentiability of $x \mapsto f(x, y_0)$ as a function of one variable.

Definition 1.44 Suppose f is a (real valued) function defined in a neigbourhood of a point (x_0, y_0). Then f is said to have the **partial derivative with respect to** x at (x_0, y_0) if $\dfrac{d}{dx} f(x, y_0)$ exists at x_0, and its value at (x_0, y_0) is denoted by $\frac{\partial f}{\partial x}(x_0, y_0)$, called the **partial derivative** of f with respect to x at (x_0, y_0).

Similarly, partial derivative of f with respect to y at (x_0, y_0) is defined by $\dfrac{d}{dy} f(x_0, y)$ at x_0, if it exists, and it is denoted by $\dfrac{\partial f}{\partial y}(x_0, y_0)$.

◊

Thus,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{d}{dx} f(x, y_0)\Big|_{x=x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \frac{d}{dy} f(x_0, y)\Big|_{y=y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y},$$

whenever they exist. One may write these also as follows, whenever they exist:

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0} = \lim_{\Delta y \to y_0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

Partial derivatives $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ are also denoted by $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ respectively. That is,

$$f_x(x_0, y_0) := \frac{\partial f}{\partial x}(x_0, y_0), \quad f_y(x_0, y_0) := \frac{\partial f}{\partial y}(x_0, y_0).$$

We denote

$$f_{xx}(x_0, y_0) := (f_x)_x(x_0, y_0), \quad f_{xy}(x_0, y_0) := (f_x)_y(x_0, y_0),$$

$$f_{yx}(x_0, y_0) := (f_y)_x(x_0, y_0), \quad f_{yy}(x_0, y_0) := (f_y)_y(x_0, y_0).$$

Note that,

$$f_{xx}(x_0, y_0) := \frac{d}{dx} f_x(x, y_0)\Big|_{x=x_0}, \quad f_{xy}(x_0, y_0) := \frac{d}{dy} f_x(x_0, y)\Big|_{y=y_0},$$

$$f_{yx}(x_0, y_0) := \frac{d}{dx} f_x(x, y_0)\Big|_{x=x_0}, \quad f_{yy}(x_0, y_0) := \frac{d}{dy} f_y(x_0, y)\Big|_{y=y_0}.$$

Remark 1.45 It should be kept in mind that $f_x(x_0, y_0)$ cannot be interpreted as $\lim_{(x,y)\to(x_0,y_0)} f_x(x, y)$ even if this limit exists. This can be explained in the case of a function of single variable itself:

Consider the function

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Clearly, $f'(x) = 0$ for every $x \neq 0$ so that $\lim_{x \to 0} f'(x) = 0$, but $f'(0)$ does not even exist. In fact, f is not continuous at 0.

Also, there are situations in which $f_x(x_0, y_0)$ exists but $\lim_{(x,y)\to(x_0,y_0)} f_x(x, y)$ need not exist, as the following example shows. ♦

EXAMPLE 1.46 Consider the function

$$f(x,y) = \begin{cases} x^2 y \sin(1/x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Then we have

$$\frac{f(x,y) - f(0,y)}{x - 0} = xy \sin(1/x).$$

Since $|xy\sin(1/x)| \leq |x^2 y| \to 0$ as $x \to 0$, we have $f_x(0,y) = 0$. Note that for $(x,y) \neq (0,0)$,

$$f_x(x,y) = 2xy \sin(1/x) - \cos(1/x)$$

so that $\lim_{x \to 0} f_x(x,y)$ does not exist. In this example, for any $(x_0, y_0) \neq 0$, we have the following:

$$\begin{aligned} \frac{\partial f}{\partial x}(x_0, y_0) &:= \frac{d}{dx} f(x, y_0)\Big|_{x=x_0} = 2x_0^2 y_0 \sin(1/x_0) - \cos(1/x_0), \\ \frac{\partial f}{\partial y}(x_0, y_0) &:= \frac{d}{dy} f(x_0, y)\Big|_{y=y_0} = x_0^2 \sin(1/x_0). \end{aligned}$$

Note that

(i) $\frac{\partial f}{\partial x}$ exists in a neighbourhood of $(0,0)$, but it is not continuous at $(0,0)$;

(ii) $\frac{\partial f}{\partial y}$ exists in a neighbourhood of $(0,0)$ and it is not continuous in that neighbourhood . □

Partial derivatives of more than two variables are also defined analogously: For example if f is a function of three variables, say (x,y,z), then we can define

$$f_{x_i}, \quad f_{x_i x_j}, \quad f_{x_i x_j x_k} \quad \text{for} \quad i,j,k \in \{1,2,3\}$$

as

$$f_{x_i} := \frac{\partial f}{\partial x_i}, \quad f_{x_i x_j} := \frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big), \quad f_{x_i x_j x_k} := \frac{\partial}{\partial x_k}\Big[\frac{\partial}{\partial x_j}\Big(\frac{\partial f}{\partial x_i}\Big)\Big]$$

respectively.

EXAMPLE 1.47 Let $f(x,y) := x^2 y + y^3$, $(x,y) \in \mathbb{R}^2$. Then

$$f_x = 2xy; \quad (f_x)_y = 2x = (f_y)_x; \quad f_{xx} = 2y; \quad f_{yy} = 6y.$$

□

EXAMPLE 1.48 Let $u = e^{xy} \sin z$. Then,

$$u_x = e^{xy} y \sin z; \quad u_y = e^{xy} x \sin z;$$

$$u_{xy} := (u_x)_y = \sin z (xy+1) e^{xy}; \quad u_{xyz} := (u_{xy})_z = \cos z (xy+1) e^{xy};$$

$$u_{yz} := (u_y)_z = e^{xy} x \cos z; \quad u_{yzx} := (u_{yz})_x = \cos z (xy+1) e^{xy};$$

□

EXAMPLE 1.49 Let

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Note that the f does not have a limit at $(0,0)$. However, $f(x,0) = 0$ and $f(0,y) = 0$ for all $x, y \in \mathbb{R}$. Hence,

$$f_x(x,0) = \frac{d}{dx} f(x,0)\Big|_{x=0} = 0, \quad f_y(0,y) = \frac{d}{dy} f(0,y)\Big|_{y=0} = 0,$$

$$g_{xx}(x,0) = 0, \quad g_{yy}(0,y) = 0 \quad \forall\, x, y \in \mathbb{R}.$$

Let us see if $g_{xy}(0,0)$ exists. Recall that $g_{xy}(0,0) := \frac{d}{dy} g_x(0,y)\big|_{y=0}$. We have to see whether the function $g_x(0,y)$ is differentiable at $y = 0$. Note that for $y \neq 0$,

$$g_x(0,y) = \lim_{\Delta x \to 0} \frac{g(\Delta x, y) - g(0,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(\Delta x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{y}{(\Delta x)^2 + y^2} = \frac{1}{y}.$$

Similarly, $g_y(x,0) = 1/x$ for all $x \neq 0$. Since, $g_x(0,y)$ is not continues at $y = 0$, $(g_x)_y(0,0)$ does not exist. Similarly, $(g_x)_y(0,0)$ does not exist. □

Remark 1.50 The above example shows that a function can have partial derivatives at a point even if it does not have a limit at that point. ♦

The following example shows that, f_{xy} **need not be equal to** f_{yx} at some point.

EXAMPLE 1.51 Let $z = f(x,y) := \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$

Note that $f(x,0) = 0$ and $f(0,y) = 0$ for all $x, y \in \mathbb{R}$. Therefore,

$$f_x(x,0) = 0, \quad f_y(0,y) \quad f_{xx}(x,0) = 0, \quad f_{yy}(0,y) \quad \forall x, y \in \mathbb{R}.$$

Now, by definition,

$$(f_x)_y(0,0) = \frac{d}{dy} f_x(0,y)\Big|_{y=0}, \quad (f_y)_x(0,0) = \frac{d}{dx} f_y(x,0)\Big|_{x=0}.$$

Note that

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0,y)}{\Delta x} = -y,$$

and

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x, \Delta y) - f(x,0)}{\Delta y} = x.$$

Thus,

$$f_{xy}(0,y) = -1, \quad f_{yx}(x,0) = 1 \quad \forall\, x, y \in \mathbb{R}.$$

In particular,

$$f_{xy}(0,0) = -1, \quad f_{yx}(0,0) = 1. \quad \Box$$

The above example shows that, in general, f_{xy} need not be equal to f_{yx}. However, under certain additional conditions they can be equal.

Theorem 1.52 *If f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined and continuous in a neighbourhood of (x_0, y_0), then $f_{xy} = f_{yx}$ in that neighbourhood .*

Sometimes a function of two variables may be defined *implicitly* by an equation, and we may have to find its partial derivatives, as in the following example.

EXAMPLE 1.53 Let $z = f(x, y)$ be defined implicitly by

$$yz - \ln z = x + y.$$

Let us try to find its partial derivatives: By product rule for differentiation,

$$\frac{\partial}{\partial x}(yz - \ln z) = \frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}(\ln z) = y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x}.$$

Also, $\frac{\partial}{\partial x}(x + y) = 1$. Hence,

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1.$$

From this we obtain

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}. \quad \square$$

We already know that first partial derivatives at a point $(u_0 := (x_0, y_0)$ correspond to the slopes of the curves cut out of the surface $z = f(x, y)$ by the planes $x = y_0$ and $y = y_0$, respectively. Let us look a the following example.

EXAMPLE 1.54 Consider the surface defined by $z = x^2 + y^3$. Let us find the slopes of the curves cut out of this surface by the planes $x = -1$ and $y = 1$, respectively, at the point $u_0 = (-1, 1)$. In this case, we have to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial x}$ at the point $u_0 = (-1, 1)$. Note that

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 3y^2.$$

Thus, the required slopes are

$$\left.\frac{\partial z}{\partial x}\right|_{x=-1} = -2, \quad \left.\frac{\partial z}{\partial y}\right|_{y=1} = 3. \quad \square$$

Exercise 1.55 In the following, find f_x and f_y at points in appropriate domains. Check also that $f_{xy} := (f_x)_y$ and $f_x := (f_y)_x$ exist, and if so, check whether they are equal.

1. $f(x, y) = x^2 - xy + y^2$.
2. $f(x, y) = \frac{x}{x^2+y^2}$.

3. $f(x,y) = \frac{x+y}{xy-1}$.

4. $f(x,y) = \ln(x+y)$.

5. $f(x,y) = \int_x^y g(t)dt$, where $g : \mathbb{R} \to \mathbb{R}$ is continuous.

6. $f(x,y) = e^x + x\ln y + y\ln x$.

7. $f(x,y) = \log_y x$.

8. $f(x,y) = \sin xy$.

9. $f(x,y) = \frac{x-y}{x^2+y^2}$.

10. $f(x,y) = y^2 e^{x^2-y}$. ◀

Exercise 1.56 Suppose the variable x is a function of two independent variables y and z according to the equation

$$xz + y\ln x - x^2 + 4 = 0.$$

Find the value of $\partial x/\partial z$ at the point $(y_0, z_0) = (-1, -3)$. ◀

1.4.1 Partial Increments, Total Increment and Differentiability

Let us recall the following from one-variable calculus [5]:

Suppose g is a function of one variable, defined in an interval I_0. Let $x_0 \in I_0$. Then, writing $y = g(x)$, $x \in I_0$, if Δx is an *increment in* x at x_0, then the *increment in* y is defined by

$$\Delta y = g(x_0 + \Delta x) - g(x_0).$$

Thus, g is differentiable at x_0 if and only if there exists some number α_0 such that

$$\varepsilon := \frac{\Delta y}{\Delta x} - \alpha_0 \to 0 \quad \text{as} \quad \Delta x \to 0.$$

Thus,

> g is differentiable at x_0 if and only if there exists some number α_0 such that
>
> $$\Delta y = \alpha\Delta x + \varepsilon\Delta x,$$
>
> where $\varepsilon \to 0$ as $\Delta x \to 0$, and this α is the derivative of f at x_0.

In the case of a function of two (or more) variables, we define *partial increments* and *total increment*, as follows:

[5] Refer: M.T.Nair, *Calculus of One Variable*, Ane Books Pvt. Ltd., 2014

Definition 1.57 Suppose f is a (real valued) function defined in a neigbourhood D_0 of a point (x_0, y_0), and let $z = f(x, y)$ for $9x, y) \in D_0$. Then

- **Partial increment of z with respect to x** is

$$\Delta_x z := f(x_0 + \Delta x, y_0) - f(x_0, y_0).$$

- **Partial increment of z with respect to y** is

$$\Delta_y z := f(x_0, y_0 + \Delta y) - f(x_0, y_0).$$

- **Total increment of z** is

$$\Delta z := f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

♢

Recall that, if f_x exists at (x_0, y_0), then

$$\lim_{\Delta x \to 0} \left[\frac{\Delta_x z}{\Delta x} - f_x(x_0, y_0) \right] = 0$$

exists and equal to $f_x(x_0, y_0)$. Thus, writing

$$\varepsilon_1 := \frac{\Delta_x z}{\Delta x} - f_x(x_0, y_0),$$

we have

$$\Delta_x z = f_x(x_0, y_0)\Delta x + \varepsilon_1 \Delta x, \tag{1}$$

where $\varepsilon_1 \to 0$ as $\Delta x \to 0$. Similarly, if f_y exists at (x_0, y_0), then taking

$$\varepsilon_2 := \frac{\Delta_y z}{\Delta y} - f_y(x_0, y_0),$$

we have

$$\Delta_y z = f_y(x_0, y_0)\Delta y + \varepsilon_2 \Delta y, \tag{2}$$

where $\varepsilon_2 \to 0$ as $\Delta y \to 0$.

The equation (1) shows that the graph of the function $f(x, y_0)$, in a neighbourhood of x_0 is approximated by a straight line segment with slope $f_x(x_0, y_0)$. Similarly, (2) shows that the graph of the function $f(x_0, y)$, in a neighbourhood of y_0 is approximated by a straight line segment with slope $f_y(x_0, y_0)$.

Now, the question is:

> What can we say about the function $f(x, y)$ in a neighbourhood of (x_0, y_0)? Can it be approximated by a *linear function* in a neighbourhood of (x_0, y_0)?

The above questions lead to the notion of **differentiability** of f at a point (x_0, y_0).

Definition 1.58 Let f be a function of two variables defined in a neighbourhood D_0 of a point $u_0 := (x_0, y_0) \in \mathbb{R}^2$. We say that f is **differentiable** at (x_0, y_0) if f_x and f_y exist at (x_0, y_0), and $\Delta z := f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ is of the form

$$\Delta z = f_x(u_0)\Delta x + f_y(u_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \qquad (*)$$

where ε_1 and ε_2 are functions of $(\Delta x, \Delta y)$ such that

$$\varepsilon_1 \to 0, \quad \varepsilon_2 \to 0 \quad \text{as} \quad (\Delta x, \Delta y) \to (0, 0).$$

The pair $(f_x(u_0), f_y(u_0))$ or the vector $f_x(u_0\vec{i} + f_y(u_0)\vec{j}$ is called the **gradient** of f at $u_0 := (x_0, y_0)$, and it is denoted by

$$(\text{grad}\, f)(u_0) \quad \text{or} \quad (\nabla f)(u_0).$$

♢

We write

$$\nabla f = (f_x, f_y) \quad \text{or} \quad \nabla f = f_x\vec{i} + f_y\vec{j}.$$

Note that the gradient of f is a vector valued function, and the equation $(*)$ can be written as

$$\Delta z = \nabla f(u_0).[\Delta x\, \vec{i} + \Delta y\, \vec{j}] + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \qquad (*)$$

or as

$$\Delta z \simeq \nabla f(u_0).[\Delta x\, \vec{i} + \Delta y\, \vec{j}] \quad \text{for} \quad \Delta x\, \vec{i} + \Delta y\, \vec{j} \simeq \vec{0}.$$

Theorem 1.59 (The increment theorem) *Let f be a function of two variables defined in a neighbourhood D_0 of a point $(x_0, y_0) \in \mathbb{R}^2$. If f_x and f_y exist in a neighbourhood of (x_0, y_0) and are continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).*

Proof. We may observe that

$$\begin{aligned} \Delta z &:= f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) \\ &= [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y)] + [f(x_0, y_0 + \Delta y) - f(x_0, y_0)]. \end{aligned}$$

Since f has partial derivatives f_x and f_y in a neighbourhood of (x_0, y_0), by mean value theorem[6],

$$\begin{aligned} f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0 + \Delta y) &= f_x(\xi, y_0 + \Delta y)\Delta x, \\ f(x_0, y_0 + \Delta y) - f(x_0, y_0) &= f_y(x_0, \eta)\Delta y \end{aligned}$$

for some ξ lies between x_0 and $x_0 + \Delta x$ and η lies between y_0 and $y_0 + \Delta y$. Thus,

$$\Delta z = f_x(\xi, y_0 + \Delta y)\Delta x + f_y(x_0, \eta)\Delta y.$$

[6]Refer: M.T.Nair, *Calculus of One Variable*, Ane Books Pvt. Ltd., 2014

Further, since f_x and f_y are continuous at (x, y),

$$\begin{aligned}\varepsilon_1 &:= f_x(\xi, y_0 + \Delta y) - f_x(x_0, y_0) \to 0,\\ \varepsilon_2 &:= f_y(x_0, \eta) - f_y(x_0, y_0) \to 0\end{aligned}$$

as $(\Delta x, \Delta y) \to (0, 0)$, i.e., as $\Delta\rho := \sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$. Thus,

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

with $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $\Delta\rho \to 0$. Thus, we have proved that f is differentiable at (x_0, y_0). □

Theorem 1.60 *If f is differentiable at a point $u_0 := (x_0, y_0)$ in its domain of definition, then it is continuous at that point.*

Proof. The proof follows from the relation $(*)$ in definition of differentiability (Definition 1.58). □

1.4.2 Chain rule

Suppose f is defined in a domain $D \subseteq \mathbb{R}^2$ and it is differentiable in a neighbourhood of a point $(x_0, y_0) \in D$. We are interested in the *rate of change of* $z = f(x, y)$, where (x, y) vary along a curve given by

$$x = x(t), \quad y = y(t)$$

for t in an interval containing t_0 with $x(t_0) = x_0$, $y(t_0) = y_0$.

