M.T.Nair: August 2010

Functional Analysis: Assignment Problems - 2

In the following $\mathcal{F}(\mathbb{N},\mathbb{K})$ denotes the space of all scalar sequences.

1. Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent norms on a vector space X, and $S\subseteq X$. Show that S is open w.r.t. $\|\cdot\|_1$ iff S is open w.r.t. $\|\cdot\|_2$.

2. For $(\alpha_1,\ldots,\alpha_n)$ and $(\beta_1,\ldots,\beta_n)$ in $\mathbb{K}^n$, and $1<p<\infty$, show that

   (i) $\displaystyle\sum_{j=1}^n |\alpha_j\beta_j| \le \Big(\sum_{j=1}^n |\alpha_j|^p\Big)^{1/p}\Big(\sum_{j=1}^n |\beta_j|^q\Big)^{1/q}, \qquad \frac{1}{p}+\frac{1}{q}=1,$

   (ii) $\displaystyle\Big(\sum_{j=1}^n |\alpha_j+\beta_j|^p\Big)^{1/p} \le \Big(\sum_{j=1}^n |\alpha_j|^p\Big)^{1/p} + \Big(\sum_{j=1}^n |\beta_j|^p\Big)^{1/p}.$

3. Let $x,y\in\ell^p$ for $1<p<\infty$. Using Problem 2(ii), show that $x+y\in\ell^p$ and
   $$\Big(\sum_{j=1}^\infty |x(j)+y(j)|^p\Big)^{1/p} \le \Big(\sum_{j=1}^\infty |x(j)|^p\Big)^{1/p} + \Big(\sum_{j=1}^\infty |y(j)|^p\Big)^{1/p}.$$

4. Suppose $1\le r\le p\le\infty$. Show the following:

   (i) $\ell^r\subseteq\ell^p$,

   (ii) $\|x\|_p\le\|x\|_r$ for all $x\in\ell^r$,

   (iii) $\lim_{p\to\infty}\|x\|_p=\|x\|_\infty$ for every $x\in c_{00}$.

   (iv) if $r<p$, then ℓ^r is not a closed subset of ℓ^p.

5. For $p>0$, let $X_p := \{x\in\mathcal{F}(\mathbb{N},\mathbb{K}) : \|x\|_p^p := \sum_{n=1}^\infty |x(n)|^p < \infty\}$. Show that, if $0<p<1$, then there exist $x,y\in X_p$ such that the relation $\|x+y\|_p\le\|x\|_p+\|y\|_p$ *does not hold.*

6. Show that, if $0<p<1$, then $x\mapsto \Big(\int_a^b |x(t)|^p dt\Big)^{1/p}$ does not define a norm on $C[a,b]$.

7. Let X be a normed linear space, and U and V be subsets of X. Show that

   (i) if one of U and V is an open set, then $U+V$ is an open set, and

   (ii) if U is compact and V is closed, then $U+V$ is closed.

8. Let X be a linear space and $\nu : X\to\mathbb{R}$ be a seminorm.

   Show that $Z_\nu=\{x\in X : \nu(x)=0\}$ is a closed subspace of X.

9. Let X be a normed linear space, X_0 be a subspace of X. For $x \in X$, define $d(x, X_0) = \inf\{\|x - y\| : \ y \in X_0\}$. Prove the following:

   (i) $d(x, X_0) = 0$ if and only if $x \in cl(X_0)$.

   (ii) If X_0 is closed in X and $x \notin X_0$, then $d(x, X_0) > 0$.

   (iii) For $x \in X$ and $\alpha \in \mathbb{K}$, $d(\alpha x, X_0) = |\alpha| d(x, X_0)$.

10. Let X_0 be a closed subspace of a normed linear space X. Prove the following:

    (i) The map $\nu : x \mapsto dist(x, X_0)$ defines a seminorm on X.

    (ii) The map $[x] \mapsto dist(x, X_0)$ defines a norm on the quotient linear space X/X_0.

11. The space $\mathcal{P}[a, b]$ is not a Banach space with respect to $\|\cdot\|_\infty$.

12. The space c_{00} is not a Banach space with respect to $\|\cdot\|_p$ for any p with $1 \leq p \leq \infty$.

13. Prove the following.

    (i) $x \mapsto \|x'\|_\infty$ is a seminorm on $C^1[a, b]$.

    (ii) $x \mapsto \|x\|_* := \|x\|_\infty + \|x'\|_\infty$, $x \in C^1[a, b]$, defines a norm on $C^1[a, b]$.

    (iii) $C^1[a, b]$ is a Banach space with respect to $\|\cdot\|_*$.

    (iii) Show that $C^1[a, b]$ is not a Banach space with respect to $\|\cdot\|_\infty$.

14. Let $X = C[a, b]$ with $\|\cdot\|_1$ and $Y = C[a, b]$ with $\|\cdot\|_\infty$. Then the (identity) map $A : X \to Y$ defined by $Ax = x$, $x \in X$, is not continuous - Why?

15. Prove the following:

    (i) c_{00} is a proper dense subspace of ℓ^p for $1 \leq p < \infty$, and not dense in ℓ^∞.

    (ii) c_{00} is a proper dense subspace of c_0 with respect to $\|\cdot\|_\infty$, and and not dense in c with respect to $\|\cdot\|_\infty$.

    (iii) $\mathcal{P}[a, b]$ is a proper dense subspace of $C[a, b]$ with respect to $\|\cdot\|_p$ for $1 \leq p \leq \infty$.

16. Let X be an inner product space and $x, y \in X$. Show that $\|x + \alpha y\| = \|x - \alpha y\| \ \forall \alpha \in \mathbb{K}$ if and only if $\langle x, y \rangle = 0$.

17. Suppose $A : X \to Y$ is a linear operator between normed linear spaces X and Y. Show that, if A is an open map, then it is onto.

18. Prove that a Banach space is finite dimensional if and only if every subspace of it is closed.