A Short Course on LINEAR ALGEBRA and its APPLICATIONS

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Vector Spaces

1.1 Introduction

The notion of a *vector space* is an abstraction of the familiar set of vectors in two or three dimensional Euclidian space. For example, let $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ be two vectors in the plane $\mathbb{R}^2$. Then we have the notion of addition of these vectors so as to get a new vector denoted by $\vec{x} + \vec{y}$, and it is defined by

$$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2).$$

This addition has an obvious geometric meaning: If O is the coordinate origin, and if P and Q are points in $\mathbb{R}^2$ representing the vectors $\vec{x}$ and $\vec{y}$ respectively, then the vector $\vec{x} + \vec{y}$ is represented by a point R in such way that OR is the diagonal of the parallelogram for which OP and OQ are adjacent sides.

Also, if α is a positive real number, then the multiplication of $\vec{x}$ by α is defined by

$$\alpha\vec{x} = (\alpha x_1, \alpha x_2).$$

Geometrically, the vector $\vec{\alpha x}$ is an elongated or contracted form of $\vec{x}$ in the direction of $\vec{x}$. Similarly, we can define $\alpha\vec{x}$ with a negative real number α, so that $\alpha\vec{x}$ represents in the negative direction. Representing the coordinate-origin by $\vec{0}$, and $-\vec{x} := (-1)\vec{x}$, we see that

$$\vec{x} + \vec{0} = \vec{x}, \qquad \vec{x} + (-\vec{x}) = \vec{0}.$$

We may denote the sum $\vec{x} + (-\vec{y})$ by $\vec{x} - \vec{y}$.

Now, abstracting the above properties of vectors in the plane, we define the notion of a vector space.

We shall denote by $\mathbb{F}$ the field of real numbers or the field of complex numbers. If special emphasis is required, then the fields of real numbers and complex numbers will be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively.

1.2 Definition and Some Basic Properties

Definition 1.1 (Vector space) A *vector space* over $\mathbb{F}$ is a nonempty set V together with two operations

(i) *addition* which associates each pair (x, y) of elements in V a unique element in V denoted by $x + y$, and

(ii) *scalar multiplication* which associates each pair (α, x) with $\alpha \in \mathbb{F}$ and $x \in V$, a unique element in V denoted by αx,

satisfying the following conditions:

(a) $x + y = y + x \quad \forall\, x, y \in V$.

(b) $(x + y) + z = x + (y + z) \quad \forall\, x, y, z \in V$.

(c) $\exists\, \theta \in V$ such that $x + \theta = x \quad \forall\, x \in V$.

(d) $\forall\, x \in V$, $\exists\, \tilde{x} \in V$ such that $x + \tilde{x} = \theta$.

(e) $\alpha(x + y) = \alpha x + \alpha y \quad \forall\, \alpha \in \mathbb{F}, \forall\, x, y \in V$.

(f) $(\alpha + \beta)x = \alpha x + \beta x \quad \forall\, \alpha, \beta \in \mathbb{F}, \forall\, x \in V$.

(g) $(\alpha\beta)x = \alpha(\beta x) \quad \forall\, \alpha, \beta \in \mathbb{F}, \forall\, x \in V$.

(h) $1x = x \quad \forall\, x \in V$.

Elements of a vector space are called **vectors**, and elements of the field $\mathbb{F}$ (over which the vector space is defined) are often called **scalars**.

Proposition 1.1 *Let V be a vector space. Then there is exactly one element $\theta \in V$ such that $x + \theta = V$ for all $x \in V$.*

Proof. Suppose there are θ_1 and θ_2 in V such that

$$x + \theta_1 = x \quad \text{and} \quad x + \theta_2 = x \quad \forall x \in X.$$

Then, using conditions (a) and (c), we have

$$\theta_2 = \theta_2 + \theta_1 = \theta_1 + \theta_2 = \theta_1.$$

This completes the proof. ∎

Definition 1.2 (zero element) Let V be a vector space. The unique element $\theta \in V$ such that $x + \theta = x$ for all $x \in V$ is called the *zero element* or simply, the *zero* in V.

Notation: The zero element in a vector space as well as the zero in the scalar field are often denoted by the same symbol 0.

