M.T.Nair: August 30, 2005

Functional Analysis: Assignment Sheet-I

In the following X denotes a linear space over $\mathbb{K}$.

1. Show that the zero element and the additive inverse of any element in a linear space are unique.

2. Show that if X contains at least one nonzero element, then it contains infinitely many elements: if x is a nonzero element in X, and if α, β are scalars such that $\alpha \neq \beta$, then show that $\alpha x \neq \beta x$. Which axiom of a vector space is used to prove this?

3. Show that if X is of dimension n, then there is a linear isomorphism from X onto $\mathbb{K}^n$. Show also that any two finite dimensional linear spaces of same dimension are linearly isomorphic.

4. Let S be a subset of X. Show that span$S$ is the smallest subspace containing S, i.e., if X_0 is a subspace containing S, then span$S \subseteq X_0$.

5. Show that if S be a subset of X, then span$S = \bigcap\{Y : Y$ is a subspace of X containing $S\}$.

6. Let X_0 be a subspace of X and $x_0 \in X \setminus X_0$. Show that for every $x \in \text{span}\{x_0; X_0\}$, there exist a unique $\alpha \in \mathbb{K}$, $y \in Y$ such that $x = \alpha x_0 + y$.

7. Suppose X_1 and X_2 are subspaces of linear space X such that $X = X_1 \oplus X_2$. Show that the map $A : X_2 \to X/X_1$ defined by $A(x) = [x]$, $x \in X_2$, is a linear isomorphism from X_2 onto X/X_1.

8. Suppose X_1 and X_2 are linear spaces, and let X be the cartesian product of X_1 and X_2, i.e., $X = X_1 \times X_2$ with vector space operations defined as in Example **??**(xii). Let $Y_1 = X_1 \times \{0\}$ and $Y_2 = \{0\} \times X_2$. Show that

   (i) $X = Y_1 \oplus Y_2$,

   (ii) Y_1 and Y_2 are linearly isomorphic with X_1 and X_2 respectively,

   (iii) X_1 and X_2 are linearly isomorphic with X/Y_2 and X/Y_1 respectively,

9. Show that a subset $\{u_1, \ldots, u_n\}$ of X is linearly dependent if and only if there exists a nonzero $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{K}^n$ such that $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$.

10. Show that a subset $\{u_1, \ldots, u_n\}$ of X is linearly independent if and only if the function $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 u_1 + \cdots + \alpha_n u_n$ from $\mathbb{K}^n$ into X is injective.

11. Let $E \subseteq X$. Show that

(i) if E is linearly dependent in X, then every superset of E is also linearly dependent, and

(ii) if E is linearly independent in X, then every subset of E is also linearly independent.

12. Show that if $\{u_1, \ldots, u_n\}$ is a linearly independent subset of X, and if Y is a subspace of X such that $\{u_1, \ldots, u_n\} \cap Y = \varnothing$, then every x in the span of $\{u_1, \ldots, u_n, Y\}$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$, $y \in Y$.

13. Show that if E_1 and E_2 are linearly independent subsets of X such that $(\text{span} E_1) \cap (\text{span} E_2) = \{0\}$, then $E_1 \cup E_2$ is linearly independent.

14. If $\mathbf{A}$ is an $m \times n$ matrix with entries from $\mathbb{K}$ and $n > m$, then show that there exists an $n \times 1$ nonzero matrix $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{0}$ is the $m \times 1$ zero matrix.

15. Let X and Y be linear spaces.

(i) Let X_0 be a subspace of X and $A_0 : X_0 \to Y$ be a linear operator. Show that there exists a linear operator $A : X \to Y$ such that $A_{|X_0} = A_0$.

(ii) Let $u_0 \in X$ and $v_0 \in Y$. Show that there exits a linear operator $A : X \to Y$ such that $Au_0 = v_0$.

16. Let X and Y be finite dimensional linear spaces with bases $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ respectively. Let $F = \{f_1, \ldots, f_n\}$ be the dual basis of $\mathcal{L}(X, \mathbb{K})$ with respect to U and $G = \{g_1, \ldots, g_n\}$ be the dual basis of $\mathcal{L}(Y, \mathbb{K})$ with respect to V. For $i = 1, \ldots, n$; $j = 1, \ldots, m$, let $A_{ij} : X \to Y$ defined by

$$A_{ij}(x) = f_j(x) v_i, \quad x \in X.$$

Show that $\{A_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$ is a basis of $\mathcal{L}(X, Y)$.

17. Let X and Y be finite dimensional linear spaces, and $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ be bases of X and Y, respectively. Show the following:

(i) If $\{g_1, \ldots, g_m\}$ is the ordered dual basis of $\mathcal{L}(Y, \mathbb{K})$ with respect to the basis V of Y, then $[A]_{U,V} = (g_i(Au_j))$.

(ii) If $A, B \in \mathcal{L}(X, Y)$ and $\alpha \in \mathbb{K}$, then

$$[A+B]_{U,V} = [A]_{U,V} + [B]_{U,V}, \quad [\alpha A]_{U,V} = \alpha [A]_{U,V}.$$

(iii) Suppose $\{E_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$ is a basis of $\mathbb{K}^{m \times n}$. If $T_{ij} \in \mathcal{L}(X, Y)$ is the linear transformation such that $[T_{ij}]_{U,V} = E_{ij}$, then $\{T_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$ is a basis of $\mathcal{L}(X, Y)$.

18. Let $A : X \to Y$ be a linear operator between linear spaces X and Y. Show the following:

    (i) If S is a spanning set of X, then $A(S) := \{Ax : x \in S\}$ is a spanning set of $R(A)$. In particular, if X is finite dimensional, then A is of finite rank.

    (ii) A is of finite rank if and only if there exists $n \in \mathbb{N}$, $\{v_1, \ldots, v_n\} \subset Y$ and $\{f_1, \ldots, f_n\} \subset \mathcal{L}(X, \mathbb{K})$ such that $Ax = \sum_{j=1}^{n} f_j(x) v_j$ for all $x \in X$.

19. Let $A : X \to Y$ be a linear operator between linear spaces X and Y. Show the following:

    (i) If E is a linearly independent subset of X and if A is injective, then the set $A(E) := \{Ax : x \in E\}$ is a linearly independent subset of Y.

    (ii) If $\{u_1, \ldots, u_n\} \subseteq X$ is such that $\{Au_1, \ldots, Au_n\}$ is a linearly independent subset of Y, then $\{u_1, \ldots u_n\}$ is a linearly independent subset of X.

20. Show that a linear functional on a linear space is completely determined by its null space and an element not in the null space, i.e., if f and g are linear functionals on a linear space X such that $N(f) = N(g)$, and $f(x_0) = g(x_0)$ for some $x_0 \in X \setminus N(f)$, then $f = g$.

21. Show that $X_0 := \{x \in C[0, 1] : x(0) = 0 = x(1)\}$ is a proper subspace of $C[0, 1]$, but not a hyperspace.