Thus, x and y are functions of another variable t. Then z can be written as

$$z = f(x(t), y(t))$$

so that it is a function of t. We are interested in finding $\frac{dz}{dt}$.

Recall that

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$. Therefore,

$$\frac{\Delta_t z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}.$$

Now, assuming that x and y are differentiable, $\Delta t \to 0$ implies $(\Delta x, \Delta y) \to (0, 0)$ so that $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$. Hence, taking limit as $\Delta t \to 0$, we obtain,

$$\lim_{\Delta t \to 0} \frac{\Delta_t z}{\Delta t} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

Thus, we obtain

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

EXAMPLE 1.61 Let $z = xy$, where $x = \cos t$, $y = \sin t$. Then we have

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = y(-\sin t) + x \cos t = -\sin^2 + \cos^2 t = \cos 2t.$$

Of course, in this case, we know, $z = xy = \cos t \sin t$ so that $z'(t) = \cos 2t$. □

Suppose $w = f(x, y, z)$, where x, y and z are functions of another variable t. Then, following the similar procedure as before, we obtain

$$\frac{dw}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}.$$

EXAMPLE 1.62 We would like to find the derivative of $w := xy + z$ as (x,y,z) varies over helix, that is,

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

In this case we have

$$\begin{aligned} \frac{dw}{dt} &= f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \\ &= y(-\sin t) + x \cos t + 1 = \cos^2 t - \sin^2 t + 1 = 1 + \cos 2t. \end{aligned}$$

□

Suppose $z = F(u, v)$ where u and v are functions of (x, y), i.e., $u = \varphi(x, y)$ and $v = \psi(x, y)$ for some functions φ and ψ. Then z itself is a function of (x, y).

Assume that F is differentiable in a neighbourhood of a point (u_0, v_0) and φ and ψ have partial derivatives at (x_0, y_0) with $u_0 = \varphi(x_0, y_0)$ and $v_0 = \psi(x_0, y_0)$. Then we have

$$\Delta_x z = F_u \Delta u + F_v \Delta v + \varepsilon_1 \Delta u + \varepsilon_2 \Delta v$$

where $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta u, \Delta v) \to (0, 0)$. From this we have

$$\frac{\Delta_x z}{\Delta x} = F_u \frac{\Delta_x u}{\Delta x} + F_v \frac{\Delta_x v}{\Delta x} + \varepsilon_1 \frac{\Delta_x u}{\Delta x} + \varepsilon_2 \frac{\Delta_x v}{\Delta x}.$$

Taking limits as $\Delta x \to 0$, we have

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x} = F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x}.$$

Hence

$$\frac{\partial z}{\partial x} = F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x}.$$

Thus we have proved the following theorem.

Theorem 1.63 *Suppose φ and ψ are defined in a neighbourhood D of a point (x_0, y_0) and $z = F(u, v)$ where $u = \varphi(x, y)$, $v = \psi(x, y)$ for $(x, y) \in D$. Assume that F_u, F_v exist and are continuous in a neighbourhood of (u_0, v_0), where $u_0 = \varphi(x_0, y_0)$, $v_0 = \psi(x_0, y_0)$. Then for all (x, y) in a neighbourhood of (x_0, y_0), we have*

$$\frac{\partial z}{\partial x} = F_u \frac{\partial u}{\partial x} + F_v \frac{\partial v}{\partial x}.$$

A special cases:

(i) Suppose f is a function of (x, y) in a nbd of a point (x_0, y_0), and x and y are functions of another variable t with $t \in [a, b]$. Then $z = f(x, y)$ is a function of t and we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}.$$

(ii) Suppose f is a function of (x, y) in a nbd of a point (x_0, y_0), and y is a function x. Then $z = f(x, y)$ is a function of x and we have

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} = f_x + f_y \frac{dy}{dx}.$$

EXAMPLE 1.64 Consider the function $z = \ln(u^2 + v^2)$ where $u = e^{x+y^2}$ and $v = x^2 + y$. Then

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \\ &= \frac{2u}{u^2 + v} e^{x+y^2} + \frac{2x}{u^2 + v} \\ &= \frac{2}{u^2 + v}\left(ue^{x+y^2} + x\right) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \\ &= \frac{2u}{u^2 + v} e^{x+y^2} 2y + \frac{1}{u^2 + v} \\ &= \frac{1}{u^2 + v}\left(4ue^{x+y^2} + 1\right) \end{aligned}$$

□

EXAMPLE 1.65 Consider the function $z = x^2 + \sqrt{y}$ where $y = \sin x, 0 < x < \pi$. Then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 2x + \frac{\cos x}{2\sqrt{y}}.$$

□

Euler's Theorem:

Theorem 1.66 *Suppose f is a homogeneous function of degree n in a domain D, i.e., $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for all $\lambda \in \mathbb{R}$, $(x, y) \in D$. Then*

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y).$$

Proof. We may write

$$f(x, y) = f\left(x, x\frac{y}{x}\right) = x^n f\left(1, \frac{y}{x}\right) = F(u, v)$$

where $u = x$, $v = y/x$ and $F(u, v) = u^n f(1, v)$. Then by Theorem 1.63, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= F_u u_x + F_v v_x \\ &= nu^{n-1} f(1, v).1 + u^n f'(1, v)(-y/x^2) \\ &= nx^{n-1} f(1, v) - yx^{n-2} f'(1, v), \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= F_u u_y + F_v v_y \\ &= nu^{n-1} f(1, v).0 + u^n f'(1, v)(1/x) \\ &= x^{n-1} f'(1, v). \end{aligned}$$

Hence

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x[nx^{n-1} f(1, v) - yx^{n-2} f'(1, v)] + y[x^{n-1} f'(1, v)] \\ &= nx^n f(1, v). \end{aligned}$$

Thus, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$. □

1.5 Derivatives of Implicitly Defined Functions

Definition 1.67 A function $y = f(x)$ is said to be **implicitly defined** on an interval J if there exists a function $z = F(x, y)$ defined on a domain D which contains the set $\{(x, 0) : x \in J\}$ such that

$$F(x, f(x)) = 0 \quad \forall\, x \in J.$$

◊

Theorem 1.68 *Suppose f is implicitly defined on an interval J of a point x_0 by an equation $F(x,y)=0$ with $y=f(x)$ for $x\in J$. Assume that F is differentiable in a neighbourhood of (x_0,y_0), where $x_0\in J$ and $y_0=f(x_0)$. If $F_y\neq 0$ at (x_0,y_0), then $F_y\neq 0$ and $f'(x)$ exists in a nbd J_0 of x_0 and*

$$f'(x) = -\frac{F_x}{F_y} \quad \forall\, x\in J_0.$$

Proof. Since F is differentiable in a neighbourhood of (x_0,y_0), from the equation $F(x,y)=0$, in a neighbourhood of (x_0,y_0), we have

$$F_x\frac{dx}{dx}+F_y\frac{dy}{dx}=0, \quad i.e., F_x+F_y\frac{dy}{dx}=0. \tag{$*$}$$

Also, since F_y is continuous in a neighbourhood of (x_0,y_0), $F_y\neq 0$ at (x_0,y_0) implies $F_y\neq 0$ ina neighbourhood of (x_0,y_0). Hence, from $(*)$, we have

$$\frac{dy}{dx}=-\frac{F_x}{F_y}$$

in a neighbourhood of x_0. □

Suppose F is defined in a neighbourhood D of a point (x_0,y_0).

Question: Does there exists a function $y=f(x)$ defined in a neighbourhood J_0 of x_0 such that $F(x,f(x))=0$ for all $x\in J_0$?

The following theorem, known as the *implicit function theorem* prescribes certain conditions on F which guarantees affirmative answer to the above question. We omit its proof. Interested reader can see the proof in any of the books on *Advanced Calculus*, for example, the book, Advanced Calculus, by D.V. Widder (Prentice-Hall of India, 1996).

Theorem 1.69 (Implicit function Theorem) *Suppose F is defined in a neighbourhood D of a point (x_0,y_0), and F_x and F_y exist and are continuous in D. If $F(x_0,y_0)=0$ and $F_y\neq 0$ at (x_0,y_0), then there exists a differentiable function $y=f(x)$ defined in a neighbourhood J of x_0 such that $(x,f(x))\in D$ and*

(i) $F(x,f(x))=0$ for all $x\in J$, and

(ii) $f'(x)=-\frac{F_x}{F_y}((x,f(x))$ for all $x\in J$.

EXAMPLE 1.70 Let $F(x,y)=x^2+y^2-1$ for $(x,y)\in\mathbb{R}^2$. Clearly, F_x and F_y exist and are continuous in $\mathbb{R}^2$. Also, $F(x,y)=0$ for every point on the circle $S:=\{(x,y): x^2+y^2=1\}$. Note that $F_y\neq 0$ whenever $(x,y)\neq(1,0)$. In this case we have

$$\frac{dy}{dx}=-\frac{x}{y}$$

at any point x with $(x,y)\in S$ and $y\neq 0$. □

EXAMPLE 1.71 Let $F(x,y) = e^y - e^x + xy$ for $(x,y) \in \mathbb{R}^2$. Clearly, $F_x := -e^x + y$ and $F_y := e^y + x$ are continuous in $\mathbb{R}^2$. Also, $F_y \neq 0$ whenever $e^y \neq -x$. In this case we have

$$\frac{dy}{dx} = -\frac{-e^x + y}{e^y + x} = \frac{e^x - y}{e^y + x}$$

whenever $e^y \neq -x$. □

The above considerations can be extended to functions of more than two variables.

So, let $z = f(x,y)$ be defined implicitly by the equation

$$F(x,y,z) = 0.$$

Then under appropriate conditions on F and (x_0, y_0, z_0), we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z},$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

EXAMPLE 1.72 Let $F(x,y,z) = x^2 + y^2 + z^2 - R^2$. Then

$$\frac{\partial z}{\partial x} = -\frac{x}{z}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{z}.$$

□

EXAMPLE 1.73 Let $F(x,y,z) = e^z + x^2y + z + 5$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy}{e^2 + 1}, \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2}{e^2 + 1}.$$

□

1.6 Directional derivative and gradient

Suppose f is a function of two variables defined in a domain $D \subseteq \mathbb{R}^2$. Recall that f_x at $P_0 = (x_0, y_0)$ (if it exists) is the slope of the curve on the surface $S : z = f(x,y)$ at the point (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$, where the curve is obtained as the intersection of the surface with the plane $y = y_0$. Similarly, f_y at $P_0 = (x_0, y_0)$ (if it exists) is the slope of the curve on the surface $S : z = f(x,y)$ at the point (x_0, y_0, z_0) with $z_0 = f(x_0, y_0)$, where the curve is obtained as the intersection of the surface with the plane $x = x_0$. Thus, we may observe that f_x and f_y are rate of changes in $f(x, y)$ in the directions of x-axis and y- axis respectively; in other words, in the directions of the vectors

$$\vec{i} := (1, 0) \quad \text{and} \quad \vec{j} := (0, 1),$$

respectively. We may also observe that any unit vector $\vec{u}$ on the plane (with initial point the origin) can be written as

$$\vec{u} = u_1\vec{i} + u_2\vec{j} \quad \text{with} \quad u_1^2 + u_2^2 = 1.$$

Now, we shall consider the rate of change of the values of f at a point (x_0, y_0) in the direction of a general unit vector $\vec{u}$ instead of $\vec{i}$ and $\vec{i}$. Note that points on the ray with initial point at $P_0 := (x_0, y_0) = x_0\vec{i} + y_0\vec{j}$ in the direction of $\vec{u} := u_1\vec{i} + u_2\vec{j}$ are given by

$$P_0 + s\vec{u}, \quad s > 0.$$

Thus, we would like to know if the following limit exists:

$$\lim_{s\to 0} \frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s}. \tag{$*$}$$

Definition 1.74 If the limit in $(*)$ exists, then the value of the limit is called the **directional derivative** of f at the point $P_0 = (x_0, y_0)$ in the direction of the unit vector $\vec{u}$. ♢

The directional derivative of f at the point $P_0 = (x_0, y_0)$ in the direction of the unit vector $\vec{u}$ is denoted by one of the following notations:

$$\frac{df}{ds}, \quad D_{\vec{u}}f, \quad \frac{\partial f}{\partial \vec{u}} \quad \text{at} \quad P_0.$$

Suppose f is differentiable at (x_0, y_0). Then we see that

$$\frac{f(x_0 + u_1 s, y_0 + u_2 s) - f(x_0, y_0)}{s} = \frac{f(x(s), y(s)) - f(x_0, y_0)}{s},$$

where

$$x(s) = x_0 + u_1 s, \quad y(s) = y_0 + u_2 s.$$

Writing

$$z(s) = f(x(s), y(s))$$

we see that, if $D_{\vec{u}}f$ at (x_0, y_0) is given by

$$\begin{aligned} D_{\vec{u}}f &= \lim_{s\to 0} \frac{f(x(s), y(s)) - f(x_0, y_0)}{s} = \frac{dz}{ds} \\ &= f_x x' + f_y y' = f_x u_1 + f_y u_2 \quad \text{at} \quad (x_0, y_0). \end{aligned}$$

Note that

$$f_x u_1 + f_y u_2 = \nabla f . \vec{u},$$

where ∇f is the **gradient** of f. Thus, if f is differentiable at (x_0, y_0), then

$$\boxed{D_{\vec{u}} f = \nabla f . \vec{u} \quad \text{at} \quad (x_0, y_0).}$$

Recall that

$$\nabla f . \vec{u} = |\nabla f| \cos \theta,$$

where θ is the angle between ∇f and $\vec{u}$. Thus, we have proved the following theorems:

Theorem 1.75 *$D_{\vec{u}} f$ is maximum at (x_0, y_0) if $\vec{u}$ is the unit vector in the direction of ∇f, that is, if*

$$\vec{u} = \frac{\nabla f}{|\nabla f|} \quad \text{at} \quad (x_0, y_0).$$

Theorem 1.76 *The maximum value of $D_{\vec{u}} f$ is $|\nabla f|$, that is,*

$$\max_{\vec{u}} D_{\vec{u}} f = |\nabla f|$$

and this above value is attained for $\vec{u} = \frac{\nabla f}{|\nabla f|}$

EXAMPLE 1.77 Let $f(x, y) = xy$. Then the directional derivative of f in the direction of $\vec{v} := (3, 4)$ at the point $P_0 = (1, 2)$ is given by $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ at P_0, where $\nabla f = (y, x)$ and $\vec{u} = \vec{v}/|\vec{v}| = \frac{1}{5}(3, 4)$. Hence, at $P_0 = (1, 2)$,

$$D_{\vec{u}} f(P_0) = \frac{1}{5}(3y + 4x)\big|_P = 2.$$

EXAMPLE 1.78 Let $f(x, y) = x^2 + y^2$. Then the directional derivative of f in the direction of $\vec{v} := (s_1, s_2)$ at the point $P = (a, b)$ is given by $D_{\vec{u}} = \nabla f \cdot \vec{u}$ at (a, b), where $\bigtriangledown f = (2x, 2y)$ and $\vec{u} = \vec{v}/|\vec{v}| = (s_1, s_2)/\sqrt{s_1^2 + s_2^2}$. Hence

$$D_{\vec{u}} f(P) = \frac{(2xs_1 + 2ys_2)\big|_P}{\sqrt{s_1^2 + s_2^2}} = \frac{2}{\sqrt{s_1^2 + s_2^2}}(as_1 + bs_2).$$

EXAMPLE 1.79 *Let $f(x, y, z^2) = x^2 + y^2 + z^2$ and $\vec{v} = 2\vec{i} + \vec{j} + 3\vec{k}$. Then the directional derivative of f in the direction of $\vec{v}$ at the point $P = (1, 1, 1)$ is:*

$$D_{\vec{u}} f = \nabla f \cdot \vec{u},$$

where

$$\nabla f = (2x, 2y, 2z), \quad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{(2, 1, 3)}{\sqrt{4 + 1 + 9}} = \frac{(2, 1, 3)}{\sqrt{14}}.$$

Thus,

$$\nabla f = \frac{4x + 2y + 6z}{\sqrt{14}}$$

so that at $P = (1, 1, 1)$*,*

$$D_{\vec{u}} f = \frac{(4 \times 2) + (2 \times 1) + (6 \times 3)}{\sqrt{14}} = \frac{12}{\sqrt{14}}.$$

If $\vec{v} = \nabla f$*,*

$$D_{\vec{u}} f(P) = | \bigtriangledown u|(P) = 2\sqrt{x^2 + y^2 + z^2}(P) = 2\sqrt{3}.$$

1.6.1 Geometrical interpretation - tangent vector and tangent plane

Definition 1.80 By a **curve** in $\mathbb{R}^k$ we mean a continuous function g from some interval I into $\mathbb{R}^k$.