Exercise 1.1 Let V be a vector space. For $x, y \in V$, show that $x + y = x$ implies $y = \theta$. ♢

Proposition 1.2 *Let V be a vector space. For each $x \in V$, there is exactly one element $\tilde{x} \in V$ such that $x + \tilde{x} = \theta$.*

Proof. Let $x \in V$. Suppose x' and x'' are in V such that

$$x + x' = \theta \quad \text{and} \quad x + x'' = \theta.$$

Then using the axioms (a), (b), (c), it follows that

$$x' = x' + \theta = x' + (x + x'') = (x' + x) + x'' = \theta + x'' = x''.$$

This completes the proof. ∎

Definition 1.3 (additive inverse) Let V be a vector space. For each $x \in V$, the unique element $\tilde{x} \in V$ such that $x + \tilde{x} = \theta$ is called the *additive inverse* of x.

Notation: For x in a vector space, the unique element $\tilde{x}$ which satisfies $x + \tilde{x} = \theta$ is denoted by $-x$, i.e.,

$$-x := \tilde{x}.$$

Proposition 1.3 *Let V be a vector space. Then, for all $x \in V$,*

$$0x = \theta \quad \textit{and} \quad (-1)x = -x.$$

Proof. Let $x \in V$. Since

$$0x = (0 + 0)x = 0x + 0x,$$

we have $0x = \theta$. Now,

$$x + (-1)x = [1 + (-1)]x = 0x = \theta$$

so that, by the uniqueness of the additive inverse of x, we have $(-1)x = -x$. ∎

Notation: For x, y in a vector space, the expression $x + (-y)$ is denoted by $x - y$, i.e.,

$$x - y := x + (-y).$$

Exercise 1.2 Show that, if $x \in V$ and $x \neq 0$, then $\alpha x \neq \beta x$ for every $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$. [Hint: Condition (h)] ♢

Remark 1.1 We observe that a vector space V, by definition, cannot be an empty set. It contains at least one element, viz., the zero element. If a vector space V contains at least one nonzero element, then it contains infinitely many nonzero elements: If x is a nonzero element in V, and if α, β are scalars such that $\alpha \neq \beta$, then $\alpha x \neq \beta x$ (see Exercise 1.2). ♢

Convention: Unless otherwise specified, we always assume that the vector space under discussion is *non-trivial*, i.e., it contains at least one nonzero element.

1.3 Examples of Vector Spaces

EXAMPLE 1.1 (Space $\mathbb{F}^n$) Consider the set $\mathbb{F}^n$ of all n–tuples of scalars, i.e.,

$$\mathbb{F}^n := \{(\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{F},\, i = 1, \ldots, n\}.$$

For $x = (\alpha_1, \ldots, \alpha_n)$, $y = (\beta_1, \ldots, \beta_n)$ in $\mathbb{F}^n$, and $\alpha \in \mathbb{F}$, define the addition and scalar multiplication coordinate-wise as

$$x + y = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \quad \alpha x = (\alpha\alpha_1, \ldots, \alpha\alpha_n).$$

Then it can be seen that $\mathbb{F}^n$ is a vector space with zero element $\theta := (0, \ldots, 0)$ and additive inverse of $x = (\alpha_1, \ldots, \alpha_n)$ as $-x = (-\alpha_1, \ldots, -\alpha_n)$. ♢

NOTATION: We shall, sometimes, denote the i^{th} coordinate of $x \in \mathbb{F}^n$ by $x(i)$ for $i \in \{1, \ldots, n\}$. Thus, if $x = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$, then $x(i) = \alpha_i$ for $i \in \{1, \ldots, n\}$.

EXAMPLE 1.2 (Space $\mathcal{P}_n$) For $n \in \{0, 1, 2, \ldots\}$, let $\mathcal{P}_n$ be the set of all polynomials of degree at most n, with coefficients in $\mathbb{F}$, i.e., $x \in \mathcal{P}_n$ if and only if x is of the form

$$x = a_0 + a_1 t + \ldots + a_n t^n$$

for some scalars $a_0, a_1 \ldots, a_n$. Then $\mathcal{P}_n$ is a vector space with addition and scalar multiplication defined as follows:

