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Functional Analysis: Assignment Problems-3

1. Suppose $A : X \to Y$ is a linear operator between normed linear spaces X and Y. Show that, if A is an open map, then it is onto.

2. Let $X = C[a,b]$ with $\|\cdot\|_1$ and $Y = C[a,b]$ with $\|\cdot\|_\infty$. Then the (identity) map $A : X \to Y$ defined by $Ax = x$, $x \in X$, is not continuous - Why?

3. Let X be an inner product space and $x, y \in X$. Prove *Pythagoras theorem*: If $x \perp y$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

4. Let X be an inner product space and S_1, S_2 are subsets of X. Show that if $S_1 \subseteq S_2$ then $S_2^\perp \subseteq S_1^\perp$.

5. Let X be an inner product space and $S \subseteq X$. Show that following:

   (i) $S^\perp$ is a closed subspace of X,

   (ii) $S \subseteq S^{\perp\perp}$ and $S^\perp = (\text{span}(S))^\perp = (\text{cl.span}(S))^\perp$,

   (iii) $S^{\perp\perp\perp} = S^\perp$.

6. Let X be an inner product space and X_0 be a subspace of X. Show that, for $x \in X$, if a best approximation of x from X_0 exists, then it is unique.

7. Let X be an inner product space, $\{u_1, \ldots, u_k\}$ be an orthonormal set and $P : X \to X$ is defined by $Px = \sum_{j=1}^k \langle x, u_j\rangle u_j$, $x \in X$. Show that

   (i) P is an orthogonal projection, i.e., $P^2 = P$ and $R(P) \perp N(P)$,

   (ii) $\langle Px, y\rangle = \langle x, Py\rangle$ for all $x, y \in X$

8. Let X be an inner product space over $\mathbb{R}$. If $x, y \in X$ are such that $\|x+y\| = \|x-y\|$, then show that $x \perp y$.

9. Let X be an inner product space over $\mathbb{R}$ and $P : X \to X$ be an orthogonal projection, i.e., $P^2 = P$ and $R(P) \perp N(P)$.

   (i) Show that $\|Px\| \le \|x\|$ and $\|(I-P)x\| \le \|x\|$ for all $x \in X$. Deduce that both P and $I - P$ are continuous linear operators. [*Hint:* use Pythagoras theorem.]

   (ii) Show that for every $x \in X$, Px is *the* unique best approximation of X from $R(P)$. [*Hint:* use Pythagoras theorem for existence, and parallelogram law for uniqueness.]

   (iii) $R(P)^\perp = N(P)$ [*Hint:* Note: $X = R(P) \oplus R(P)^\perp$, $X = R(P) \oplus N(P)$.]

10. Suppose X is a Hilbert space, and $P : X \to X$ and $Q : X \to X$ are orthogonal projections. Show that $R(P) \perp R(Q) \Leftrightarrow PQ = 0 = QP$.

11. Let X be an inner product space and $P : X \to X$ be a projection, i.e., $P^2 = P$. Show that P is an orthogonal projection (i.e., $R(P) \perp N(P)$) if and only if $R(P)^\perp = N(P)$.

12. Suppose X is a an inner product space and $P : X \to X$ and $Q : X \to X$ are projections, i.e., $P^2 = P$ and $Q^2 = Q$.

    (i) Show that $R(P) \subseteq R(Q) \implies QP = P$.

    (ii) If in addition P and Q satisfy $R(P)^\perp = N(P)$ and $R(Q)^\perp = N(Q)$, then show that $R(P) \subseteq R(Q) \Leftrightarrow PQ = P = QP$. [*Hint:* Use Problem 4 and (i) above.]