Department of Mathematics, IIT Madras

MA 5450: Functional Analysis
Assignment Sheet-II

Date: October 5, 2015

1. Let X be an inner product space. Prove the following:

   (a) If X is separable, then every orthonormal set in X is countable.

   (b) Converse of (a) is true if X is a Hilbert space.

2. Let X be a Hilbert space. Show an orthonormal basis of X is a Hamel basis if and only if X is finite dimensional.

3. Let X be an inner product space and and $P : X \to X$ be an orthogonal projection, i.e., P is a linear operator satisfying $P^2 = P$ and $R(P) \perp N(P)$. Then prove the following:

   (a) P is a bounded linear operator with $\|P\| = 1$.

   (b) $\langle Px, y\rangle = \langle x, Py\rangle \, \forall \, x, y \in X$.

   (c) $\|x - Px\| = \inf_{v \in R(P)} \|x - v\|$.

4. Let X be an inner product space and let $E = \{u_1, u_2, \ldots\}$ be a denumerable orthonormal set. Prove the following.

   (a) For every $x \in X$, the sequence $(\langle x, u_1\rangle, \langle x, u_2\rangle, \ldots)$ belongs to ℓ^2.

   (b) The map $x \mapsto (\langle x, u_1\rangle, \langle x, u_2\rangle, \ldots)$ from X to ℓ^2 is injective if and only if E is an orthonormal basis.

   (c) For each $n \in \mathbb{N}$, the map $P_n : X \to X$ defined by $P_n x = \sum_{j=1}^{n} \langle x, u_j\rangle u_j$, $x \in X$, is an orthogonal projection.

   (d) If X is a Hilbert space, then for each $x \in X$, the series $\sum_{j=1}^{\infty} \langle x, u_j\rangle u_j$ converges in X and the map $P : X \to X$ defined by $Px = \sum_{j=1}^{\infty} \langle x, u_j\rangle u_j$, $x \in X$, is an orthogonal projection and $R(P) = \overline{\text{span}(E)}$.

5. Let $p, r \in [1, \infty]$ with $p \leq r$. Prove the following:

   (a) ℓ^p is a dense subspace of ℓ^r.

   (b) The inclusion operator $I : \ell^p \to \ell^r$ is a bounded linear operator.

   (c) The inclusion operator $I : \ell^r \to \ell^p$ is a not a bounded linear operator.

6. For $1 \leq p \leq \infty$, let $X_p := C[a,b]$ with $\|\cdot\|_p$. Let $p, r \in [1, \infty]$ with $p \leq r$. Prove the following:

   (a) The identity operator $I : X_r \to X_p$ is a bounded linear operator.

   (b) The identity operator $I : X_p \to X_r$ is a not a bounded linear operator.

7. For $1 \leq p \leq \infty$, let $X_p := C[a,b]$ with $\|\cdot\|_p$. Let $k(\cdot,\cdot) \in C([a,b] \times [a,b])$, and for $x \in C[a,b]$ let $(Ax)(s) = \int_a^b k(s,t)x(t)dt$, $s \in [a,b]$. Prove the following:

   (a) The map $A : X_\infty \to X_\infty$ is a bounded linear operator and $\|A\| \leq \sup_{s\in[a,b]} \int_a^b |k(s,t)|dt$.

   (b) For any $p, r \in [1, \infty]$, $A : X_p \to X_r$ is a bounded linear operator

8. Let $u \in C[a,b]$ and $A : C[a,b] \to C[a,b]$ be defined by $(Ax)(t) = u(t)x(t)$ for $t \in [a,b]$, $x \in C[a,b]$. Prove that A is a bounded linear operator w.r.t. the norm $\|\cdot\|_\infty$ on $C[a,b]$, and $\|A\| = \|u\|_\infty$.

9. Let (λ_n) be a bounded sequence in $\mathbb{K}$. For $1 \leq p \leq \infty$, let $A : \ell^p \to \ell^p$ be defined by $(Ax)(i) = \lambda_i x(i)$ for $i \in \mathbb{N}$, $x \in \ell^p$. Prove that A is a bounded linear operator and $\|A\| = \sup_{n\in\mathbb{N}} |\lambda_n|$.

10. Let X and Y be normed liner spaces and let X_0 be a subspace of X. Prove the following:

    (a) If $A : X \to Y$ is a bounded linear operator, then the restriction operator $A_0 : X_0 \to Y$ defined by $A_0 x = Ax$ for all $x \in X_0$ is a bounded linear operator, and $\|A_0\| \leq \|A\|$.

    (b) Suppose $A_0 : X_0 \to Y$ is a bounded linear operator, X_0 is dense in X and Y is a Banach space. Then A_0 has a unique norm-preserving bounded linear extension to all of X, i.e., there exists a unique bounded linear operator $A : X \to Y$ such that $Ax = A_0 x$ for all $x \in X_0$ and $\|A\| = \|A_0\|$.

11. Let $X = c_{00}$ with usual inner produc, i.e., $\langle x, y \rangle := \sum_{j=1}^\infty x(j)\overline{y(j)}$. Let $f : X \to \mathbb{K}$ be defined by $f(x) = \sum_{j=1}^\infty \frac{x(j)}{j}$. Show that

    (a) f is a bounded linear functional.

    (b) There is no $y \in X$ such that $f(x) = \langle x, y \rangle$ for all $x \in X$..

12. Let $X = c_{00}$ with inner product $\langle x, y \rangle := \sum_{j=1}^\infty x(j)\overline{y(j)}$. Let $X_0 := \{x \in c_{00} : \sum_{j=1}^\infty \frac{x(j)}{j} = 0\}$. Show that

    (a) X_0 is a closed subspace of X.

    (b) $X_0^\perp = \{0\}$.

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