For $x = a_0 + a_1 t + \ldots a_n t^n,\ y = b_0 + b_1 t + \ldots + b_n t^n$ in $\mathcal{P}_n$ and $\alpha \in \mathbb{F}$,

$$x + y = (a_0 + b_0) + (a_1 + b_1)t + \ldots + (a_n + b_n)t^n,$$

$$\alpha x = \alpha a_0 + \alpha a_1 t + \ldots + \alpha a_n t^n.$$

The zero polynomial, i.e., the polynomial with all its coefficients zero, is the zero element of the space, and

$$-x = -a_0 - a_1 t - \ldots - a_n t^n.$$

$\diamondsuit$

EXAMPLE 1.3 (Space $\mathcal{P}$) Let $\mathcal{P}$ be the set of all polynomials with coefficients in $\mathbb{F}$, i.e., $x \in \mathcal{P}$ if and only if $x \in \mathcal{P}_n$ for some $n \in \{0, 1, 2, \ldots\}$. For $x, y \in \mathcal{P}$, let n, m be such that $x \in \mathcal{P}_n$ and $y \in \mathcal{P}_m$. Then we have $x, y \in \mathcal{P}_k$, where $k = \max\{n, m\}$. Hence we can define $x + y$ and αx for $\alpha \in \mathbb{F}$ as in $\mathcal{P}_k$. With this addition and scalar multiplication, it follows that $\mathcal{P}$ is a vector space. $\diamondsuit$

EXAMPLE 1.4 (Space $\mathbb{F}^{m\times n}$) Let $V = \mathbb{F}^{m\times n}$ be the set of all $m \times n$ matrices with entries in $\mathbb{F}$. If A is a matrix with its ij-th entry a_{ij}, then we shall write $A = [a_{ij}]$. It is seen that V is a vector space with respect to the addition and scalar multiplication defined as follows: For $A = [a_{ij}]$, $B = [b_{ij}]$ in V, and $\alpha \in \mathbb{F}$,

$$A + B := [a_{ij} + b_{ij}], \qquad \alpha A := [\alpha a_{ij}].$$

In this space, $-A = [-a_{ij}]$, and the matrix with all its entries are zeroes is the zero element. $\diamondsuit$

EXAMPLE 1.5 (Space $\underline{\mathbb{F}}^k$) This example is a special case of the last one, namely,

$$\underline{\mathbb{F}}^k := \mathbb{F}^{k\times 1}.$$

This vector space is in one-one correspondence with $\mathbb{F}^k$. One such correspondence is given by $F : \mathbb{F}^k \to \underline{\mathbb{F}}^k$ defined by

$$F((x_1, \ldots, x_k)) = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_k \end{bmatrix}, \qquad (x_1, \ldots, x_k) \in \mathbb{F}^k.$$

$\diamondsuit$

NOTATION: If $\underline{x} \in \mathbb{F}^n$, we shall denote its j^{th} entry or coordinate by $\underline{x}_j$.

EXAMPLE 1.6 (Sequence space) Let V be the set of all scalar sequences. For (α_n) and (β_n) in V, and $\alpha \in \mathbb{F}$, we define

$$(\alpha_n) + (\beta_n) = (\alpha_n + \beta_n), \quad \alpha(\alpha_n) = (\alpha\alpha_n).$$

With this addition and scalar multiplication, V is a vector space with its zero element as the sequence of zeroes, and $-(\alpha_n) = (-\alpha_n)$. $\Diamond$

Exercise 1.3 Verify that the sets considered in Examples 1.1 – 1.6 are indeed vector spaces. $\Diamond$

EXAMPLE 1.7 (Space $C(I)$) Let I be an interval and $C(I)$ be the set of all real valued continuous functions defined on I. For $x, y \in C(I)$ and $\alpha \in \mathbb{F}$, we define $x + y$ and αx point-wise, i.e.,

$$(x + y)(t) = x(t) + y(t), \quad (\alpha x)(t) = \alpha x(t), \quad t \in I.$$

Then it can be shown that $x + y$ and αx are in $C(I)$, and $C(I)$ is a vector space over $\mathbb{R}$. The zero element is the zero function, and the additive inverse of $x \in C(I)$ is the function $-x$ defined by $(-x)(t) = -x(t),\ t \in I$. $\Diamond$

EXAMPLE 1.8 (Space $\mathcal{R}[a, b]$) Let $\mathcal{R}[a, b]$ be the set of all real valued Riemann integrable functions on $[a, b]$. From the theory of Riemann integration, it follows that if $x, y \in \mathcal{R}[a, b]$ and $\alpha \in \mathbb{F}$, then $x + y$ and αx defined pointwise belongs to $\mathcal{R}[a, b]$. It is seen that (*Verify*) $\mathcal{R}[a, b]$ is a vector space over $\mathbb{R}$. $\Diamond$