Sometimes the range of such a function γ is also called the curve. $\Diamond$

Let γ be a curve in $\mathbb{R}^2$. Then it can be represented by $t \mapsto (x(t), y(t))$ or in vector notation as

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}, \quad t \in I.$$

Note that

$$\lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0}$$

exists, then it is equal to

$$x'(t_0)\vec{i} + y'(t_0)\vec{j}.$$

1.6.2 Tangent vector and its normal direction

Definition 1.81 If $\vec{r}(t) := x(t)\vec{i} + y(t)\vec{j}$ is a curve in $\mathbb{R}^2$ and if $t \mapsto x(t)$ and $t \mapsto y(t)$ are differentiable at t_0, then the vector

$$\vec{\tau} = x'(t_0)\vec{i} + y'(t_0)\vec{j}$$

is called the **tangent vector** of the curve at $(x_0, y_0) := (x(t_0), y(t_0))$. $\Diamond$

Now, suppose that $t \mapsto (x(t), y(t))$ is a level curve of a function $z = f(x, y)$ in a neighbourhood D_0 of (x_0, y_0), say

$$f(x, y) = c$$

and suppose both $x(t)$ and $y(t)$ are differentiable in an open interval I_0 containing t_0, with $x_0 = x(t_0)$, $y_0 = y(t_0)$.Then we have

$$f(x(t), y(t)) = c \quad \forall t \in I_0$$

so that

$$f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) = 0.$$

This shows that

$$(\nabla f)(x_0, y_0) := f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$$

is perpendicular to the tangent vector $x'(t_0)\vec{i} + y'(t_0)\vec{j}$. That is,

$$(\nabla f)(P_0).\vec{r}(P_0) = 0, \quad P_0 := (x_0, y_0).$$

In other words, $(\nabla f)(P_0)$ is in the direction of the normal to the curse $\vec{r}(t)$.

1.6.3 Tangent plane and its normal direction

Next, consider a function f of three variables. Let S be a level surface of a function $g(x, y, z)$, i.e.,

$$S = \{(x, y, z) : g(x, y, z) = 0\}.$$

Let $\vec{r}(t) := x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, $t \in I$, be a curve C lying on S. Then we have

$$g(x(t), y(t), z(t)) = 0 \quad \forall\, t \in I.$$

Assume that g is differentiable on its domain, and C be a smooth curve, that is, $x(t), y(t), z(t)$ are differentiable. Then, $\varphi(t) := g(x(t), y(t), z(t)) = 0$ implies $\varphi'(t) = 0$, that is,

$$g_x(P_0)x'(t_0) + g_y(P_0)y'(t_0) + g_z(P_0)z'(t_0) = 0,$$

where $P_0 = (x_0, y_0, z_0) = (x(t_0), y(t_0), z(t_0)$ for some $t_0 \in I$. Thus,

$$(\nabla g)(P_0).\vec{r}\,'(t_0) = 0,$$

where $\vec{r}\,'(t_0)$ is the tangent vector to C at the point P_0. This is true for any smooth curve in S passing through the point $P_0 \in S$. Thus, $(\nabla g)(P_0)$ is normal to tangent vectors of all smooth curve in S passing through the point $P_0 \in S$. Hence, there there is a plane touching the surface at the point P_0, and $(\nabla g)(P_0)$ is normal to this plane.

Definition 1.82 The plane containing tangent vectors of all smooth curve in S passing through the point $P_0 \in S$ is called the **tangent plane** of the surface S at the point P_0. $\Diamond$

Thus, $(\nabla g)(P_0)$ is normal to the tangent plane of the surface S at the point P_0. Hence, the equation of the tangent plane at the point $P_0 \in S$ is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

Also, the equation of the normal to S at $P_0 \in S$ is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)}.$$

Equivalently,

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t$$

for $t \in \mathbb{R}$.

1.7 Extreme values

Let f be a function of two variables defined on a region D.

Definition 1.83 Let $(x_0, y_0) \in D$.

1. The function f is said to have a **local maximum** at $(x_0, y_0) \in D$ if

$$(f(x, y) \leq f(x_0, y_0)$$

   for all $(x, y) \in D_0 \cap D$, where D_0 is a neighbourhood of (x_0, y_0), and in that case $f(x_0, y_0)$ is called a **local maximum value** of f.

2. The function f is said to have a **local minimum** at $(x_0, y_0) \in D$ if

$$(f(x, y) \geq f(x_0, y_0)$$

   for all $(x, y) \in D_0 \cap D$, where D_0 is a neighbourhood of (x_0, y_0), and in that case $f(x_0, y_0)$ is called a **local minimum value** of f..

3. The function f is said to have a **local extremum** at $(x_0, y_0) \in D$ if it has a local maximum or local minimum at (x_0, y_0).

$\Diamond$

A necessary condition for local extrema:

Suppose φ is a function of one variable. Let us recall the following:

Suppose φ is differentiable at a point t_0 and φ attains local maximum t_0. Then for t in an open interval I_0 containing t_0 with $t \neq t_0$, we have

$$\frac{\varphi(t) - \varphi(t_0)}{t - t_0} \geq 0 \quad \text{for} \quad t < t_0 \text{ and } t \text{ close to } t_0,$$

$$\frac{\varphi(t) - \varphi(t_0)}{t - t_0} \leq 0 \quad \text{for} \quad t > t_0 \text{ and } t \text{ close to } t_0.$$

Taking limit as $t \to t_0$, we obtain $\varphi'(t_0) = 0$. Similarly, if φ is differentiable at a point t_0 and φ attains local minimum t_0, then $\varphi'(t_0) = 0$.

Theorem 1.84 *Suppose f has a local extremum at an interior point (x_0, y_0), and f has partial derivatives f_x and f_y at (x_0, y_0). Then*

$$f_x(x_0, y_0) = 0, \quad f_y(x_0, y_0) = 0.$$

Proof. Let $g(x) = f_x(x, y_0)$. Then g has local extremum at x_0. Hence, $g'(x_0) = 0$. But, $f_x(x_0, y_0) = g'(x_0)$. Thus, $f_x(x_0, y_0) = 0$. Similarly, considering the function $h(y) := f(x_0, y)$, we obtain $f_y(x_0, y_0) = h'(y_0) = 0$. $\square$

Suppose f has partial derivatives f_x and f_y at (x_0, y_0). Recall that, the equation of the tangent plane to the surface $z = f(x, y)$ at $P_0 = (x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$ is:

$$f_x(P_0)(x - x_0 + f_y(P_0)(y - y_0) + (-1)(z - z_0) = 0.$$

Suppose that f has a local extremum at (x_0, y_0). Then, the equation of the tangent plane at (x_0, y_0, z_0) takes the form $0.(x - x_0) + 0.(y - y_0 + (-1)(z - z_0) = 0$. That is

$$z = z_0.$$

In other words, the tangent plane at (x_0, y_0) is parallel to the xy-plane.

Definition 1.85 Let f be defined in a domain $D \subseteq \mathbb{R}^2$. An interior point P_0 of D is called a **critical point**[7] of f if either atleast one of f_x and f_y does not exist at P_0 or both f_x and f_y exist at P_0 and are zero at P_0, that is, $f_x(P_0) = 0$ and $f_y(P_0) = 0$. ♢

Now we prescribe a sufficient condition for a function to have a maximum or a minimum at a point. Its proof will be given in the next section after defining Taylor's formula.

Theorem 1.86 *Suppose f is continuous and has continuous first and second partial derivatives in a neighbourhood of a point $P_0 \in \mathbb{R}^2$ and $f_x(P_0) = 0 = f_y(P_0)$. Then we have the following:*

(i) f has a local maximum at P_0, if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \& \quad f_{xx} < 0 \quad at \quad P_0.$$

(ii) f has a local minimum at P_0 if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \& \quad f_{xx} > 0 \quad at \quad P_0.$$

(iii) f has a saddle point at P_0 if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at P_0.

Note that

$$f_{xx}f_{yy} - f_{xy}^2 = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

The matrix $H_f := \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$ is called the **Hessian matrix** of f and its determinant is called the **discriminant** or **Hessian** of f.

Proof of the above theorem can be deduced using the Tayor's formula which we shall derive in the next section.

[7]In some books critical points are defined as those points at which f_x and f_y exist and are zero.

1.8 Taylor's Formula

Recall mean value theorem (MVT) from school calculus: Suppose $f : [a, b] \to \mathbb{R}$ is continuous and is differentiable in interior of $[a, b]$. Then there exists c with $a < c < b$ such that

$$f(b) = f(a) + f'(c)(b - a).$$

We may state the conclusion in the above slightly differently: For any x, x_0 in $[a, b]$ there exists a point c_x lies in the open interval with endpoints x and x_0 such that

$$f(x) = f(x_0) + f'(c_x)(x - x_0).$$

Note that c_x can be written as $x_0 + t(x - x_0)$ for some t with $0 < t < 1$. Thus, the above relation is:

$$f(x) = f(x_0) + f'(c)(x - x_0)$$

where $c = x_0 + t(x - x_0)$ for some t with $0 < t < 1$.

This result is a particular case of a general result, known as **Taylor's formula**:

Theorem 1.87 (Taylor's formula) *Suppose f is a function of one variable having continuous derivatives up to the order $n + 1$ in a neighbourhood I_0 of a point x_0. Then for any x in $\bar{I}_0$,*

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(k+1)}(\xi_x)}{k!}(x - x_0)^{k+1}$$

for some ξ_x in the open interval with end points x_0 and x.

Writing $\Delta x = x - x_0$, the above formula can be written as

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{1}{k!} \left(\Delta x \frac{d}{dx}\right)^k f\,(x_0) + \frac{1}{(n+1)!} \left(\Delta x \frac{d}{dx}\right)^{n+1} f(x_0 + \xi \Delta x),$$

where ξ is such that $x_0 + \xi \Delta x$ lies between x and x_0.

We derive similar formula for functions of two variables as well.

Let us assume that f is a function of two variables defined in a neighbourhood D_0 of a point $P_0 = (x_0, y_0)$. Assume that, in D_0, f is continuous and have continuous partial derivatives of order upto $(n + 1)$ for soem $n \in \mathbb{N}$, that is, for all i, j with $0 \le i + j \le n + 1$,

$$\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(P) \quad \text{exists} \quad \forall\, P \in D_0.$$

For $x \in D_0$, let

$$\varphi(t) = f(x_0 + t\Delta x, y_0 + t\Delta y).$$

Then we have

$$\varphi'(t) = \Delta x f_x + \Delta y f_y \quad \text{at} \quad P(t) := (x_0 + t\Delta x, y_0 + t\Delta y).$$

By the same argument,

$$\begin{aligned}\varphi''(t) &= \Delta x[f_{xx}\Delta x + f_{xy}\Delta y] + \Delta y[f_{yx}\Delta x + f_{yy}\Delta y]\\ &= (\Delta x)^2 f_{xx} + 2(\Delta x)(\Delta y) f_{xy} + (\Delta y)^2 f_{yy}\\ &= (\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y})^2 f \quad \text{at} \quad P(t).\end{aligned}$$

Thus, taking derivative of φ corresponds to applying the operator

$$\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}$$

on f and evaluating at $P(t)$. Thus, for any $k \le n+1$,

$$\varphi^{(k)}(t) = \left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^k f \quad \text{at} \quad P(t). \tag{$*$}$$

Now, using the Taylor's formula for functions of one variable (Theorem 1.87) we have

$$\varphi(1) = \varphi(0) + \sum_{k=1}^{n} \frac{\varphi^{(k)}(0)}{k!} + \frac{\varphi^{(k+1)}(\xi)}{k!}$$

for some ξ between 0 and 1. Thus, we have the following theorem:

Theorem 1.88 **(Taylor's formula)** *Suppose f is a function of two variables having continuous partial derivatives up to the order $n+1$ in a neighbourhood D_0 of a point (x_0, y_0). Then the Taylor's formula for $f(x,y)$ for any $(x,y) \in D_0$ is given by*

$$\begin{aligned}f(x,y) &= f(x_0,y_0) + \sum_{k=1}^{n} \frac{1}{k!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^k f(x_0,y_0)\\ &\quad + \frac{1}{(n+1)!}\left(\Delta x\frac{\partial}{\partial x} + \Delta y\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\xi\Delta x, y_0+\xi\Delta y).\end{aligned}$$

for some ξ with $0<\xi<1$.

Note that, if $n = 1$ and $f_x = 0 = f_y$ at P_0, then the above formula takes the form:

$$f(x,y) - f(x_0,y_0) = \frac{1}{2}[(\Delta x)^2 f_{xx} + 2(\Delta x)(\Delta y) f_{xy} + (\Delta y)^2 f_{yy}]_{(P(\xi))}.$$

Let

$$Q(t) := [(\Delta x)^2 f_{xx} + 2(\Delta x)(\Delta y) f_{xy} + (\Delta y)^2 f_{yy}]_{(P(t))}.$$

Thus,

$$f(x,y) - f(x_0,y_0) = \frac{1}{2}Q(\xi).$$

Note that, by the continuity of $Q(t)$, $Q(\xi)$ and $Q(0)$ have the same same sign for

ξ close to 0, and we know that $f(x,y) - f(x_0,y_0)$ and $Q(\xi)$ have the same sign. Hence, $f(x,y) - f(x_0,y_0)$ and $Q(0)$ have the same same sign for ξ close to 0.

We observe that

$$\begin{aligned} f_{xx}Q(0) &= f_{xx}[(\Delta x)^2 f_{xx} + 2(\Delta x)(\Delta y)f_{xy} + (\Delta y)^2 f_{yy}]_{(P_0)} \\ &= [(\Delta x)f_{xx} + (\Delta y)f_{xy}]^2 + (\Delta y)^2[f_{xx}f_{yy} - f_{xy}^2] \quad \text{at} \quad P_0. \end{aligned}$$

Suppose $f_{xx}f_{yy} - f_{xy}^2 > 0$. Then, from the above it can be inferred that:

1. $f_x x < 0$ implies that $Q(0) < 0$ so that f has local maximum at P_0.
2. $f_x x > 0$ implies that $Q(0) > 0$ so that f has local minimum at P_0.

Other inferences in second-derivative test can be made by considering different small values of Δx and Δy.

EXAMPLE 1.89 Let $f(x,y) = (x-1)^2 + (y-2)^2 - 1$, $(x,y) \in \mathbb{R}^2$. Then we see that $f(x,y) > f(1,2) = -1$ for all $(x,y) \neq (1,2)$. Thus, f attains minimum at $(1,2) \in \mathbb{R}^2$. □

EXAMPLE 1.90 Let $f(x,y) = \frac{1}{2} - \sin(x^2+y^2)$, $(x,y) \in \mathbb{R}^2$. Then we see that $f(x,y) < f(0,0) = \frac{1}{2}$ for all $(x,y) \neq (0,0)$ in a nbd of $(0,0)$. Thus, f attains maximum at $(1,2) \in \mathbb{R}^2$. □

EXAMPLE 1.91 Let $f(x,y) = x^2 - y^2$. Then $f_x = 0 = f_y$ at $(0,0) \in \mathbb{R}^2$, but, f attains neither maximum nor minimum at $(0,0)$; in fact, in every nbd of the origin, there are points at which f takes positive as well as negative values. Thus, $(0,0)$ is a critical point which is a saddle point of f. □

EXAMPLE 1.92 Find the distance between the straight lines given by the equations

$$(i) \quad \frac{x-1}{1} = \frac{y}{2} = \frac{z}{1} \quad \text{and} \quad (ii) \quad \frac{x}{1} = \frac{y}{1} = \frac{z}{1}.$$

Note that an arbitrary point on the first line given is by $(\lambda+1, 2\lambda, \lambda)$ whereas an arbitrary point on the second line is given by (μ,μ,μ). Thus, we have to minimize the function

$$f(\lambda,\mu) := (\lambda+1-\mu)^2 + (2\lambda-\mu)^2 + (\lambda-\mu)^2.$$

Note that

$$f_\lambda = 2(\lambda+1-\mu) + 4(2\lambda-\mu) + 2(\lambda-\mu) = 12\lambda - 8\mu + 2,$$

$$f_\mu = -2(\lambda+1-\mu) - 2(2\lambda-\mu) - 2(\lambda-\mu) = -8\lambda + 6\mu - 2 =$$

Hence, $f_\lambda = 0$, $f_\mu = 0$ give $\lambda = 1/2$, $\mu = 1$. It can be seen that

$$f_{\lambda\lambda}f_{\mu\mu} - f_{\lambda\mu}^2 > 0, \quad f_{\lambda\lambda} > 0 \quad \text{at} \quad (\lambda,\mu) = (1/2,1).$$

Hence, the minimum distance is $\sqrt{f(1/2,1)} = 1/\sqrt{2}$. □

EXAMPLE 1.93 Let $f(x,y) = x^2 + y^2 - xy + 3x - 2y + 1$. Then

$$f_x = 2x + -y + 3, \quad f_y = 2y - x - 2.$$

Hence

$$f_x = 0 = f_y \iff (x,y) = (-4/3, 1/3)$$

and

$$f_{xx}f_{yy} - f_{xy}^2 = 3 \quad \text{and} \quad f_{xx} = 2 > 0 \text{ at } (-4/3, 1/3).$$

Hence, by Theorem 1.86, f has a minimum at $(-4/3, 1/3)$.

