Department of Mathematics, IIT Madras

MA 5450: Functional Analysis
Assignment Sheet-I

Date: August 25, 2015

1. Let X be a linear space and $x \mapsto \|x\|$ satisfy the properties:

   (a) For $x \in X$, $\|x\| = 0 \implies x = 0$

   (b) $\|x + y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in X$

   (c) $\|\alpha x\| = |\alpha|\, \|x\| \quad \forall\, x \in X,\ \alpha \in \mathbb{K}$.

   Deduce the following:

   (a) $\|0\| = 0$

   (b) $\|x\| \geq 0 \quad \forall\, x \in X$

   (c) $\|x - y\| \geq \|x\| - \|y\| \quad \forall\, x, y \in X$.

2. Show that the norm on a linear space X is a uniformly continuous function from X to $\mathbb{R}$.

3. Let X be a normed linear space and for $x_0 \in X$ and $r > 0$, let and $B(x_0, r) := \{x \in X : \|x - x_0\| < r\}$. Show that closure of $B(x_0, r)$ is $\{x \in X : \|x - x_0\| \leq r\}$.

4. For $1 \leq p < \infty$ and $x = (x(1), \dots, x(n)) \in \mathbb{K}^n$, let $\|x\|_p := \left(\sum_{k=1}^{n} |x(k)|^p\right)^{1/p}$. Show that $x \mapsto \|x\|_p$ defines a norm on $\mathbb{K}^n$.

5. For $1 \leq p < \infty$ and $x \in C[a, b]$, let $\|x\|_p := \left(\int_a^b |x(t)|^p dt\right)^{1/p}$. Show that $x \mapsto \|x\|_p$ defines a norm on $C[a, b]$.

6. For $1 \leq p < \infty$ let $\ell^p := \{x : \mathbb{N} \to \mathbb{K} \,\big|\, \sum_{k=1}^{\infty} |x(k)|^p < \infty\}$. Show that ℓ^p is a linear space and $x \mapsto \|x\|_p := \left(\sum_{k=1}^{\infty} |x(k)|^p\right)^{1/p}$ is a norm on ℓ^p.

7. Let Ω be a nonempty set and $B(\Omega)$ be the linear space of all bounded $\mathbb{K}$-valued functions on Ω. Show that $x \mapsto \|x\|_\infty := \sup_{t \in \Omega} |x(t)|$ is a norm on $B(\Omega)$, and $B(\Omega)$ with $\|\cdot\|_\infty$ is a Banach space.

8. Let Ω be a nonempty set and $C_b(\Omega)$ be the linear space of all bounded $\mathbb{K}$-valued continuous functions on Ω. Show that $C_b(\Omega)$ is a closed subspace of $B(\Omega)$ w.r.t. the norm $\|\cdot\|_\infty$. Deduce that $C[a, b]$ is a Banach space w.r.t. the norm $\|\cdot\|_\infty$.

9. For $p, r \in [1, \infty]$, show that the norms $\|\cdot\|_p$ and $\|\cdot\|_r$ on $\mathbb{K}^n$ are equivalent.

10. For $p \in [1, \infty)$, show that the norm $\|\cdot\|_p$ on $C[a, b]$ is not equivalent to $\|\cdot\|_\infty$.

11. For $p \in [1, \infty)$, find a Cauchy sequence in $C[a, b]$ which is not convergent in $C[a, b]$ w.r.t. $\|\cdot\|_p$.

12. For $x : [0,1] \to \mathbb{K}$, let $\nu(x) = |x(0)| + |x(1/2)| + |x(1)|$. Check whether ν is a norm on the spaces (a) $C[0,1]$, (b) $\mathcal{P}_2[0,1]$.

13. Let X be a normed linear space and X_0 be a subspace of X. Show that for every $x \in X$, $u \in X_0$ and $\alpha \in \mathbb{K}$,

$$\mathrm{dist}(\alpha x, X_0) = |\alpha| \mathrm{dist}(x, X_0), \quad \mathrm{dist}(x + u, X_0) = \mathrm{dist}(x, X_0).$$

14. For $x \in C^1[0,1]$, define $\|x\|_0 = |x(1/2)| + \|x'\|_\infty$. Show that

    (a) $x \mapsto \|x\|_0$ is a norm on $C^1[0,1]$,

    (b) $C^1[0,1]$ with $\|\cdot\|_0$ is a Banach space.

    Is $\|\cdot\|_0$ equivalent to the norm $\|\cdot\|_\infty$ on $C^1[0,1]$? - Why?

15. For $1 \le p < \infty$ and $x \in \ell^p$, define $x_n := \sum_{j=1}^n x(j) e_j$, $n \in \mathbb{N}$. Show that the sequence (x_n) converges to x with respect to $\|\cdot\|_p$. Is $\{e_1, e_1, \ldots\}$ a basis of ℓ^1? Why?

16. Let X be a normed linear space and let $S = \{x \in X : \|x\| = 1\}$. Show that the following are equivalent:

    (a) X is finite dimensional.

    (b) S is compact.

    (c) For any $r > 0$, $\{x \in X : \|x - x_0\| \le r\}$ is compact.

    (d) For any $r > 0$, $\{x \in X : \|x - x_0\| < r\}$ is totally bounded.

17. Let X be a Banach space. Show that the following are equivalent:

    (a) X is finite dimensional.

    (b) Every subspace of X is closed.

18. Let X be an inner product space and $x, y \in X$. Prove the following:

    (a) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

    (b) $x \perp y \implies \|x + y\|^2 = \|x\|^2 + \|y\|^2$.

19. Let X be an inner product space and E be an orthonormal set. Show that E is an orthonormal basis if and only if $E^\perp = \{0\}$.

20. Let X be an inner product space. Prove the following:

    (a) If X is finite dimensional, then every orthonormal basis of X is a basis.

    (b) If X is infinite dimensional, then an orthonormal basis of X need not be a basis.

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