EXAMPLE 1.9 (Product space) Let $V_1, \ldots, V_n$ be vector spaces. Then the *cartesian product*

$$V = V_1 \times \cdots \times V_n,$$

the set of all of ordered n-tuples $(x_1, \ldots, x_n)$ with $x_j \in V_j$ for $j \in \{1, \ldots, n\}$, is a vector space with respect to the addition and scalar multiplication defined by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n),$$

$$\alpha(x_1, \ldots, x_n) := (\alpha x_1, \ldots, \alpha x_n)$$

with zero element $(0,\ldots,0)$ and additive inverse of $x=(x_1,\ldots,x_n)$ defined by $-x=(-x_1,\ldots,-x_n)$.

This vector space is called the *product space* of $V_1,\ldots V_n$.

As a particular example, the space $\mathbb{F}^n$ can be considered as the product space $V_1\times\cdots\times V_n$ with $V_j=\mathbb{F}$ for $j=1,\ldots,n$. ◊

Exercise 1.4 In each of the following, a set V is given and some operations are defined. Check whether V is a vector space with these operations:

(i) Let $V=\{x=(x_1,x_2)\in\mathbb{R}^2 : x_1+x_2=1\}$ with addition and scalar multiplication as for $\mathbb{R}^2$.

(ii) Let $V=\mathbb{R}^2$, $\mathbb{F}=\mathbb{R}$. For $x=(x_1,x_2)$, $y=(y_1,y_2)$, let $x+y:=(x_1+y_1,x_2+y_2)$ and for all $\alpha\in\mathbb{R}$,

$$\alpha x := \begin{cases} (0,0) & \alpha=0, \\ (\alpha x_1, x_2/\alpha), & \alpha\neq 0. \end{cases}$$

(iii) Let $V=\mathbb{C}^2$, $\mathbb{F}=\mathbb{C}$. For $x=(x_1,x_2)$, $y=(y_1,y_2)$, let

$$x+y:=(x_1+2y_1,x_2+3y_2) \quad \text{and} \quad \alpha x:=(\alpha x_1,\alpha x_2) \quad \forall \alpha\in\mathbb{C}.$$

(iv) Let $V=\mathbb{R}^2$, $\mathbb{F}=\mathbb{R}$. For $x=(x_1,x_2)$, $y=(y_1,y_2)$, let

$$x+y:=(x_1+y_1,x_2+y_2) \quad \text{and} \quad \alpha x:=(x_1,0) \quad \forall \alpha\in\mathbb{R}.$$

◊

Exercise 1.5 Let Ω be a nonempty set and W be a vector space. Let $\mathcal{F}(\Omega,W)$ be the set of all functions from Ω into W. For $f,\ g\in\mathcal{F}(\Omega,W)$ and $\alpha\in\mathbb{F}$, let $F+G$ and αF be defined point-wise, i.e.,

$$(f+g)(s)=f(s)+g(s), \quad (\alpha f)(s)=\alpha f(s), \quad s\in\Omega.$$

Let $-f$ and θ be defined by

$$(-f)(s)=-f(s), \quad \theta(s)=0, \quad s\in S.$$

Show that $\mathcal{F}(\Omega,W)$ is a vector space over $\mathbb{F}$ with the above operations. ◊

Exercise 1.6 In Exercise 1.5, let $\Omega=\{1,\ldots,n\}$ and $W=\mathbb{F}$. Show that the map $T:\mathcal{F}(S,\mathbb{F})\to\mathbb{F}^n$ defined by

$$T(f)=(f(1),\ldots,f(n)), \quad f\in\mathcal{F}(\Omega,\mathbb{F}),$$

is bijective.

Also show that for every f, g in $\mathcal{F}(\Omega, \mathbb{F})$ and $\alpha \in \mathbb{F}$,

$$T(f+g) = T(f) + T(g), \quad T(\alpha f) = \alpha T(f).$$

Such a map is called a *linear transformation* or a *linear transformation*. Linear transformations will be considered in more detail in the next chapter. ♢

1.4 Subspace and Span

1.4.1 Subspace

We observe that

- $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$, which is a subset of $\mathbb{R}^2$ is a vector space with respect to the addition and scalar multiplication as in $\mathbb{R}^2$.