Remark 1.94 In Example 1.93, although $f_{xx}f_{yy} - f_{xy}^2 = 3$ and $f_{xx} = 2$ at every $(x,y) \in \mathbb{R}^2$, the function has only one extremum. ♦

EXAMPLE 1.95 Let us find the shortest distance from the origin to the plane $x - 2y - 2z = 1$ (by finding extrema of certain functions).

Well, you already know that the shortest distance from the origin to the plane given by $ax = by + cz = d$ is $|d|/\sqrt{a^2 + b^2 + c^2}$. Thus, the answer required in the present example is 1/3. Let us find the same by finding the minimum of certain functions.

Recall that the distance from the origin to any point (x,y,z) is given by $\sqrt{x^2 + y^2 + z^2}$. Thus, the problem reduces to minimizing the function $x^2 + y^2 + z^2$ when (x,y,z) varies over the plane given by $x - 2y - 2z = 1$.

Since $x - 2y - 2z = 1$ implies $z = \frac{1}{2}(x - 2y - 1)$, it is enough to find the minimum value of

$$f(x,y) := x^2 + y^2 + z^2 = x^2 + y^2 + \frac{1}{4}(x - 2y - 1)^2.$$

Note that

$$f_x = 2x + \frac{1}{2}(x - 2y - 1) = \frac{5}{2}x - y - \frac{1}{2},$$

$$f_y = 2y - (x - 2y - 1) = 4y - x + 1.$$

Thus,

$$f_x = 0 \quad \& \quad f_y = 0 \iff x = \frac{1}{9} \quad \& \quad y = -\frac{2}{9},$$

and in that case $z = -\frac{2}{9}$. Since the function f does not attain maximum at any point, the minimum of f is $f(\frac{1}{9}, -\frac{2}{9}) = \frac{1}{9}$ so that the required minimum distance is $\sqrt{f(\frac{1}{9}, -\frac{2}{9})} = \frac{1}{3}$. □

Exercise 1.96 Show that the shortest distance from the origin to the plane given by $ax = by + cz = d$ is $\frac{|d|}{\sqrt{a^2+b^2+c^2}}$, by finding minimum of certain function. ◀

Exercise 1.97 Show that the rectangle having a maximum area for a fixed perimeter is a square. ◀

1.8.1 Method of Lagrange Multipliers

In Example 1.95 what we have found is the minimum of a function

$$f(x,y,z) := x^2 + y^2 + z^2 \qquad (i)$$

when (x,y,z) varies over the set of points such that

$$\varphi(x,y,z) := x - 2y - 2z - 1 = 0. \qquad (ii)$$

The method we adopted was that, from the equation (ii), we expressed z as a function of (x,y), namely

$$z = g(x,y) := \frac{1}{2}(x - 2y - 1)$$

and substituted the same into (i) to obtain a function of two variables, namely

$$h(x,y) := f(x,y,g(x,y)) := x^2 + y^2 + x^2 + y^2 + \frac{1}{4}(x - 2y - 1)^2$$

and used the method of finding minimum of h.

In general, suppose the problem is to find an extremum of a function

$$f(x,y,z) \qquad (iii)$$

when (x,y,z) varies over the set of points such that

$$\varphi(x,y,z) = 0. \qquad (iv)$$

If we are able to find a function $z = g(x,y)$ such that

$$\varphi(x,y,g(x,y)) = 0,$$

then we could find extrema of

$$h(x,y) := f(x,y,g(x,y)).$$

But, if the function φ is not simple enough to find such a function g, then the above procedure cannot be adopted. Now we describe a procedure, called *method of Lagrange multipliers*, which does not involve such a function g explicitly.

We shall consider the method of Lagrange multipliers in the case of two variables. The procedure can be extended to any finite number of variables.

Suppose we would like to find maximum or minimum of a function f which is defined in some open set $D \subseteq \mathbb{R}^2$, subject to the condition $\varphi(x,y) = 0$, where φ is also defined in D. Suppose f attains maximum at a point $P_0 := (x_0, y_0)$ as (x,y) varies over the set

$$S_f := \{(x,y) : \varphi(x,y) = 0\}.$$

Let us also assume that $f_x, f_y, \varphi_x, \varphi_y$ exist in a nbd of P_0, and there exists a function

$y = g(x)$ defined in a nbd J_0 of x_0 such that $\varphi(x, g(x)) = 0$ for all $x \in J_0$. Then $u = f(x, y)$ is a function of a single variable $x \in J_0$. Thus, u attains maximum at P_0 so that derivative of u w.r.t. x is 0 at x_0. Thus, we have the following necessary conditions:

$$\frac{du}{dx} := f_x + f_y y' = 0, \quad \varphi_x + \varphi_y y' = 0 \quad \text{at} \quad P_0.$$

Hence, we must have

$$(f_x + f_y y') + \lambda(\varphi_x + \varphi_y y') = 0 \quad \text{at} \quad P_0 \quad \forall \lambda \in \mathbb{R},$$

i.e,

$$(f_x + \lambda\varphi_x) + (f_y + \lambda\varphi_y)y' = 0 \quad \text{at} \quad u_0 \quad \forall \lambda \in \mathbb{R},$$

Choosing λ_0 such that $f_y + \lambda_0\varphi_y = 0$ at (x_0, y_0), we obtain

$$\varphi = 0, \quad f_x + \lambda\varphi_x = 0, \quad f_y + \lambda\varphi_y = 0 \quad \text{for} \quad (\lambda, x, y) = (\lambda_0, x_0, y_0). \qquad (*)$$

Note that, writing

$$F(x, y, \lambda) := f(x, y) + \lambda\varphi(x, y)$$

the conditions in $(*)$ is same as

$$\varphi = 0, \quad F_x = 0 = F_y \quad \text{at} \quad (\lambda_0, x_0, y_0).$$

The parameter λ above is called the **Lagrange multiplier**, and the method using Lagrange multiplier is the procedure of finding (λ, x, y) such that

$$\varphi = 0, \quad F_x = 0 = F_y$$

so that the required point at which f attains an extremum in S_f is one among these points.

EXAMPLE 1.98 Among all rectangles with a given perimeter ℓ, let us find the one having maximum area. Thus, the problem is to find the point (x_0, y_0) at which the function $f(x, y) := xy$ attains maximum subject to the constraint $\varphi(x, y) := 2(x + y) - \ell = 0$. We consider the equation

$$\varphi = 2(x + y) - \ell = 0, \quad f_x + \lambda\varphi_x := y + 2\lambda = 0, \quad f_y + \lambda\varphi_y = x + 2\lambda = 0.$$

Thus,

$$x = -2\lambda = y, \quad \ell = 2(x + y) = 4x$$

so that $x = y = \ell/4$. □

EXAMPLE 1.99 We show that among all rectangular parallelepiped inscribed in a given sphere, cube has the maximum volume. To see this, let x, y, z be the sides of the parallelepiped. Clearly, we must have $x^2 + y^2 + z^2 = d^2$, where d is the diameter of the sphere. So, we must find the maximum of the function $f(x, y, z) := xyz$

subject to the condition $\varphi(x,y,z) := x^2+y^2+z^2-d^2$. Hence, the equations to be solved are:

$$\begin{aligned}\varphi(x,y,z) := x^2+y^2+z^2-d^2 &= 0\\ f_x+\lambda\varphi_x := yz+\lambda(2x) &= 0\\ f_y+\lambda\varphi_y := xz+\lambda(2y) &= 0\\ f_x+\lambda\varphi_z := xy+\lambda(2z) &= 0.\end{aligned}$$

From the above equations, it follows that

$$x(f_x+\lambda\varphi_y)+y(f_y+\lambda\varphi_y)+z(f_z+\lambda\varphi_z)=0,$$

i.e.,

$$3xyz+2\lambda(x^2+y^2+z^2)=0.$$

Thus,

$$3xyz+2\lambda d^2=0.$$

Hence,

$$0=yz+2\lambda 2x=yz-\frac{3}{d^2}x^2yz=yz\Big(1-\frac{3}{d^2}x^2\Big).$$

Thus, $x=d/\sqrt{3}$. Similarly, $y=d/\sqrt{3}$ and $z=d/\sqrt{3}$. □

EXAMPLE 1.100 Let us find the parelleloeiped of maximum volume with a given surface area:

The function to be maximized is

$$f(x,y,z)=xyz$$

subject to the condition

$$2(xy+yz+zx)=A.$$

Thus,

$$\varphi(x,y,z)=2(xy+yz+zx)=A.$$

Note that

$$\begin{aligned}f_x+\lambda\varphi_x=0 &\iff yz+2\lambda(y+z)=0, &(i)\\ f_y+\lambda\varphi_y=0 &\iff zx+2\lambda(z+x)=0, &(ii)\\ f_z+\lambda\varphi_z=0 &\iff xy+2\lambda(x+y)=0. &(iii)\end{aligned}$$

Multiplying (i), (ii) and (iii) by x, y and z respectively and adding we get

$$3xyz+2\lambda A=0.$$

Thus, $\lambda=-3xyz/2A$. Hence, from (i),

$$yz-\frac{3}{A}xyz(y+z)=0,\quad \text{i.e.,}\quad 3x(y+z)=A,\quad \text{i.e.,}\quad 3(xy+xz)=A,$$

i.e.,

$$3(A-2yz)=2A,,\quad \text{i.e.,}\quad 6yz=A,,\quad \text{i.e.,}\quad yz=A/6.$$

Similarly, $zx=A/6$ and $xy=A/6$. Thus, so that $z=A/6y$ and hence, $x=A/6z=y$. Hence, $x^2=A/6$ and $x=\sqrt{A/6}$. Similarly, $y=\sqrt{A/6}$ and $z=\sqrt{A/6}$. □

EXAMPLE 1.101 Suppose a wire of length ℓ is cut into three pieces and are bent them to form a circle, a square, and an equilateral triangle. Let us find the length of these pieces so that that the total areas inscribed by these figures is minimum:

The function to be minimized is

$$f(x,y,z) := \pi x^2 + y^2 + \frac{\sqrt{3}}{4} z^2$$

subject to the constrained

$$\varphi(x,y,z) := 2\pi x + 4y + 3z - \ell = 0.$$

Now, we may apply Lagrange multiplier method. The equations to be solved are:

$$\begin{aligned} \varphi(x,y,z) := 2\pi x + 4y + 3z - \ell &= 0 \\ f_x + \lambda \varphi_x := 2\pi x + \lambda(2\pi) &= 0 \\ f_y + \lambda \varphi_y := 2y + \lambda(4) &= 0 \\ f_z + \lambda \varphi_z := \frac{\sqrt{3}}{2} z + \lambda(3) &= 0. \end{aligned}$$

Thus,

$$x = -\lambda, \quad y = -2\lambda, \quad z = -\lambda \frac{6}{\sqrt{3}} \tag{$*$}$$

and

$$\varphi(x,y,z) := 2\pi x + 4y + 3z - \ell = 2\pi(-\lambda) + 4(-2\lambda) + 3\Big(-\lambda \frac{6}{\sqrt{3}} \Big) - \ell = 0$$

Hence

$$\lambda \Big[-2\pi - 8 - 3\Big(\frac{6}{\sqrt{3}}\Big)\Big] = \ell. \tag{$**$}$$

From $(**)$, we get value of λ, and then obtain the values of x, y, z, from the previous three equations:

$$f(x,y,z) = \pi x^2 + y^2 + \frac{\sqrt{3}}{4} z^2 = \pi \lambda^2 + 4\lambda^2 + 3\sqrt{3}\lambda^2 = (\pi + 4 + 3\sqrt{3})\lambda^2.$$

□

Exercise 1.102 *Find the maximum of* $f(x,y,z) := x^2 y^2 z^2$ *subject to the constraint that* $x^2 + y^2 + y^2 = 1$.

Exercise 1.103 *Among all parallelepiped of a given volume* V, *find the one which has minimum surface area.*

Exercise 1.104 *Find points on the surface given by* $z^2 = xy + 4$ *subject closest to the origin.*

Exercise 1.105 *Show that, among all parallelepiped with the given surface area* A, *the cube has the maximum volume.*

Exercise 1.106 *Show that, of all triangles inscribed in a circle, the equilateral triangle has the greatest area.*

1.9 Problems

1. Give an example of a set which is neither open or closed.

2. Show that the limit of a convergent sequence is unique.

3. Let (X_n) be a sequence in $\mathbb{R}^2$ with $X_n := (x_n, y_n)$ for $n \in \mathbb{N}$ and $U := (x, y) \in \mathbb{R}^2$. Show that

   (i) $X_n \to X$ if and only if $x_n \to x$ and $y_n \to y$, and

   (ii) $X_n \to X$ if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ and $|y_n - y| < \varepsilon$ whenever $n \geq N$.

4. Let (X_n) be a sequence in $\mathbb{R}^2$. Show that (X_n) converges if and only if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|X_n - X_m| < \varepsilon$ whenever $n, m \geq N$.

5. Show that a point $X_0 \in \mathbb{R}^2$ is a limit point of $D \subseteq \mathbb{R}^2$ if and only if there exists a sequence (X_n) of distinct points in D which converges to X_0.

6. For the function $f(x, y) := xy\dfrac{x^2 - y^2}{x^2 + y^2}$ defined on $D := \{(x, y) : x^2 + y^2 \neq 0\}$, show that $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.

7. Show that for the function $z = f(x, y) := \dfrac{x^2 y}{x^4 + y^2}$ defined on $D := \{(x, y) : x^2 + y^2 \neq 0\}$, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

8. Let $f(x, y) := \dfrac{(y - x)(1 + x)}{(y + x)(1 + y)}$ defined on $D := \{(x, y) : x + y \neq 0\}$. Check whether $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y)$ and $\lim\limits_{x\to 0} \lim\limits_{y\to 0} f(x, y)$ exist. Are they same?

9. For $f(x, y) := x \sin \dfrac{1}{y} + y \sin \dfrac{1}{x}$ defined on $D := \{(x, y) : xy \neq 0\}$, show that $\lim_{(x,y)\to(0,0)} f(x, y) = 0$. Check whether $\lim\limits_{y\to 0} \lim\limits_{x\to 0} f(x, y)$ and $\lim_{x\to 0} \lim_{y\to 0} f(x, y)$ exist.

10. For the following functions, examine whether f_x, f_y, f_{xx}, f_{xy}, f_{yy} exist at $(0, 0)$, and if so find their values.

    (i) $g(x, y) := \begin{cases} f(x, y), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

    (ii) $z = f(x, y) := \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

11. Find the constants a, b so that the surface given by $ax^2 - byz = (a + z)x$ will be orthogonal to the surface given by $4x^2 y + z^3 = 4$ at the point $(1, -1, 2)$.

12. In the following cases, find the *directional derivative* of the function f at the point P_0 in the direction of $\vec{s}$:

    (i) $f(x, y, z) = x^2 + y^2 - z$; $P_0 = (1, 1, 2)$; $\vec{s} = 4\vec{i} + 4\vec{j} - 2\vec{k}$.

    (ii) $f(x, y, z) = x^2y - 3xyz + +z^3$; $P_0 = (3, 1, 2)$; $\vec{s} = 3\vec{i} - 2\vec{j} + 6\vec{k}$.

    (iii) $f(x, y, z) = x^2yz^2$; $P_0 = (1, 2, 3) \in \mathbb{R}^3$; $\vec{s} = -2\vec{i} + 3\vec{j} - 6\vec{k}$.

    Also, Calculate the maximum rate of change of f at P_0.

13. Find $\frac{\partial f}{\partial s}$ at $P_0 = (2, -1, 1)$ if $f(x, y, z) = xy^2 + yz^3$ and $\vec{s}$ is perpendicular to the surface given by $x\nu(z) - y^2 = -4$ at $(-1, 2, 1)$.

