Consequences of Coercivity of Operators

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Definition 1. Let $\mathcal{D}$ be a subspace of a Hilbert space $\mathcal{H}$. A linear operator $A : \mathcal{D} \to \mathcal{H}$ satisfying

$$|\langle Au, u\rangle| \geq \gamma \|u\|^2 \quad \forall\, u \in \mathcal{D}$$

for some $\gamma > 0$ is called a **coercive operator**, and the constant γ in (1) is called a **coefficient of coercivity**. $\diamondsuit$

THEOREM 2. *Let $\mathcal{D}$ be a subspace of a Hilbert space $\mathcal{H}$ and $A : \mathcal{D} \to \mathcal{H}$ be a closed linear operator satisfying*

$$|\langle Au, u\rangle| \geq \gamma \|u\|^2 \quad \forall\, u \in \mathcal{D} \tag{1}$$

for some $\gamma > 0$. Thus for every $f \in \mathcal{H}$, there exists a unique $u \in \mathcal{D}$ such that

$$Au = f,$$

and in that case

$$\|u\| \leq \frac{\|f\|}{\gamma}.$$

Proof. By (1) we have

$$\|Au\| \geq \gamma \|u\| \quad \forall\, u \in \mathcal{D}, \tag{2}$$

so that A is injective, $A^{-1} : R(A) \to X$ is continuous and

$$\|A^{-1}\| \leq \frac{1}{\gamma}. \tag{3}$$

Now, we show that $R(A) = \mathcal{H}$:

Let (u_n) in $\mathcal{D}$ be such that $Au_n \to v$ for some $v \in \mathcal{H}$. Then (Au_n) is a Cauchy sequence so that by (2), (u_n) is also a Cauchy sequence. Let $u \in \mathcal{H}$ be such that $u_n \to u$. Since A is a closed operator,

$$u \in \mathcal{D} \quad \text{and} \quad Au = v.$$

Thus, $R(A)$ is closed. Now, using (1), we have

$$u \in R(A)^\perp \implies u = 0$$

so that $R(A)^\perp = \{0\}$. Hence, by Projection Theorem (cf. [1]), $R(A)$ is dense in $\mathcal{H}$. Since $R(A)$ is already a closed subspace, $R(A) = \mathcal{H}$. Thus A is bijective so that for every $f \in \mathcal{H}$, there exists a unique $u \in \mathcal{D}$ such that $Au = f$, and in that case, by (3),

$$\|u\| = \|A^{-1}f\| \leq \|A^{-1}\|\,\varphi\| \leq \frac{\|f\|}{\gamma}.$$

$\square$

PROPOSITION 3. *Let A be as in Theorem 3. Assume that*

$$\langle Au, u\rangle \in \mathbb{R} \quad \textit{and} \quad \langle Au, u\rangle \geq 0 \quad \forall\, u \in \mathcal{D},$$

$$\langle Au, v\rangle = \langle u, Av\rangle \quad \forall\, u, v \in \mathcal{D}. \tag{4}$$

Then

$$\langle u, v\rangle_1 = \langle Au, v\rangle, \quad u \in \mathcal{D},$$

defines an inner product on $\mathcal{D}$.

Proof. We observe that for every $u, v, w \in \mathcal{D}$,

- $\langle u, u\rangle_1 = \langle Au, u\rangle \in \mathbb{R}$ and $\langle u, u\rangle_1 \geq 0$ for all $u \in \mathcal{D}$,
- coercivity of A implies $\langle u, u\rangle_1 = 0 \iff u = 0$,
- $\langle u, v\rangle_1 = \langle Au, v\rangle = \langle u, Av\rangle = \overline{\langle Av, u\rangle} = \overline{\langle v, u\rangle_1}$.
- linearity of $u \mapsto \langle Au, v\rangle$ for each $u \in \mathcal{D}$ implies

$$\langle u + v, w\rangle_1 = \langle u, w\rangle_1 + \langle v, w\rangle_1, \quad \langle \alpha u, w\rangle_1 = \alpha\langle u, v\rangle_1.$$

□

Let $\mathcal{H}_1$ be the completion of $\mathcal{D}$ with respect to the norm $\|\cdot\|_1$. In applications, specially in PDEs, the subspace $\mathcal{D}$ is dense in $\mathcal{H}$, $\mathcal{H}_1 \subseteq \mathcal{H}$ and the inclusion $I_0 : \mathcal{H}_1 \to \mathcal{H}$ is a compact operator. Now, by Theorem 2, we know that A is bijective and $A^{-1} : \mathcal{H} \to \mathcal{H}$ is continuous, satisfying

$$\|A^{-1}f\| \leq \frac{1}{\gamma}\|f\| \quad \forall\, f \in \mathcal{H}.$$

Note that, for every $f \in \mathcal{H}$,

$$\|A^{-1}f\|_1^2 = \langle A(A^{-1}f), A^{-1}f\rangle = |\langle f, A^{-1}f\rangle| \leq \|f\|\,\|A^{-1}f\| \leq \frac{1}{\gamma}\|f\|^2.$$

Hence, $A^{-1} : \mathcal{H} \to \mathcal{H}_1$ is continuous. Since the inclusion $I_0 : \mathcal{H}_1 \to \mathcal{H}$ is a compact operator, we can conclude that $B = I_0 A^{-1} : \mathcal{H} \to \mathcal{H}$ is a compact operator. Note that

$$Bf = A^{-1}f \quad \forall\, f \in \mathcal{H}.$$

In view of this, one uses the symbol A^{-1} for the compact operator B also. Using the property (**??**), we see that A^{-1} is self-adjoint as well.

By *spectral theorem*, there exists orthonromal sequence (u_n) of vectors in $\mathcal{H}$ and a null sequence (λ_n) of positive real numbers such that

$$A^{-1}f = \sum_n \lambda_n \langle f, u_n\rangle u_n, \quad f \in \mathcal{H}.$$

Hence,

$$Au = \sum_n \mu_n \langle u, u_n \rangle u_n, \quad u \in \mathcal{D},$$

where $\mu_n = 1/\lambda_n \to \infty$ as $n \to \infty$. Note also that $\lambda_n u_n = A^{-1} u_n \in \mathcal{D}$ so that $u_n \in \mathcal{D}$ for every $n \in \mathbb{N}$.

References

[1] M.T. Nair, *Functional Analysis: A First Course*, New Delhi: Printice-Hall of India, 2002 (Third Print, PHI Learning, 2010).