- $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : 2x_1 + 3x_2 = 0\}$ which is a subset of $\mathbb{R}^2$ is a vector space with respect to the addition and scalar multiplication as in $\mathbb{R}^2$.

- $\mathcal{P}_n$ which is a subset of the vector space $\mathcal{P}$ is also a vector space,

These examples motivate the following definition.

Definition 1.4 (Subspace) Let V_0 be a subset of a vector space V. If V_0 is a vector space with respect to the operations of addition and scalar multiplication as in V, then V_0 is called a *subspace* of V.

The following theorem is very useful for checking whether a subset of a vector space is a subspace or not.

Theorem 1.4 *Let V be a vector space, and V_0 be a subset of V. Then V_0 is a subspace of V if and only if for every x, y in V_0 and $\alpha \in \mathbb{F}$,*

$$x + y \in V_0 \quad \textit{and} \quad \alpha x \in V_0.$$

Proof. Clearly, if V_0 is a subspace of V, then $x + y \in V_0$ and $\alpha x \in V_0$ for all $x, y \in V_0$ and for all $\alpha \in \mathbb{F}$.

Conversely, suppose that $x + y \in V_0$ and $\alpha x \in V_0$ for all $x, y \in V_0$ and for all $\alpha \in \mathbb{F}$. Then, for any $x \in V_0$,

$$\theta = 0\,x \in V_0 \quad \text{and} \quad -x = (-1)x \in V_0.$$

Thus, conditions (c) and (d) in the definition of a vector space are satisfied for V_0. All the remaining conditions can be easily verified as elements of V_0 are elements of V as well. ∎

EXAMPLE 1.10 The space $\mathcal{P}_n$ is a subspace of $\mathcal{P}_m$ for $n \leq m$. ♢

EXAMPLE 1.11 The space $C[a, b]$ is a subspace of $\mathcal{R}[a, b]$. ♢

EXAMPLE 1.12 (Space $C^k[a, b]$) For $k \in \mathbb{N}$, let $C^k[a, b]$ be the set of all $\mathbb{F}$-valued functions defined on $[a, b]$ such that the j-th derivative $x^{(j)}$ of x exists and $x^{(j)} \in C[a, b]$ for each $j \in \{1, \ldots, k\}$. It can be seen that $C^k[a, b]$ is a subspace of $C[a, b]$. ♢

EXAMPLE 1.13 For $n \in \mathbb{N}$ and $(a_1, \ldots, a_n) \in \mathbb{F}^n)$, let

$$V_0 = \{(x_1, \ldots, x_n) \in \mathbb{F}^n : a_1x_1 + \ldots + a_nx_n = 0\}.$$

Then V_0 is a subspace of $\mathbb{F}^n$.

Recall from school geometry that, for $\mathbb{F} = \mathbb{R}$ and $n = 3$, the subspace

$$V_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : a_1x_1 + +a_2x_2 + a_3x_3 = 0\}$$

is a plane passing through the origin. ♢

Theorem 1.5 *Suppose V_1 and V_2 are subspaces of a vector space. Then $V_1 \cap V_2$ is a subspace of V.*

Proof. Suppose $x, y \in V_1 \cap V_2$ and $\alpha \in \mathbb{F}$. Then $x, y \in V_1$ and $x, y \in V_2$. Since V_1 and V_2 are subspaces, it follows that $\alpha x, x+y \in V_1$ and $\alpha x, x + y \in V_2$ so that $\alpha x, x + y \in V_1 \cap V_2$. Thus, by Theorem 1.4, $V_1 \cap V_2$ is a subspace. ∎

Union of two subspaces need not be a subspace. To see this consider the subspaces

$$V_1 := \{(x_1, x_2) : x_2 = x_1\}, \qquad V_2 := \{(x_1, x_2) : x_2 = 2x_1\}$$

of the space $\mathbb{R}^2$. Note that $x = (1, 1) \in V_1$ and $y = (1, 2) \in V_2$, but $x + y = (2, 3) \not\in V_1 \cup V_2$. Hence $V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$.