14. Find the values of the constants a, b, c so that the directional derivative of $f(x, y, z) := axy^2 + byz + cz^2x^3$ at $P_0 = (1, 2, -1)$ has a maximum magnitude of 64 in the direction parallel to z-axis.

15. Find the maximal directional derivative of x^3y^2z at $(1, -2, 3)$.

16. Find the direction in which the directional derivative of x^2yz^3 is maximum at a point $P_0 = (a, b, c)$.

17. If u is a function of x implicitly defined by $u = f(x, u)$, find $\frac{du}{dx}$.

    Hint: Express the given equation in the form $F(x, u) = 0$.

18. Suppose u is a function of (x, y) implicitly defined by $u = f(g(x, u), h(y, u))$, where f, g and h are known functions. Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

    Hint: Express the given equation in the form $F(x, y, u) = 0$.

19. If $u = \log(x + u)$, find $\frac{du}{dx}$.

20. If $\log uy + y \log u = x$, find $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial x}$ and $\frac{\partial^2 y}{\partial x^2}$.

21. If $\sin zy = \cos zx$, compute $\frac{\partial z}{\partial x}$ at $(x, y, z) = (\frac{1}{3}, \frac{1}{6}, \pi)$.

22. If $z(z^2 + 3x) + 3y = 0$, prove that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2z(x-1)}{(z^2+x)^3}.$$

23. If $u = f(x + u, yu)$, find $\dfrac{\partial y}{\partial x}, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial y}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial x}$.

24. Using Taylor's formula for functions of two variables, find an expression for

    (i) $f(x, y) := x^2 + xy - y^2$ in terms of powers of $x - 1$ and $y + 2$;

    (ii) $f(x, y) := (1 - 3x + 2y)^3$ in terms of powers of $x - 1$ and $y + 1$.

25. Find Taylor expansion for $x^3 - 2xy^2$ at the point $(x_0, y_0) = (1, -1)$.

26. Expand $x^2y + \sin y + e^x$ in powers of $(x - 1)$ and $(y - \pi)$ through quadratic terms and write the reminder.

27. Test the following functions for maximum, minimum and saddle points:

    (i) $x^4 + y^4 - x^2 - y^2 + 1$.

    (ii) $x^2 + 2y^2 + 3z^2 - 2xy - 2yz - 2$.

28. Find the extrema of

    (i) $(x^2 + y^2)e^{6x+2x^2}$.

    (ii) $\sin x + \sin y + \sin(x + y)$.

29. Find the shortest distance from the origin to the plane $x - 2y - 2z = 1$.

30. Find the minimum value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz = p$, where a, b, c, p are given numbers.

31. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Chapter 2

Multiple Integrals

In this chapter we consider integrals of functions of more than one variables.

2.1 Review on integral of functions of one variable

Let f be a real valued function defined on a *closed and bounded interval* $[a, b]$. Suppose $f(x) \geq 0$ for all $x \in [a, b]$. Corresponding to this f, we would like to associate a number γ, which should be the candidate for the *area* of the region bounded below the graph of f and above the x-axis. Of course, we want only one such γ. For this purpose we consider the following procedure, without restricting the function to be non-negative.

For a given $n \in \mathbb{N}$, consider $x_0, x_1, x_2, \ldots, x_n$ in $[a, b]$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Such a set $\mathcal{P} := \{x_i\}_{i=0}^n$ of points in $[a, b]$ is called a partition of $[a, b]$. Thus, given a partition $\mathcal{P}$ of $[a, b]$, we have a *decomposition* of $[a, b]$ into subintervals $I_i := [x_{i-1}, x_i]$, $i = 1, \ldots, n$. Take a set $\mathcal{T} := \{t_i\}_{i=1}^n$ of points, called *tag points*, in $[a, b]$ such that $t_i \in I_i$ for $i = 1, \ldots, n$, and consider the sum

$$S(f, \mathcal{P}) := \sum_{i=1}^{n} f(t_i)\ell(I_i),$$

where $\ell(I_i)$ is the length of the interval $[x_{i-1}, x_i]$, that is, $\ell(I_i) := x_i - x_{i-1}$ for $i = 1, \ldots, n$. If the sum $S(f, \mathcal{P})$ approaches a unique number γ as $\mu((\mathcal{P}) := \max_{1 \leq i \leq n} \ell(I_i)$ approaches 0, then we say that f is *Riemann integrable* and the number γ is called the *Riemann integral* of f, denotes by $\int_a^b f(x)dx$. More precisely, we have the following definition:

Definition 2.1 The function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** if there is a unique number γ such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(f, \mathcal{P}) - \gamma| < \varepsilon$$

for every partition $\mathcal{P}$ of $[a, b]$ with $\mu((\mathcal{P}) < \delta$ and for every set $\mathcal{T}$ of *tag points* associated with $\mathcal{T}$, and in that case γ is denoted by $\int_a^b f(x)dx$, called the **Riemann integral** of f over $[a, b]$. $\Diamond$

Note that if $f \geq 0$, then $f(t_i)\ell(I_i)$ is the area of the rectangle with height $f(t_i)$ and base length $\ell(I_i)$, so that $S(f, \mathcal{P})$ represents the total areas of such rectangles, which may be considered as an approximation of γ for small enough $\mu(\mathcal{P})$.

It can be shown that:

- If f is Riemann integrable, then for any sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ of partitions of $[a, b]$ such that $\mu(\mathcal{P}_k) \to 0$ as $k \to \infty$, then

$$S(f, \mathcal{P}_k) \to \int_a^b f(x)dx \quad \text{as} \quad k \to \infty.$$

It is known that:

- Every continuous function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

In fact:

- Every bounded function $f : [a, b] \to \mathbb{R}$ having only a finite number of points of discontinuities is Riemann integrable.

Here are two examples of functions which are not Riemann integrable.

(1) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{if}\, 0 < x \leq 1, \\ 1 & \text{if}\, x = 0. \end{cases}$$

It has discontinuity only at $x = 0$. If we take the partition points as $x_i = \frac{i}{n}$ for $i = 0, 1, \ldots, n$, and tag points as $t_i = i/n$, $i = 1, \ldots, n$, then we have

$$S(f, \mathcal{P}_n) = \sum_{i=1}^{n} f(t_i)\ell(I_i) = \sum_{i=1}^{n} \frac{n}{i}\Big(\frac{i}{n} - \frac{i-1}{n}\Big) = \sum_{i=1}^{n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + \cdots \frac{1}{n}.$$

Thus, $S(f, \mathcal{P}_n) \to \infty$ as $n \to \infty$. Hence, f is not Riemann integrable.

(2) Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if}\, x \in \mathbb{Q}, \\ 1 & \text{if}\, x \notin \mathbb{Q}. \end{cases}$$

This function is not continuous at any point. Let $\mathcal{P}$ be any partition $\mathcal{P}$ of $[0, 1]$. If we take tag points as $t_i \in \mathbb{Q}$, then the corresponding Riemann sum is 0, and if we take tag points $t_i \notin \mathbb{Q}$, then the corresponding Riemann sum is 1. Thus, by the definition, f is not Riemann integrable.

Note that, in (1) above, the function is not bounded, and in (2), the points at which the function is discontinuous is uncountable.

2.2 Double Integrals

In this section we consider integrals of functions of two variables.

First recall that an open subset D of $\mathbb{R}^2$ is a *domain* if any two points in it can be joined by a finite number of line segments, that is, if $u, v \in D$, then there exists $u_1, u_2, \ldots, u_k$ in D such that $\{(1-t)u_{i-1} + tu_i : 0 \le t \le 1\} \subseteq D$ for $i = 1, 2, \ldots, k$ with $u_0 = u$ and $u_k = v$.

A domain together with its boundary points is called a *closed domain*.

Let D be a closed domain in $\mathbb{R}^2$ and $f : D \to \mathbb{R}$ be a function. Suppose D is decomposed into a finite number of subregions $D_1, D_2, \ldots, D_n$ such that no two of them have common interior points. Such a decomposition $\mathcal{P} := \{D_i\}_{i=1}^n$ of D is called a **partition** of D.

Let $\mathcal{P} := \{D_i\}_{i=1}^n$ be a partition of D, $P_i \in D_i$ and ΔA_i be the area of D_i for $i = 1, 2, \ldots, n$. Consider the sum

$$S(f, \mathcal{P}) := \sum_{i=1}^{n} f(P_i)\Delta A_i.$$

The sum $S(f, \mathcal{P})$ is called a **Riemann sum** of f corresponding to the partition $\mathcal{P}$ and the *tag points* P_i for $i = 1, \ldots, n$.

If $f(x, y) \ge 0$ for $(x, y) \in D$, then each $f(P_i)\Delta A_i$ is the volume of the cylinder with base D_i, generators parallel to z-axis, and height $f(P_i)$, and $S(f, \mathcal{P})$ is the sum of all such volumes.

Let

$$\mu(\mathcal{P}) := \max_{1 \le i \le n} A_i.$$

The quantity $\mu(\mathcal{P})$ is called the **mesh** or **norm** of the partition $\mathcal{P}$.

Definition 2.2 We say that f is **integrable** over D if $S(f, \mathcal{P})$ approaches some number, say γ, as $\mu(\mathcal{P})$ approaches zero, irrespective of the manner in which the partition $\mathcal{P} := \{D_i\}_{i=1}^n$ and the set $\mathcal{P} = \{P_i\}_{i=1}^n$ are taken.

More precisely, f is **integrable** over D, if there exists $\gamma \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a $\delta > 0$ satisfying

$$|S(f, \mathcal{P}) - \gamma| < \varepsilon$$

for every partition $\mathcal{P}$ with $\mu(\mathcal{P}) < \delta$ and for every tag points P_i, $i = 1, \ldots, n$ associated with $\mathcal{P}$, and in that case, the quantity γ is called the **integral** of f over D, and we write γ as

$$\iint_D f(P)dA \quad \text{or} \quad \iint_D f(x, y)dxdy.$$

◊

Theorem 2.3 *If f is continuous, then the integral of f over D exists.*

In the following we shall take functions for which integrals over D exist.

Theorem 2.4 *For functions f and g and real number α, the following hold.*

$$\iint\limits_D [f(x,y)+g(x,y)]dxdy = \iint\limits_D f(x,y)dxdy + \iint\limits_D g(x,y)dxdy,$$

$$\iint\limits_D \alpha\, f(x,y)dxdy = \alpha \iint\limits_D f(x,y)dxdydy.$$

Theorem 2.5 *Suppose D is decomposed into two subregions D_1 and D_2 (having no common interior points). Then $\iint\limits_D f(x,y)dxdy$ exists $\iint\limits_{D_1} f(x,y)dxdydy$ and $\iint\limits_{D_2} f(x,y)dxdydy$ exist, and in that case*

$$\iint\limits_D f(x,y)dxdy = \iint\limits_{D_1} f(x,y)dxdydy + \iint\limits_{D_2} f(x,y)dxdydy.$$

2.2.1 Calculating double integrals

Suppose D is a rectangular region:

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

Suppose f is a continuous function defined on R. Then it is known[1] that

$$\boxed{\iint\limits_D f(x,y)dxdy = \int_a^b \Big(\int_c^d f(x,y)\,dy\Big)dx = \int_c^d \Big(\int_a^b f(x,y)\,dx\Big)dy.}$$

Suppose the region D is of the form

$$D = \{(x,y) \in \mathbb{R}^2 : \varphi_1(x) \le y \le \varphi_2(x);\ a \le x \le b\}, \tag{2.1}$$

where φ_1 and φ_2 are continuous functions on $[a,b]$ such that $\varphi_1(x) \le \varphi_2(x)$ for $a < x < b$. Then we would like to see if we have

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy\right] dx. \tag{2.2}$$

[1]This result is called *Fubini's theorem.*

A domain D of the above form is called **regular in y-direction**. Similarly, domain D is said to be **regular in x-direction** if there exist continuous functions ψ_1 and ψ_2 on $[c, d]$ such that $\varphi_1(y) \le \varphi_2(y)$ for $c < y < d$ and

$$D = \{(x, y) \in \mathbb{R}^2 : \psi_1(y) \le x \le \psi_2(y);\ c \le y \le d\}. \tag{2.3}$$

In this case, we would like to see if

$$\iint\limits_D f(x, y)dxdy = \int_c^d \left[\int_{\psi_1(y)}^{\varphi_1(y)} f(x, y)dx \right] dy. \tag{2.4}$$

A domain which regular in both y and x direction is called a **regular domain**.

The integrals on the right hand sides of (i) and (ii) are called *two-fold iterated iterated integrals.*

If D is regular in y-direction and given as in (2.1), then we say that D is bounded by the curves

$$x = a, \quad x = b, \quad y = \varphi_1(x), \quad y = \varphi_2(x); \quad a < x < b.$$

Similarly, if D is regular in x-direction and given as in (2.3), then we say that D is bounded by the curves

$$y = c, \quad y = d, \quad x = \psi_(y), \quad x = \psi_2(y); \quad c < y < d.$$

EXAMPLE 2.6 let us calculate the two-fold iterated integrals in (i) and (ii) for the function $f(x, y) = x^2 + y^2$ if the domain D is the region bounded by the curves $y = x^2$, $x = 1$ and $y = 0$:

$$\int_0^1 \left(\int_0^{x^2} (x^2 + y^2)dy \right) dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_0^{x^2} dx = \int_0^1 \left(x^4 + \frac{x^6}{3} \right) dx = \frac{26}{105}.$$

$$\begin{aligned}
\int_0^1 \left(\int_0^{\sqrt{y}} (x^2 + y^2)dx \right) dy &= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_{\sqrt{y}}^1 dy \\
&= \int_0^1 \left[\left(\frac{1}{3} + y^2 \right) - \left(\frac{y^{3/2}}{3} + y^{5/2} \right) \right] dy \\
&= \left[\left(\frac{y}{3} + \frac{y^3}{3} \right) - \left(\frac{y^{3/2}}{3} + y^{5/2} \right) \right]_0^1 dy \\
&= \left[\frac{2}{3} - \frac{44}{105} \right] = \frac{26}{105}.
\end{aligned}$$

□

Suppose D is regular in y-direction. Let us denote the integral in (2.2) by I_D^y, that is,

$$I_D^y :== \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy\right] dx.$$

The following theorem can be proved using the following facts:

If D is regular in y-direction given as in (2.2) then

$$Area(D) = \int_a^b [\varphi_2(x) - \varphi_1(x)]dx.$$

Similarly, if If D is regular in x-direction given as in (2.4), then

$$Area(D) = \int_c^d [\psi_2(y) - \psi_1(y)]dy.$$

Theorem 2.7 *Suppose D is regular in y-direction given as in (2.2), and $S = Area(D)$. Let m, M be such that*

$$m \leq f(x,y) \leq M \quad \forall x \in D.$$

Then

$$mS \leq \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy\right] dx \leq MS.$$

By Theorem 2.7 and using the *intermediate value property* of continuous functions:

If f is continuous on a closed and bounded subset of $\mathbb{R}^2$ and c_1 and c_2 are values of f, i.e., there exists P_1 and P_2 in D such that $f(P_1) = c_1$ and $f(P_2) = c_2$, and if $c_1 < c_2$ and c such that $c_1 < c_0 < c_2$, then there exists P_0 in D such tha $f(P_0) = c_0$.

we obtain the following.

Theorem 2.8 (Mean-value theorem) *Suppose D is regular in y-direction given as in (2.2), and $S = Area(D)$. If f is continuous, then there exists $P \in D$ such that*

$$\frac{1}{S}\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy\right] dx = f(P).$$

Similarly, if D is regular in x-direction given as in (2.4) and f is continuous, then there exists $Q \in D$ such that

$$\frac{1}{S}\int_a^b \left[\int_{\psi_1(y)}^{\psi_1(y)} f(x,y)dx\right] dy = f(Q).$$

Suppose D is decomposed into two subregions D_1 and D_2 by using a straight line parallel to either x-axis or y-axis. Then it can be seen that

$$I_D^y = I_{D_1}^y + I_{D_2}^y$$

by using appropriate curves bounding D_1 and D_2.

Similarly, if D is regular in x-direction, then we can define I_D^x and show that

$$I_D^x = I_{D_1}^x + I_{D_2}^x.$$

More generally, we have following:

Theorem 2.9 *If D is regular in y-direction and if it is decomposed into two subregions $D_1, D_2, \ldots, D_k$ using a straight line parallel to either x-axis or y-axis, then*

$$I_D^y = I_{D_1}^y + I_{D_2}^y + \ldots + I_{D_k}^y.$$

Similarly, if D is regular in x-direction and if it is decomposed into two subregions $D_1, D_2, \ldots, D_k$ using a straight line parallel to either x-axis or y-axis, then

$$I_D^x = I_{D_1}^x + I_{D_2}^x + \ldots + I_{D_k}^x.$$

Now, Theorem 2.9 and Theorem 2.8 give the following:

Theorem 2.10 *Suppose f is a continuous function defined on D given as in (2.2). If D is regular in y-direction given as in (2.2), then*

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy\right] dx.$$

Similarly, if D is regular in y-direction given as in (2.4), then

$$\iint\limits_D f(x,y)dxdy = \int_a^b \left[\int_{\psi_1(y)}^{\psi_1(y)} f(x,y)dx\right] dy.$$

EXAMPLE 2.11 Let us calculate $\iint\limits_D xy\,dxdy$, where D is a region bounded by the curves $x=1$, $x=2$, $y=x$, $y=\sqrt{3}x$. In this case,

$$\iint\limits_D xy\,dxdy = \int_1^2 x\left(\int_x^{\sqrt{3}x} y\,dy\right) dx = \int_1^2 x^3 dx = \frac{15}{4}.$$

□

Remark 2.12 If $f \geq 0$ on D, then we can have geometric meaning of the the above theorem by observing that for each x, the integral

$$\int_{\varphi_1(x)}^{\varphi_1(x)} f(x,y)dy$$

is the area of a section of the solid bounded by

$$z = 0, \quad z = f(x,y), \quad y = \varphi_1(x), \quad y = \varphi_2(x).$$

♦

In view of the remarks in the beginning of this section, if f is a non-negative function, then the integral of f over D can be thought of as the **volume** of the solid bounded by the cylindrical surface with generators parallel to z-axis, bases D and bounded from the top by the surface $z = f(x,y)$, $(x,y) \in D$.

Let D be a closed domain in $\mathbb{R}^2$, and a solid region W in $\mathbb{R}^3$ is the one common to the region below the graph of $z = f(x,y)$ and above the graph of $z = g(x,y)$ where f and g are continuous functions defined on D, i.e.,

$$W = \{(x,y,z) : g(x,y) \leq z \leq f(x,y); (x,y) \in D\}.$$

Then volume of W is given by

$$vol(W) = \iint_D [f(x,y) - g(x,y)]dxdy.$$

2.2.2 Calculating double integrals using polar coordinates

Suppose a region D in $\mathbb{R}^2$ is described by polar coordinates as

$$D = \{(\theta,\rho) \in \mathbb{R}^2 : \Phi_1(\theta) \leq \rho \leq \Phi_2(\theta);\ \alpha \leq \theta \leq \beta\},$$

where Φ_1 and Φ_2 are continuous functions on $[\alpha,\beta]$ such that $\Phi_1(\theta) \leq \Phi_2(\theta)$ for $\alpha < \theta < \beta$. Such a region is called **regular with respect to rays** or simply a **regular domain in polar coordinates**. Note that if D is regular, then each ray from the origin passing through an interior point of D will cut the boundary of D exactly at two different points.

Let D be a domain regular in polar coordinates, and let $F(\theta,\rho)$ be a continuous function defined on D. We would like to express the double integral

$$\iint_D F(\theta,\rho)d\theta\, d\rho$$

as iterated integrals with respect to θ and ρ.

Recall that $\iint\limits_D F(\theta,\rho)d\theta\, d\rho$ is a number to which a Riemann sum

$$S(F,\mathcal{P}) := \sum_{i=1}^{n} F(P_i)\Delta s_i$$

corresponding to partitions $\mathcal{P}$ approach as the mesh $\mu(\mathcal{P})$ approach 0 irrespective of the manner in which the domain is partitioned and the points $P_1,\ldots,P_n$ are taken. Now, consider the family of partitions $\mathcal{P}_{m,n} := \{D_{ij} : i = 1,\ldots,n; j = 1,\ldots,m\}$ of D by using the concentric circles $\rho = \rho_i$, $i = 1,\ldots,n$ and rays $\theta = \theta_j$, $j = 1,\ldots,m$ so that $\mu(\mathcal{P}_{m,n}) \to 0$ as $m,n \to \infty$. Then $S(F,\mathcal{P}_{m,n})$ is of the form

$$S(F,\mathcal{P}_{m,n}) = \sum_{j=1}^{m}\sum_{i=1}^{n} F(P_{ij})\Delta s_{ij}.$$

Let i,j be such that the domain $D_{ij} \subseteq D$. Then

$$\begin{aligned}\Delta s_{ij} &= \frac{1}{2}\rho_i^2\Delta\theta_j - \frac{1}{2}\rho_{i-1}^2\Delta\theta_j \\ &= \frac{(\rho_i+\rho_{i-1})}{2}\Delta\rho_i\Delta\theta_j \\ &= \rho_i^*\Delta\rho_i\Delta\theta_j.\end{aligned}$$

Let $P_{ij} = (\theta_j^*, \rho_i^*)$ where $\theta_{j-1} \le \theta_j^* \le \theta_j$. Then we see that

$$S(F,\mathcal{P}_{m,n}) := \sum_{j=1}^{m}\sum_{i=1}^{n} F(P_{ij})\Delta s_{ij} = \sum_{j=1}^{m}\sum_{i=1}^{n} F(\theta_j^*,\rho_i^*)\rho_i^*\Delta\rho_i\Delta\theta_j - \varepsilon_{m,n},$$

where $\varepsilon_{m,n} \to 0$ as $m,n \to \infty$. Note that, for each j,

$$\sum_{i=1}^{n} F(\theta_j^*,\rho_i^*)\Delta\rho_i \to \int_{\Phi_1(\theta_j^*)}^{\Phi_2(\theta_j^*)} F(\theta_j^*,\rho)\rho\, d\rho \quad \text{as} \quad n \to \infty.$$

Writing

$$g(\theta) := \int_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta,\rho)\rho\, d\rho$$

and using the fact that

$$\sum_{i=1}^{m} g(\theta_j^*)\Delta\theta_j \to \int_\alpha^\beta g(\theta)\, d\theta \quad \text{as} \quad m \to \infty,$$

it follows that

$$S(F,\mathcal{P}_{m,n}) := \sum_{j=1}^{m}\sum_{i=1}^{n} F(P_{ij})\Delta s_{ij} \to \int_\alpha^\beta g(\theta)\, d\theta \quad \text{as} \quad m \to \infty,$$

i.e.,

$$S(F, \mathcal{P}_{m,n}) \to \int_\alpha^\beta \left(\int_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta,\rho)\rho\, d\rho \right) d\theta \quad \text{as} \quad m \to \infty.$$

Thus,

$$\iint\limits_D F(\theta,\rho) d\theta\, d\rho = \int_\alpha^\beta \left(\int_{\Phi_1(\theta)}^{\Phi_2(\theta)} F(\theta,\rho)\rho\, d\rho \right) d\theta.$$

EXAMPLE 2.13 Let us find the volume V of the solid bounded by the spherical surface $x^2+y^2+z^2=4a^2$ and the cylindrical surface $x^2+y^2-2ay=0$.

Clearly, $V = 4V_0$, where V_0 is the volume of the part in the first octant. Note that the function to be integrated is

$$f(x,y) = \sqrt{4a^2 - x^2 - y^2}$$

and the domain D_0 over which integral is to be taken is bounded by the curves: $y = 0$, $y = 2a$; $x = \sqrt{2ay - y^2}$. Thus, in cartesian coordinates,

$$V_0 = \int\limits_{D_0}\int f(x,y)\, dx\, dy = \int_0^{2a} \left(\int_0^{\sqrt{2ay-y^2}} \sqrt{4a^2 - x^2 - y^2}\, dx \right) dy.$$

This integral is too complicated to evaluate as it is. Now let us compute it using polar coordinates, $x = \rho\cos\theta$ and $y = \rho\sin\theta$. Note that

$$x^2+y^2-2ay=0 \iff \rho^2 - 2a\rho\sin\theta = 0 \iff \rho = 2a\sin\theta,$$

$$F(\theta,\rho) := f(\rho\cos\theta, \rho\sin\theta) = \sqrt{4a^2-\rho^2}$$

with

$$D : 0 \le \theta \le \pi/2, \quad 0 \le \rho \le 2a\sin\theta.$$

Then we have

$$\begin{aligned} V_0 &= \int_0^{\pi/2} \left(\int_0^{2a\sin\theta} (\sqrt{4a^2-\rho^2})\rho\, d\rho \right) d\theta \\ &= \int_0^{\pi/2} \left[-\frac{(4a^2-\rho^2)^{3/2}}{3} \right]_0^{2a\sin\theta} d\theta \\ &= \frac{8a^3}{3} \int_0^{\pi/2} (1-\cos^3\theta)\, d\theta \\ &= \frac{4}{9}a^3(3\pi - 4). \end{aligned}$$

□

For the next example, let us recall the definition of *improper integrals* over unbounded intervals;

Definition 2.14 Let $a, b \in \mathbb{R}$.

1. For $a \in \mathbb{R}$, let $f : [a, \infty) \to \mathbb{R}$ be continuous and $\lim_{b\to\infty} \int_a^b f(x)dx$ exists. Then $\int_a^\infty f(x)dx$ is defined by

$$\boxed{\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.}$$

2. For $b \in \mathbb{R}$, let $f : (-\infty, b] \to \mathbb{R}$ be continuous and $\lim_{a\to-\infty} \int_a^b f(x)dx$ exists. Then $\int_{-\infty}^b f(x)dx$ is defined by

$$\boxed{\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.}$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $\int_{-\infty}^c f(x)dx$ and $\int_c^\infty f(x)dx$ are well-defined for some $c \in \mathbb{R}$, then $\int_{-\infty}^\infty f(x)dx$ is defined by

$$\boxed{\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx.}$$

$\Diamond$

It can be shown that

- If J is any of the intervals $[a, \infty)$, $(-\infty, b]$ for some $a, b \in \mathbb{R}$ or $(-\infty, \infty)$, and if $f : J \to \mathbb{R}$ is continuous and $\int_J |f(x)|dx$ is well defined, then $\int_J f(x)dx$ is well defined and

$$\boxed{\left|\int_J f(x)dx\right| \le \int_J |f(x)|dx.}$$

EXAMPLE 2.15 Let us evaluate the Poisson integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

Recall that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \lim_{a\to\infty}\int_{-a}^{a} e^{-x^2}\,dx.$$

Note that

$$\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dxdy = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2. \tag{1}$$

Thus, we can conclude that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \lim_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2 = \lim_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dxdy.$$

Thus, it is enough to calculate $\lim\limits_{a\to\infty}\int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dxdy$. Note that

$$\iint\limits_{D_r} e^{-(x^2+y^2)}\,dxdy \le \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy \le \iint\limits_{D_R} e^{-(x^2+y^2)}\,dxdy, \tag{2}$$

where D_r and D_R are circular regions with origin as center and r and R as radii with $r < a$, $R \ge 2\sqrt{a}$. Now, using polar coordinates,

$$\iint\limits_{D_r} e^{-(x^2+y^2)}\,dxdy = \int_0^{2\pi}\left(\int_0^{r} e^{-\rho^2}\rho\,d\rho\right)d\theta = \pi[1-e^{-r^2}]. \tag{3}$$

Similarly,

$$\iint\limits_{D_R} e^{-(x^2+y^2)}\,dxdy = \int_0^{2\pi}\left(\int_0^{r} e^{-\rho^2}\rho\,d\rho\right)d\theta = \pi[1-e^{-R^2}]. \tag{4}$$

Thus, from (1)-(4),

$$\pi[1-e^{-r^2}] \le \left(\int_{-a}^{a} e^{-x^2}\,dx\right)^2 \le \pi[1-e^{-R^2}].$$

Letting $r\to\infty$, $R\to\infty$, we have

$$\pi \le \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 \le \pi.$$

Thus, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$. □

2.2.3 Multiple integrals using change of variables

In the last section, we considered the polar coordinates

$$x = \rho\cos\theta, \quad y = \rho\sin\theta$$

for evaluating double integrals. Now, we consider a general change of variables of the form

$$x = \varphi(u, v), \quad y = \psi(u, v)$$

where (u, v) varies over a domain $\widetilde{D}$ in such a way that the map

$$(u, v) \mapsto (x, y)$$

is a one-one correspondence between $\widetilde{D}$ and D. Moreover, we assume that the functions φ and ψ defined on $\widetilde{D}$ has continuous partial derivatives

$$\frac{\partial\varphi}{\partial u}, \quad \frac{\partial\varphi}{\partial v}, \quad \frac{\partial\psi}{\partial u}, \quad \frac{\partial\psi}{\partial v}$$

on $\widetilde{D}$. Suppose f is a continuous real valued function defined on D. Then we obtain the function

$$\tilde{f}(u, v) := f(\varphi(u, v), \psi(u, v)$$

defined on $\widetilde{D}$.

Suppose the the domain $\widetilde{D}$ is divided into sub-domains using straight lines parallel to coordinate axes. Now, lt us take a rectangle lying in $\widetilde{D}$ with its vertices given by

$$A_1 = (u_0, v_0), \quad A_2 = (u_0+\Delta u, v_0), \quad A_3 = (u_0+\Delta u, v_0+\Delta v), \quad A_4 = (u_0+\Delta u, v_0+\Delta v).$$

Let the corresponding images under the transformation

$$(u, v) \mapsto (x, u) = (\varphi(u, v), \psi(u, v))$$

be B_1, B_2, B_3, B_4 respectively. Let $B_i = (x_i, y_i)$ for $i = 1, 2, 3, 4$. Then we have

$$\begin{aligned} x_1 &= \varphi(u_0, v_0), \\ x_2 &= \varphi(u_0 + \Delta u, v_0) \approx \varphi(u_0, v_0) + \varphi_u\Delta u, \\ x_3 &= \varphi(u_0 + \Delta u, v_0 + \Delta v) \approx \varphi(u_0, v_0) + \varphi_u\Delta u + \varphi_v\Delta v, \end{aligned}$$

and

$$\begin{aligned} y_1 &= \psi(u_0, v_0), \\ y_2 &= \psi(u_0 + \Delta u, v_0) \approx \psi(u_0, v_0) + \psi_u\Delta u, \\ y_3 &= \psi(u_0 + \Delta u, v_0 + \Delta v) \approx \psi(u_0, v_0) + \psi_u\Delta u + \psi_v\Delta u. \end{aligned}$$

Now,

$$Area(\Delta B_1B_2B_3) \approx \frac{1}{2}|(x_3 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_3 - y_1)|.$$

Thus, we see that

$$\begin{aligned} Area(B_1B_2B_3B_4) &\approx |(x_3 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_3 - y_1)| \\ &= |J(\varphi, \psi)|\Delta u \Delta v, \end{aligned}$$

where $J(\varphi, \psi)$ is the determinant of the **Jacobian matrix**

$$\begin{bmatrix} \dfrac{\partial \varphi}{\partial u} & \dfrac{\partial \varphi}{\partial v} \\ \\ \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \psi}{\partial v} \end{bmatrix}$$

Thus, it can be seen that

$$\iint\limits_D f(x,y)\, dx\, dy = \int\limits_{\widetilde{D}}\int \tilde{f}(u,v)|J(\varphi,\psi)|\, du\, dv,$$

Note that, in the case of polar coordinates,

$$x = \varphi(\theta, \rho) = \rho\cos\theta, \quad y = \psi(\theta,\rho) = \rho\sin\theta$$

so that

$$J(\varphi,\psi) = \det \begin{bmatrix} \dfrac{\partial \varphi}{\partial u} & \dfrac{\partial \varphi}{\partial v} \\ \\ \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \psi}{\partial v} \end{bmatrix} = \det \begin{bmatrix} -\rho\sin\theta & \cos\theta \\ \\ \rho\cos\theta & \sin\theta \end{bmatrix} = -\rho.$$

Hence, in this case,

$$\iint\limits_D f(x,y)\, dx\, dy = \int\limits_{\widetilde{D}}\int \tilde{f}(\theta,\rho)\rho\, d\theta\, d\rho,$$

EXAMPLE 2.16 Let us evaluate the double integral $\iint\limits_D (y-x)dxdy$ where D is the region bounded by the straight lines

$$y = x+1, \quad y = x-3, \quad y = -\frac{1}{3}x + \frac{7}{3}, \quad y = -\frac{1}{3}x + 5$$

by appropriate change of variables.

Note that

$$y - x = 1, \quad y - x = -3, \quad y + \frac{1}{3}x = \frac{7}{3}, \quad y + \frac{1}{3}x = -5.$$

The above representation prompts us to take the variables:

$$u = y - x = 1 \quad \text{and} \quad v = y + \frac{1}{3}x$$

so that it can be seen that the domain in the uv-plane is bounded by the straight lines

$$u = 1, \quad u = -3, \quad v = \frac{7}{3}, \quad v = -5.$$

The change of variables formulas are given by

$$x = \varphi(u, v) := -\frac{3}{4}u + \frac{3}{4}v, \quad y = \psi(u, v) := \frac{1}{4}u + \frac{3}{4}v.$$

Hence,

$$J(\varphi, \psi) = \det \begin{bmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{bmatrix} = \det \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} \\ \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} = -\frac{3}{4}.$$

Thus,

$$\iint_D (y - x) dxdy = \iint_{\tilde{D}} \tilde{f}(u, v)|J(\varphi, \psi)|dudv = \int_{7/3}^{5} \left(\int_{-3}^{1} \frac{3}{4} udu \right) dv = -8.$$

□

EXAMPLE 2.17 Let us write the integral $\iint_D f(x, y)dxdy$ in terms of the variables u, v using the change of variables $x = u - uv$, $y = uv$, where D is bounded by the curves $y = \alpha x$, $y = \beta x$ and $x = c$ with $0 < \alpha < \beta$ and $c > 0$.