Exercise 1.7 Let V_1 and V_2 be subspaces of a vector space. Prove that $V_1 \cup V_2$ is a subspace if and only if either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$. ♢

Exercise 1.8 Let A be an $m \times n$ matrix of scalars. Show that the sets

$$V_1 = \{x \in \underline{\mathbb{F}}^n : Ax = 0\},$$

$$V_2 = \{y \in \underline{\mathbb{F}}^m : y = Ax \text{ for some } x \in \underline{\mathbb{F}}^m\}$$

are subspaces of $\underline{\mathbb{F}}^n$. $\Diamond$

Exercise 1.9 Let $\mathcal{P}[a,b]$ be the vector space $\mathcal{P}$ over $\mathbb{R}$ taking its elements as continuous real valued functions defined on $[a,b]$. Then the space $\mathcal{P}[a,b]$ is a subspace of $C^k[a,b]$ for every $k \geq 1$. $\Diamond$

Exercise 1.10 Suppose V_0 is a subspace of a vector space V, and V_1 is a subspace of V_0. Then show that V_1 is a subspace of V. $\Diamond$

Exercise 1.11 Show that

$$V_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0,\ x_1 + 2x_2 + 3x_3 = 0\}$$

is a subspace of $\mathbb{R}^3$. Observe that V_0 is the intersection of the subspaces

$$V_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$$

and

$$V_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + 2x_2 + 3x_3 = 0\}.$$

Note that V_1 and V_2 are planes through the origin and hence V_0 is a straight line passing through the origin. $\Diamond$

Exercise 1.12 Suppose Λ is a set, and for each $\lambda \in \Lambda$ let V_λ be a subspace of a vector space V. Then $\cap_{\lambda \in \Lambda} V_\lambda$ is a subspace of V. $\Diamond$

Exercise 1.13 In each of the following vector space V, see if the subset V_0 is a subspace of V:

(i) $V = \mathbb{R}^2$ and $V_0 = \{(x_1, x_2) : x_2 = 2x_1 - 1\}$.

(ii) $V = C[-1,1]$ and $V_0 = \{f \in V : f \text{ is an odd function}\}$.

(iii) $V = C[0,1]$ and $V_0 = \{f \in V : f(t) \geq 0\, \forall t \in [0,1]\}$.

(iv) $V = \mathcal{P}_3$ and $V_0 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0 = 0\}$.

(v) $V = \mathcal{P}_3$ and $V_0 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_2 = 0\}$.

◊

Exercise 1.14 Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin. ◊

Exercise 1.15 Let V be a vector space and $u_1, \ldots, u_n$ be in V. Show that

$$V_0 := \{\alpha_1 u_1 + \ldots + \alpha_n u_n : \alpha_i \in \mathbb{F},\, i = 1, \ldots, n\}$$

is a subspace of V. ◊

1.4.2 Linear Combination and Span

Definition 1.5 (Linear combination) Let V be a vector space and $u_1, \ldots, u_n$ belongs to V. Then, by a *linear combination* of $u_1, \ldots, u_n$, we mean an element in V of the form $\alpha_1 u_1 + \cdots + \alpha_n u_n$ with $\alpha_j \in \mathbb{F}$, $j = 1, \ldots, n$.

Definition 1.6 (Span) Let V be a vector space and $u_1, \ldots, u_n$ belongs to V. Then the set of all linear combinations of $u_1, \ldots, u_n$ is called the *span* of $u_1, \ldots, u_n$, we write it as

$$\operatorname{span}\{u_1, \ldots, u_n\}.$$

In view of Exercise 1.15, if $u_1, \ldots, u_n$ belongs to a vector space V, then $\operatorname{span}\{u_1, \ldots, u_n\}$ is a subspace of V.

More generally, we have the following definition.

Definition 1.7 (Span) Let S be a subset of V. Then the set of all linear combinations of elements of S is called the *span* of S, and is also denoted by $\operatorname{span}(S)$.

Thus, for $S \subseteq V$, $x \in \operatorname{span} S$ if and only if there exists $x_1, \ldots, x_n$ in S and scalars $\alpha_1, \ldots, \alpha_n$ such that $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$.

As a convention, span of the empty set is taken to be the singleton set $\{0\}$.

Remember! By a linear combination, we always mean a linear combination of a *finite number* of elements in the space. An expression of the form $\alpha_1 x_1 + \alpha_n x_n + \cdots$ with $x_1, x_2, \ldots$ in V and $\alpha_1, \alpha_2, \ldots$ in $\mathbb{F}$ has no meaning in a vector space, unless there is some additional structure which allows such expression.