It is seen that the boundary of the region determined by the lines $y = \alpha x$, $y = \beta x$, $x = c$ correspond to the lines

$$v = \frac{\alpha}{1 + \alpha}, \quad v = \frac{\beta}{1 + \beta}, \quad v = 1 - \frac{c}{u},$$

together with $u = 0$ in the uv-plane, and the Jacobian of the transformation is

$$det \begin{bmatrix} 1 - v & -u \\ v & u \end{bmatrix}$$

Thus,

$$\iint_D f(x, y)dxdy = \int_{\frac{\alpha}{1+\alpha}}^{\frac{\beta}{1+\beta}} \int_0^{1-\frac{c}{u}} \tilde{f}(u, v)\, dvdu.$$

□

EXAMPLE 2.18 Let us evaluate $\iint\limits_D \cos\left(\frac{x-y}{x+y}\right) dxdy$ using the change of variables $u = x - y$, $v = x + y$, where D is the region bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.

Note that the transformation $(u, v) \mapsto (x, y)$ is given by

$$x = \frac{1}{2}(u + v), \quad x = \frac{1}{2}(v - u),$$

and the Jacobian of the transformation is $1/2$. Also, the lines $x = 0$, $y = 0$, $x+y = 1$ correspond to the lines $u = -v$, $u = v$, $v = 1$ respectively. Hence,

$$\iint\limits_D \cos\left(\frac{x-y}{x+y}\right) dxdy = \frac{1}{2}\iint\limits_{\tilde{D}} \cos\left(\frac{u}{v}\right) dudv = \frac{1}{2}\int_0^1 \left(\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\right) dv = ...\frac{\sin 1}{2}.$$

□

2.2.4 Surfaces and their representations

A surface in the space $\mathbb{R}^3$ can be given in various forms. Most common among them are as follows:

(i) A set of the form

$$S := \{(x, y, z) \in \mathbf{D} : \varphi(x, y, z) = 0\},$$

where $\mathbf{D}$ is a domain in $\mathbb{R}^3$ and $F : \mathbf{D} \to \mathbb{R}$ is a continuous function.

(ii) A set of the form

$$S := \{\Phi(u, v) : (u, v) \in D\},$$

where D is a region[2] in the plane $\mathbb{R}^2$ and $\Phi : D \to \mathbb{R}^3$ is a continuous function. Functions of the form Φ are gives as

$$\Phi(u, v) := (x(u, v), x(u, v), x(u, v)), \quad (u, v) \in D.$$

We shall also consider surfaces which are finite combination of surfaces of the forms given in (i) and/or (ii).

In case (i), the surface is a **level surface**, and in case (ii), the surface S is said to be **parametrized** by Φ.

As a special case of the above two representations, a surface S may be given by the set of all points (x, y, z) satisfying the equation

$$z = g(x, y), \quad (x, y) \in D,$$

for some continuous function $f : D \to \mathbb{R}$. Clearly, by taking

[2] A region consists of a connected open set together with some or all of its boundary points.

(a) $\varphi(x, y, z) := g(x, y) - z$ and

(b) $\Phi(x, y) := (x, y, g(x, y))$

we see that this surface has the forms in (i) and (ii) respectively.

Suppose a surface S is given as in (i), and at each $P \in S$, the gradient $\nabla\varphi$ exists and nonzero. Then $\nabla\varphi$ is a normal to the surface S, in the sense that the vector $\vec{n}(P)$ is perpendicular to the tangents to all smooth curves lying in S and passing through P. Indeed, if a curve lying on S is given by

$$\gamma(t) := (x(t), y(t), z(t)), \quad a \leq t \leq b,$$

and $P_0 = (x(t_0), y(t_0), z(t_0)$ then

$$F(x(t), y(t), z(t)) = 0 \quad \forall\, t \in [a, b],$$

so that

$$\varphi_x x'(t) + \varphi_y y'(t) + \varphi_z z'(t) = 0 \quad \text{at} \quad t = t_0.$$

Thus,

$$\nabla\varphi.\mathbf{v}_\gamma = 0,$$

where $\mathbf{v}_\gamma := (x'(t_0), y'(t_0), z'(t_0))$, the tangent vector to the curve at P_0. In such case we say that S is a **smooth surface**. The vector

$$\vec{n}(P) := \frac{\nabla\varphi}{|\nabla\varphi|}(P)$$

is a unit normal to S. Clearly, -$\vec{n}(P)$ is also a unit normal to S. The above vector $\vec{n}(P)$ is called the **canonical unit normal** to S. The plane pasing through the point P with $\vec{n}(P)$ as normal is called the **tangent plane** to S at P.

Suppose a surface S is represented **parametrically** by a function $\Phi : D \to \mathbb{R}^3$. Then S is a set of all points of the form $\Phi(u, v) := (x(u, v), y(u, v), z(u, v))$ with (u, v) varying in D. Suppose that

$$\Phi_u := x_u(u, v)\vec{i} + y_u(u, v)\vec{j} + z_u(u, v)\vec{k},$$
$$\Phi_v := x_v(u, v)\vec{i} + y_v(u, v)\vec{j} + z_v(u, v)\vec{k}$$

exist and are continuous. Further, assume that the vectors Φ_u and Φ_v are linearly independent; equivalently, $\Phi_u \times \Phi_v \neq \vec{0}$. Then, at each point $P \in S$, the vector

$$\vec{n}(P) = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|}(P)$$

is a unit normal to the surface S at P. We observe that

$$\begin{aligned}
|\Phi_u \times \Phi_v|^2 &= |\Phi_u|^2|\Phi_v|^2 \sin^2\theta \\
&= |\Phi_u|^2|\Phi_v|^2(1 - \cos^2\theta) \\
&= |\Phi_u|^2|\Phi_v|^2(1 - \frac{(\Phi_u.\Phi_v)^2}{|\Phi_u|^2|\Phi_v|^2} \\
&= |\Phi_u|^2|\Phi_v|^2 - (\Phi_u.\Phi_v)^2 \\
&= EG - F^2,
\end{aligned}$$

where

$$E := |\Phi_u|^2, \quad G := |\Phi_v|^2, \quad F := \Phi_u . \Phi_v. \tag{2.5}$$

If a surface S is given by an equation $z = g(x, y)$ such that partial derivatives g_x and g_y exist and are continuous on S, then

$$\frac{\nabla\varphi}{|\nabla\varphi|} = \frac{\Phi_u \times \Phi_v}{|\Phi_u \times \Phi_v|} == \frac{g_x\vec{i} + g_y\vec{j} - \vec{k}}{\sqrt{1 + g_x^2 + g_y^2}},$$

where $\varphi(x, y, z) := g(x, y) - z$ and $\Phi(x, y) := (x, y, g(x, y))$.

EXAMPLE 2.19 Consider the surface S given by the equation: $x^2 + y^2 + z^2 = R^2$. We may take $\varphi(x, y, z) = x^2 + y^2 + z^2 - R^2$. Then we have

$$\nabla\varphi = \varphi_x\vec{i} + \varphi_y\vec{j} + \varphi_z\vec{k} = 2(x\vec{i} + y\vec{j} + z\vec{k}).$$

Hence

$$\vec{n}(\vec{P}) = \frac{\nabla\varphi}{|\nabla\varphi|}(\vec{P}) = \frac{1}{R}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\vec{P}}{R}.$$

□

EXAMPLE 2.20 Consider the representation

$$z = \frac{h}{R}\sqrt{x^2 + y^2}$$

of a right circular cone of hight h and base radius R with vertex at the origine and z-axis as the axis. In this case, we can take $\varphi(x, y, z) = h\sqrt{x^2 + y^2} - Rz$. Then we have

$$\nabla\varphi = \varphi_x\vec{i} + \varphi_y\vec{j} + \varphi_z\vec{k} = h\frac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}} - R\vec{k} = \frac{h^2}{R}\left(\frac{x\vec{i} + y\vec{j}}{z}\right) - R\vec{k},$$

$$|\nabla\varphi|^2 = \varphi_x^2 + \varphi_y^2 + \varphi_z^2 = h^2 + R^2$$

so that

$$\vec{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{1}{\sqrt{h^2 + R^2}}\left[\frac{h^2}{R}\left(\frac{x\vec{i} + y\vec{j}}{z}\right) - R\vec{k}\right].$$

□

EXAMPLE 2.21 Consider the surface $S : x^2 + y^2 + z^2 = R^2$. This surface can also be represented by

$$S : \Phi(\theta, \phi) := (R\sin\phi\cos\theta,\ R\sin\phi\sin\theta,\ R\cos\phi)$$

with $0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$. Then

$$\Phi_\theta := -R\sin\phi\sin\theta\vec{i} + R\sin\phi\cos\theta\vec{j} + 0\vec{k},$$

$$\Phi_\phi := R\cos\phi\cos\theta\vec{i} + R\cos\phi\sin\theta\vec{j} - R\sin\phi\vec{k}.$$

Hence,

$$\begin{aligned}\Phi_\theta \times \Phi_\phi &= -R\sin\phi[R\sin\phi\cos\theta\vec{i} + R\sin\phi\sin\theta\vec{j} + R\cos\phi\vec{k}] \\ &= -R\sin\phi[x\vec{i} + \vec{j} + \vec{k}],\end{aligned}$$

$$|\Phi_\theta \times \Phi_\phi| = R^2|\sin\phi|,$$

so that for a point $\vec{P} \in S$,

$$\frac{\Phi_\theta \times \Phi_\phi}{|\Phi_\theta \times \Phi_\phi|} = \begin{cases} \vec{P}/R, & \sin\phi > 0, \\ -\vec{P}/R, & \sin\phi < 0 \end{cases}$$

□

Definition 2.22 A surface S is said to be **orientable** if at each point $P \in S$, a normal vector $\vec{n}(P)$ can be specified in such a way that it can be continued to the entire surface in a unique and continuous manner. ◊

For an orientable surface, if the chosen unit normal varies along a closed curve, then its direction will not be reversed as it comes back. **Moebius strip** is an example of a non-orientable surface.

2.2.5 Area of a surface using double integrals

Suppose a surface S is given by the equation

$$z = g(x, y), \quad (x, y) \in D.$$

Let $\Pi = \{D_i\}_{i=1}^n$ be a partition of D, and let $P_i := (x_i, y_i) \in D_i$ for $i = 1, \ldots, n$.

$$M_i := (x_i, y_i, z_i) \in D_i; \quad z_i := f(x_i, y_i), \quad i = 1, \ldots, n.$$

Let S_i be that part of the tangent plane at M_i intersected by the cylinder with base D_i with generators parallel to z-axis. Let $\Delta\sigma_i$ for $i = 1, \ldots, n$. Then the area of S is defined as the limit of the sum

$$\sum_{i=1}^n \Delta\sigma_i$$

as $\mu(\Pi) \to 0$. Now, if we denote $\Delta s_i = Area(D_i)$, then it can be seen that

$$\Delta s_i \approx \Delta\sigma_i \cos\theta_i$$

where θ_i is the acute angle between D_i and S_i. Since

$$\nabla\varphi = (g_x, g_y, -1)$$

it follows that

$$\cos\theta_i = \frac{1}{\sqrt{1+g_x^2+g_y^2}} \quad \text{at} \quad P_i.$$

Thus,

$$\Delta\sigma_i \approx \frac{\Delta s_i}{\cos\theta_i} = \Delta s_i\sqrt{1+g_x^2+g_y^2} \quad \text{at} \quad P_i$$

and hence

$$Area(S) = \lim_{\mu(\Pi)\to 0}\sum_{i=1}^{n}\sqrt{1+g_x^2+g_y^2}\Delta s_i = \iint_D \sqrt{1+g_x^2+g_y^2}dxdy.$$

Similarly, if the surface S is given by the equation

$$x = g(y,z), \quad (y,z)\in D$$

then

$$Area(S) = \iint_D \sqrt{1+g_y^2+g_y^2}dxdy$$

and if the surface S is given by the equation

$$y = h(z,x), \quad (z,x)\in D$$

then

$$Area(S) = \iint_D \sqrt{1+h_z^2+h_x^2}dxdy$$

EXAMPLE 2.23 Let us calculate the area of the surface S of a sphere of radius R using double integrals: Let us take the centre of the sphere as the origin of the coordinate axes. Then the equation of the sphere is given by

$$x^2+y^2+z^2 = R^2.$$

Taking

$$D := \{(x,y) : x^2+y^2 \le R^2\}$$

and

$$z = g(x,y) := \sqrt{R^2-(x^2+y^2)}, \quad (x,y)\in D$$

we obtain

$$Area(S) = 2\iint_D \sqrt{1+g_x^2+g_y^2}dxdy.$$

Note that

$$g_x = \frac{-x}{\sqrt{R^2-(x^2+y^2)}}, \quad f_y = \frac{-y}{\sqrt{R^2-(x^2+y^2)}},$$

and hence

$$\sqrt{1+g_x^2+g_y^2} = \frac{R}{\sqrt{R^2-(x^2+y^2)}}.$$

Thus,

$$\begin{aligned} Area(S) &= 2\iint_D \sqrt{1+g_x^2+g_y^2}dxdy \\ &= 2\iint_D \frac{R}{\sqrt{R^2-(x^2+y^2)}}dxdy \\ &= 2\int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2-\rho^2}}\rho\,\delta\rho\,d\theta \\ &= 4\pi R^2. \end{aligned}$$

□

EXAMPLE 2.24 Let us calculate the area of that part of the cylinder $x^2+y^2=a^2$ which is cut out by the cylinder $x^2+z^2=a^2$. In this case we may take

$$y = g(z,x) := \sqrt{a^2-x^2)}, \quad (z,x) \in D$$

with

$$D : \{(z,x) : z^2+x^2 \le a^2,\ z \ge 0,\ x \ge 0\}.$$

Then

$$Area(S) = 8\iint_D \sqrt{1+g_z^2+g_x^2}dxdy.$$

Note that

$$g_x = \frac{-x}{\sqrt{a^2-x^2)}}, \quad g_z = 0, \quad \sqrt{1+g_x^2+g_y^2} = \frac{a}{\sqrt{a^2-x^2}}.$$

Thus,

$$\begin{aligned} Area(S) &= 8\iint_D \sqrt{1+f_z^2+f_x^2}dxdy \\ &= 8a\int_0^a\left(\int_0^{\sqrt{a^2-x^2}} \frac{1}{\sqrt{a^2-x^2}}dx\right)dx \\ &= 8a^2. \end{aligned}$$

□

EXAMPLE 2.25 Let us calculate the area of the surface of the paraboloid $y^2 + z^2 = 4x$ which lies between the parabolic cylinder $y^2 = 2x$ and the plane $x = 2$. In this case we may take

$$z = g(x, y) := \sqrt{4x - y^2}, \quad (x, y) \in D$$

with $D \subseteq \mathbb{R}^2$ as the domain bounded by the curves $y^2 = 2x$ and $x = 2$. Then

$$Area(S) = 2 \iint_D \sqrt{1 + g_z^2 + g_x^2} dxdy.$$

Note that

$$g_x = \frac{2}{\sqrt{4x - y^2}}, \quad g_y = \frac{-y}{\sqrt{4x - y^2}},$$

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{\frac{4x + 4}{4x - y^2}}.$$

Thus,

$$\begin{aligned} Area(S) &= 2 \iint_D \sqrt{1 + g_x^2 + g_y^2} dxdy \\ &= 2 \iint_D \sqrt{\frac{4x+4}{4x - y^2}} dxdy \\ &= 4 \int_0^2 \sqrt{4x+4} \left(\int_0^{\sqrt{2x}} \frac{dy}{\sqrt{4x - y^2}} dy \right) dx \\ &= \pi \int_0^2 \sqrt{4x+4} dx \\ &= \frac{8}{3} \pi (4 - \sqrt{2}). \end{aligned}$$

□

2.2.6 Mass, Moment of inertia and Centre of gravity

2.2.6.1 (i) Mass:

We would like to find a formula or representation for the **mass** of a material planar region provided the density distribution of the material is known.

Let us assume that a material region occupy a domain D in $\mathbb{R}^2$. Suppose that the density at each point $P \in D$ is given as $f(P)$.

Suppose $\Pi := \{D_i\}_{i=1}^n$ is a partition of D, $P_i \in D_i$ and $\Delta s_i := Area(D_i)$ for $i = 1, \ldots, n$. Then, for $\mu(\Pi)$ small enough, the mass of D_i can be thought of as approximately equal to $f(P_i)\Delta s_i$, so that the mass of D is approximately equal to

$$\sum_{i=1}^n f(P_i)\Delta s_i.$$

Motivated by this observation, we define

$$\text{Massof } D = \lim_{\mu(\Pi)\to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i = \iint\limits_{D} f(x,y)dxdy.$$

EXAMPLE 2.26 Consider a material disc of radius R. Suppose that the surface density $f(P)$ at a point P is proportional to its distance from the center O, i.e., $f(P) = k|OP|$ for some constant k. Taking O as the coordinate origin, the mass of the disc is

$$\iint\limits_{D} k\sqrt{x^2+y^2}ds = k\int_0^{2\pi}\int_0^R \rho.\rho d\rho\delta\theta = \frac{2}{3}k\pi R^3.$$

□

2.2.6.2 (ii) Moment of inertia:

We would like to find a formula or representation for the **moment of inertia** of a material planar region with respect to a point or a line, provided the density distribution of the material is known.

Let us assume that a material region occupy a domain D in $\mathbb{R}^2$. Suppose that the density at each point $P \in D$ is given as $f(P)$. Suppose we want to find the moment of inertia of D w.r.t. a point Q.

Suppose $\Pi := \{D_i\}_{i=1}^n$ is a partition of D, $P_i \in D_i$ and $\Delta s_i := Area(D_i)$ for $i = 1, \ldots, n$. Then, for each i, the moment of inertia of P_i w.r.t. Q is given by

$$m_i|P_iQ|^2,$$

where m_i is the mass of the material point at P_i. For $\mu(\Pi)$ small enough, the above quantity is approximately equal to

$$f(P_i)\Delta s_i|P_iQ|^2.$$

Thus, for $\mu(\Pi)$ small enough, the moment of inertia of the material points at $P_1, \ldots, P_n$ put together is approximately equal to

$$\sum_{i=1}^{n} f(P_i)\Delta s_i|P_iQ|^2.$$

Motivated by the above, we define the **moment of inertia of D w.r.t. a point** $Q = (\alpha, \beta)$ as

$$M.I(D;Q) := \lim_{\mu(\Pi)\to 0} \sum_{i=1}^{n} f(P_i)\Delta s_i|P_iQ|^2 = \iint\limits_{D} [(x-\alpha)^2 + (y-\beta)^2]f(x,y)dxdy.$$

In particular, moment of inertia of D w.r.t. the origin $O := (0,0)$ is

$$M.I(D;O) == \iint\limits_{D} (x^2+y^2)f(x,y)dxdy.$$

Since distance from a point $P_0 = (x_0, y_0)$ to a line $\mathcal{L}$ given by the equation $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

we define the moment of inertial of D w.r.t. the line $\mathcal{L}$ given by the equation $ax + by + c = 0$ as

$$M.I(D; \mathcal{L}) == \iint\limits_D \frac{(ax + by + c)^2}{a^2 + b^2} f(x, y) dxdy.$$

Clearly, if the line is given by

$$y = \sigma x + c \quad \text{or} \quad x \sin\theta - y\cos\theta = d = 0,$$

where σ is the slop of the line and θ is the angle that it makes with the positive side of the x-axis, then the formula for the moment of inertia will take the form

$$M.I(D; \mathcal{L}) == \iint\limits_D \frac{(y - mx - c)^2}{1 + \sigma^2} f(x, y) dxdy$$

and

$$M.I(D; \mathcal{L}) == \iint\limits_D (x \sin\theta - y\cos\theta + d)^2 f(x, y) dxdy,$$

respectively. In particular, we have the following:

(i) If $\mathcal{L}$ is the y-axis, i.e., equation of the line is $x = 0$, then

$$I_{yy} := M.I(D; \mathcal{L}) = \iint\limits_D x^2 f(x, y) dxdy.$$

(ii) If $\mathcal{L}$ is the x-axis, i.e., equation of the line is $y = 0$, then

$$I_{xx} := M.I(D; \mathcal{L}) = \iint\limits_D y^2 f(x, y) dxdy,$$

(iii) if $\mathcal{L}$ is any line through the origin i.e., equation of the line is $x \sin\theta - y\cos\theta = 0$, then

$$I_\theta := M.I(D; \mathcal{L}_0) = \iint\limits_D (x \sin\theta - y\cos\theta)^2 f(x, y) dxdy.$$

Ellipse of inertia: Note that in (iii) above, we have

$$I_\theta = \sin^2\theta I_{yy} + \cos^2\theta I_{xx} - 2\cos\theta \sin\theta I_{xy},$$

where

$$I_{xy} = \iint\limits_D xy f(x, y) dxdy.$$

Thus, writing

$$X := \frac{\cos\theta}{\sqrt{I_\theta}} \quad \text{and} \quad Y := \frac{\sin\theta}{\sqrt{I_\theta}},$$

we have

$$I_{xx}X + I_{yy}Y - 2I_{xy}XY = 1$$

Since

$$I_{xy}^2 \le I_{xx}I_{yy}, \qquad \text{(Cauchy} - \text{Schwarz inequality)}$$

it follows that the equation $I_{xx}X + I_{yy}Y - 2I_{xy}XY = 1$ represents an ellipse, which is the locus of points $P_\theta := (\cos\theta/\sqrt{I_\theta}, \sin\theta/\sqrt{I_\theta})$ on the line as θ varies. This ellipse is called the **ellipse of inertia** of the domain D.

EXAMPLE 2.27 Let us find the moment of inertia (M.I) of a material plate of unit constant density occupying the area bounded by the lines

$$x = 0, \quad x = a, \quad y = 0, \quad y = b$$

w.r.t. to the coordinate origin: By definition,

$$M.I = \iint_D (x^2 + y^2)dxdy = \int_0^b \left(\int_0^a (x^2 + y^2)dy \right) dx = \frac{ab}{3}(a^2 + b^2).$$

□

EXAMPLE 2.28 Let us find the moment of inertia (M.I) of a material disc of unit constant density occupying the area bounded by the circle

$$\rho = 2a\cos\theta.$$

w.r.t. to the coordinate origin: Note that the representation of the given circle in polar coordinate is $(x-a)^2 + y^2 = a^2$. Thus,

$$\begin{aligned} M.I &= \iint_D (x^2 + y^2)dxdy = \int_{-\pi/2}^{\pi/2} \left(\int_0^{2a\cos\theta} \rho^2 \rho d\rho \right) d\theta \\ &= \int_{-\pi/2}^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^{2a\cos\theta} = 4a^4 \int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{3\pi a^4}{2}. \end{aligned}$$

□

EXAMPLE 2.29 Let us find the moment of inertia (M.I) of a material disc of unit constant density occupying the area bounded by the circle

$$(x-a)^2 + (y-b)^2 = 2a^2$$

w.r.t. to the y-axis. Taking the coordinate transformations:

$$x = a + \rho\cos\theta, \quad y = b + \rho\sin\theta, \quad 0 \le \theta \le 2\pi,\ 0 \le \rho \le \sqrt{2},$$

we have

$$M.I = \iint_D x^2 dxdy = \int_0^{2\pi} \left(\int_0^{\sqrt{2}a} (a + \rho\cos\theta)^2 \rho d\rho \right) d\theta = 3\pi a^4.$$

□

2.2.6.3 (iii) Centre of gravity:

Recall that the coordinates of the **centre of gravity** of a finite number of material points with masses $m_1, m_2, \dots, m_n$ located at the points $P_1, P_2, \dots, P_n$ respectively, with coordinates $P_i = (x_i, y_i)$, $i = 1, 2, \dots, n$, are given by

$$x_C := \frac{\sum_{i=1}^n x_i m_i}{\sum_{i=1}^n m_i}, \quad y_C := \frac{\sum_{i=1}^n y_i m_i}{\sum_{i=1}^n m_i}.$$

Note that $\sum_{i=1}^n m_i$ is the total mass of the points $P_1, P_2, \dots, P_n$. Motivated by this we define the coordinates x_C and y_C of the **centre of gravity** of a material plate occupying a region $D \subset \mathbb{R}^2$ of point density $\gamma(x, y)$ is as follows:

Suppose $\Pi := \{D_i\}_{i=1}^n$ is a partition of D, $P_i \in D_i$ and $\Delta s_i := Area(D_i)$ for $i = 1, \dots, n$. Since the coordinates of the centre of gravity of the points $P_1, P_2, \dots, P_n$ are approximately equal to

$$\tilde{x} = \frac{\sum_{i=1}^n x_i \gamma(x_i, y_i)\Delta s_i}{\sum_{i=1}^n \gamma(x_i, y_i)\Delta s_i}, \quad \tilde{y} := \frac{\sum_{i=1}^n y_i \gamma(x_i, y_i)\Delta s_i}{\sum_{i=1}^n \gamma(x_i, y_i)\Delta s_i},$$

we define the coordinates x_C and y_C of the **centre of gravity** of D as

$$x_C = \lim_{\mu(\Pi)\to 0} \frac{\sum_{i=1}^n x_i \gamma(x_i, y_i)\Delta s_i}{\sum_{i=1}^n \gamma(x_i, y_i)\Delta s_i}, = \frac{\iint_D x\gamma(x, y)dxdy}{\iint_D \gamma(x, y)dxdy} = \frac{1}{M}\iint_D x\gamma(x, y)dxdy,$$

$$y_C = \lim_{\mu(\Pi)\to 0} \frac{\sum_{i=1}^n y_i \gamma(x_i, y_i)\Delta s_i}{\sum_{i=1}^n \gamma(x_i, y_i)\Delta s_i}, = \frac{\iint_D y\gamma(x, y)dxdy}{\iint_D \gamma(x, y)dxdy} = \frac{1}{M}\iint_D y\gamma(x, y)dxdy.$$

where $M := \iint_D \gamma(x, y)dxdy$ is the mass of the plate.

EXAMPLE 2.30 Let us find the coordinates x_C, y_C of the centre of gravity of the part of a circular plate (of unit density) of radius a in the upper half plane. Clearly, $x_C = 0$. By definition

$$y_C = \frac{1}{M}\iint_D ydxdy,$$

where the mass M is given by

$$M = \iint_D dxdy = \frac{\pi a^2}{2}$$

and

$$\iint_D ydxdy = \int_0^\pi \int_0^a \rho \sin\theta \rho d\rho d\theta = \Big[-\cos\theta\Big]_0^\pi \Big[\frac{\rho^3}{3}\Big]_0^a = 2\frac{a^3}{3}.$$

Thus, $y_C = \dfrac{4a}{3\pi}$ □

EXAMPLE 2.31 Let us find the coordinates x_C, y_C of the centre of gravity of the part of an elliptical plate (of unit density) in the first quadrant, where the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

By definition

$$x_C = \frac{1}{M} \iint_D x dx dy,$$

where the mass M is given by

$$M = \iint_D dx dy = \int_0^a \left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy \right) dx = \frac{b}{a} \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{\pi ab}{4},$$

and

$$\iint_D x dx dy = \int_0^a \left(\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x dy \right) dx = \frac{b}{a} \int_0^a x\sqrt{a^2-x^2} dx = \frac{ba^2}{3}.$$

Thus,

$$x_C = \frac{4a}{3\pi}.$$

Similarly, we get

$$y_C = \frac{4b}{3\pi}.$$

□

2.3 Triple Integrals

In this section we consider integrals of functions of three variables. Let f be a function defined on a domain D in the space $\mathbb{R}^3$. We would like to define the integral of f over D.

Let D be a closed domain in $\mathbb{R}^3$ and $f : D \to \mathbb{R}$ be a function. Consider a partition $\mathcal{P} := \{D_i\}_{i=1}^n$ of D. Let $P_i \in D_i$ and Δv_i denotes the volume of D_i for $i = 1, 2, \ldots, n$. Let

$$\mu(\mathcal{P}) := \max\{\Delta v_i : i = 1, \ldots, n\},$$

the **mesh** of the partition $\mathcal{P}$.

Definition 2.32 We say that f is **integrable** over D if

$$S(f, \mathcal{P}) := \sum_{i=1}^{n} f(P_i)\Delta v_i \tag{1}$$

approaches a value, say γ, as $\mu(\mathcal{P})$ approaches zero, irrespective of the manner in which the partition $\{D_i\}_{i=1}^n$ and the set $\{P_i\}_{i=1}^n$ are taken, and in that case γ is called the **integral** of f over V, and it is denoted by

$$\iiint\limits_D f(P)dv \quad \text{or} \quad \iiint\limits_D f(x,y,z)dxdydz.$$

$\diamondsuit$

We may observe that if $f(P) = 1$ for all P in D, then the sum in (1) is nothing but the volume of D, irrespective of the way the domain is partitioned. Thus,

$$vol(D) = \sum_{i=1}^n \Delta v_i = \iiint\limits_V dv.$$

Let (n_k) be an increasing sequence of positive integers. For each $k \in \mathbb{N}$, let $\mathcal{P}_k = \{D_i^{(k)}\}_{i=1}^{n_k}$ be a partition of D, and let $\Delta v_i^{(k)} := vol(D_i^{(k)})$ and

$$S(f, \mathcal{P}_k) := \sum_{i=1}^{n_k} f(P_i^{(k)}) \Delta v_i^{(k)}.$$

It can be shown that if $\mu(\mathcal{P}_k) \to 0$ as $k \to \infty$, then

> f is integrable $\iff$ $\{S(f, \mathcal{P}_k)\}$ converges as $k \to \infty$, and in that case, the limit is the integral of f over D.

Triple integrals share all that properties that double integrals have. Now, we consider methods of evaluation of triple integrals using single and double integrals.

2.3.1 Calculating triple integrals

Suppose V consists of the the set of all (x, y, z) such that

$$(x, y) \in D_0, \quad \psi_1(x, y) \le z \le \psi_2(x, y),$$

where D_0 is a closed domain in the plane $\mathbb{R}^2$. Then, it can be seen that

$$\iiint\limits_D (f(x,y,z)dv = \iint\limits_{D_0} \int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dzdxdy. \tag{2}$$

Note that, for a fixed $(x, y) \in D_0$, $g(x, y) := \displaystyle\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x, y, z)dz$ is a function of two variables defined on D. Thus integral on the right-hand side of (2) above can be evaluated using methods for double integrals.

EXAMPLE 2.33 Find the volume of the solid D with boundary as the ellipsoid:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

In this case,

$$vol(D) = \iint\limits_{D_0} \left(\int_{\psi_1(x,y)}^{\psi_2(x,y)} f(x,y,z)dz \right) dxdy,$$

where D_0 is the region in $\mathbb{R}^2$ with boundary $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and

$$\psi_1(x,y) = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad \text{and} \quad \psi_1(x,y) = +c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}.$$

Thus,

$$vol(D) = \iint\limits_{D_0} 2c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} dxdy.$$

Now, taking the coordinate transformations $x = a\rho\cos\theta$, $y = b\rho\sin\theta$ we have $|J| = ab\rho$ so that

$$vol(D) = 2abc \int_0^{2\pi} \int_0^1 \sqrt{1-\rho^2}\rho d\rho d\theta = \frac{4}{3}\pi abc.$$

□

EXAMPLE 2.34 Let us find the mass of a hemisphere of radius R if the density at each point P is proportional to the distance of P from the base: Thus, density is given by $\gamma(x,y,z) = kz$ and hence

$$Mass = \iiint\limits_V kzdx\,dy\,dz = k \int_0^{2\pi} \int_0^R \int_0^{\sqrt{R^2-\rho^2}} \rho z\,d\rho\,d\theta = \frac{k\pi R^4}{4}.$$

□

2.4 Applications

As in the case of double integrals, **moment of inertia** and **centre of gravity** of a domain V in the space can be defined analogously using triple integrals:

(i). Moment of inertia of a domain V in $\mathbb{R}^3$ w.r.t. a point $Q = (x_0, y_0, z_0)$ is defined as

$$M.I(V;Q) = \iiint\limits_V [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]\gamma(x,y,z)dx\,dy\,dz,$$

where $\gamma(x, y, z)$ is the point density function. In particular, moment of inertia of D w.r.t. the origin $O := (0, 0)$ is

$$M.I(D; O) = \iiint\limits_V (x^2 + y^2 + z^2)\gamma(x, y, z)dx\,dy\,dz.$$

Moment of inertia of a domain V in $\mathbb{R}^3$ w.r.t. the coordinate planes are given by

$$I_{xy} = \iiint\limits_V z^2)\gamma(x, y, z)dx\,dy\,dz,$$

$$I_{yz} = \iiint\limits_V x^2\gamma(x, y, z)dx\,dy\,dz,$$

$$I_{zx} = \iiint\limits_V y^2\gamma(x, y, z)dx\,dy\,dz,$$

respectively.

(i). Coordinates x_C, y_C, z_C of the **centre of gravity** of V are defined as

$$x_C = \frac{1}{M}\iiint\limits_V x\gamma(x, y, z)\,dx\,dy\,dz,$$

$$y_C = \frac{1}{M}\iiint\limits_V y\gamma(x, y, z)\,dx\,dy\,dz,$$

$$z_C = \frac{1}{M}\iiint\limits_V y\gamma(x, y, z)\,dx\,dy\,dz.$$

where $M := \iiint\limits_V \gamma(x, y, z)\,dx\,dy\,dz$ is the mass of the solid.

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