Exercise 1.16 Let V be a vector space, and $S \subseteq V$. Then $\operatorname{span}(S)$ is a subspace of V, and $\operatorname{span}(S)$ is the smallest subspace containing

S, in the sense that, if if V_0 is a subspace of V such that $S \subset V_0$, then $\operatorname{span}(S) \subseteq V_0$. $\Diamond$

Exercise 1.17 Let S be a subset of a vector space V. Show that S is a subspace if and only if $S = \operatorname{span} S$. $\Diamond$

Exercise 1.18 Let V be a vector space. Show that the the following hold.

(i) Let S be a subset of V. Then

$$\operatorname{span} S = \bigcap\{Y : Y \text{ is a subspace of } V \text{ containing } S\}.$$

(ii) Suppose V_0 is a subspace of V and $x_0 \in V \setminus V_0$. Then for every $x \in \operatorname{span}\{x_0; V_0\} := \operatorname{span}(\{x_0\} \cup V_0)$, there exists a unique pair $(\alpha, y) \in \mathbb{F} \times V_0$ such that $x = \alpha x_0 + y$. $\Diamond$

NOTATION (Kronecker[1] delta): For $(i, j) \in \mathbb{N} \times \mathbb{N}$, let

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

EXAMPLE 1.14 Let $V = \mathbb{F}^n$ and for each $j \in \{1, \ldots, n\}$, let $e_j \in \mathbb{F}^n$ be such that its i-th coordinate is δ_{ij}. Then $\mathbb{F}^n$ is the span of $\{e_1, \ldots, e_n\}$. $\Diamond$

EXAMPLE 1.15 For $1 \leq k < n$, let

$$V_0 := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \alpha_j = 0, j = k + 1, \ldots, n\}.$$

Then it is seen that V_0 is the span of $\{e_1, \ldots, e_k\}$. $\Diamond$

EXAMPLE 1.16 Let $V = \mathcal{P}$, and $u_j = t^{j-1}$, $j \in \mathbb{N}$. Then $\mathcal{P}_n$ is the span of $\{u_1, \ldots, u_{n+1}\}$, and $\mathcal{P} = \operatorname{span}\{u_1, u_2, \ldots\}$. $\Diamond$

EXAMPLE 1.17 (Space c_{00}) Let V be the set of all sequences with real entries. For $n \in \mathbb{N}$, let

$$e_n = (\delta_{n1}, \delta_{n2}, \ldots).$$

Then $\operatorname{span}\{e_1, e_2, \ldots\}$ is the space of all scalar sequences with only a finite number of nonzero entries. The space $\operatorname{span}\{e_1, e_2, \ldots\}$ usually denoted by c_{00}. $\Diamond$

[1] German mathematician Leopold Kronecker (December 7, 1823 December 29, 1891). He was quoted as having said, "God made integers; all else is the work of man".

Exercise 1.19 Consider the system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \dots & + & \dots & + & \dots & + & \dots & = & \dots \\ a_{m1}x_1 & + & a_{m1}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Let

$$u_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix}, u_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix}, \dots, u_n := \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}.$$

Show that the above system has a solution vector $x = [x_1, \dots, x_n]^T$ if and only if $b = [b_1, \dots, b_n]^T$ is in the span of $\{u_1, \dots, u_n\}$. $\Diamond$

Exercise 1.20 Let $u_j(t) = t^{j-1}$, $j \in \mathbb{N}$. Show that span of $\{u_1, \dots, u_{n+1}\}$ is $\mathcal{P}_n$, and span of $\{u_1, u_2, \dots\}$ is $\mathcal{P}$. $\Diamond$

Exercise 1.21 Let $u_1(t) = 1$, and for $j = 2, 3, \dots$, let $u_j(t) = 1 + t + \dots + t^j$. Show that span of $\{u_1, \dots, u_n\}$ is $\mathcal{P}_n$, and span of $\{u_1, u_2, \dots\}$ is $\mathcal{P}$. $\Diamond$

Definition 1.8 (Sum of subsets) Let V be a vector space, $x \in V$, and E, E_1, E_2 be subsets of V. Then we define the following:

$$x + E := \{x + u : u \in E\},$$

$$E_1 + E_2 := \{x_1 + x_2 : x_1 \in E_1,\ x_2 \in E_2\}.$$

The set $E_1 + E_2$ is called the *sum of the subsets* E_1 and E_2.

Theorem 1.6 *Suppose V_1 and V_2 are subspaces of V. Then $V_1 + V_2$ is a subspace of V. In fact,*

$$V_1 + V_2 = \text{span}\,(V_1 \cup V_2).$$

Proof. Let $x, y \in V_1 + V_2$ and $\alpha \in \mathbb{F}$. Then, there exists $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ such that $x = x_1 + y_1$, $y = y_1 + y_2$. Hence,

$$x + y = (x_1 + y_1) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2) \in V_1 + V_2,$$

$$\alpha(x + y) = \alpha(x_1 + y_1) = (\alpha x_1 + \alpha y_1) \in V_1 + V_2.$$

Thus, $V_1 + V_2$ is a subspace of V.

Now, since $V_1 \cup V_2 \subseteq V_1 + V_2$, and since $V_1 + V_2$ is a subspace, we have $\text{span}\,(V_1 \cup V_2) \subseteq V_1 + V_2$. Also, since $V_1 \subseteq \text{span}\,(V_1 \cup V_2)$, $V_2 \subseteq \text{span}\,(V_1 \cup V_2)$, and since $\text{span}\,(V_1 \cup V_2)$ is a subspace, we have $V_1 + V_2 \subseteq \text{span}\,(V_1 \cup V_2)$. Thus,

$$V_1 + V_2 \subseteq \text{span}\,(V_1 \cup V_2) \subseteq V_1 + V_2,$$

which proves the last part of the theorem. ∎

Exercise 1.22 Suppose V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$. Show that every $x \in V_1 + V_2$ can be written *uniquely* as $x = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$. ♢

Exercise 1.23 Suppose V_1 and V_2 are subspaces of a vector space V. Show that $V_1 + V_2 = V_1$ if and only if $V_2 \subseteq V_1$. ♢

1.5 Basis and Dimension

Definition 1.9 (Linear dependence) Let V be a vector space. A subset E of V is said to be *linearly dependent* if there are $u_1, \ldots, u_n$, $n \geq 2$, in E such that at least one of them is a linear combination of the remaining ones.

Definition 1.10 (Linear independence) Let V be a vector space. A subset E of V is said to be *linearly independent* in V if it is not linearly dependent.

Exercise 1.24 Let E be a subset of of a vector space V. Then prove the following.

(i) E is linearly dependent if and only if there exists $u_1, \ldots, u_n$ in E and scalars $\alpha_1, \ldots, \alpha_n$, with at least one of them nonzero, such that $\alpha_1 u_1 + \cdots + \alpha_n x_n = 0$,

(ii) E is linearly independent if and only if for every finite subset $\{u_1, \ldots, u_n\}$ of E,

$$\alpha_1 u_1 + \cdots + \alpha_n x_n = 0 \implies \alpha_i = 0 \quad \forall\, i = 1, \ldots, n.$$

♢

If $\{u_1, \ldots, u_n\}$ is a linearly independent (respectively, dependent) subset of a vector space V, then we may also say that $u_1, \ldots, u_n$ are linearly independent (respectively, dependent) in V.

Note that a linearly dependent set cannot be empty. In other words, the empty set is linearly independent!

Remark 1.2 If $u_1, \ldots, u_n$ are such that at least one of them is not in the span of the remaining, then we *cannot* conclude that $u_1, \ldots, u_n$ are linearly independent. For the linear independence of $\{u_1, \ldots, u_n\}$, it is required that $u_i \notin \operatorname{span}\{u_j : j \neq i\}$ for every $i \in \{1, \ldots, n\}$.

Also, if $\{u_1, \ldots, u_n\}$ are linearly dependent, then it *does not imply* that any one of them is in the span of the rest.

To illustrate the above points, consider two linearly independent vectors u_1, u_2. Then we have $u_1 \notin \operatorname{span}\{u_2, 3u_2\}$, but $\{u_1, u_2, 3u_2\}$ is linearly dependent, and $\{u_1, u_2, 3u_2\}$ is linearly dependent, but $u_1 \notin \operatorname{span}\{u_2, 3u_2\}$. $\Diamond$

Exercise 1.25 Let V be a vector space.

(i) Show that a subset $\{u_1, \ldots, u_n\}$ of V is linearly dependent if and only if there exists a nonzero $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{F}^n$ such that $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$.

(ii) Show that a subset $\{u_1, \ldots, u_n\}$ of V is linearly independent if and only if the function $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 u_1 + \cdots + \alpha_n u_n$ from $\mathbb{F}^n$ into V is injective.

(iii) Show that if $E \subseteq V$ is linearly independent in V, then $0 \notin E$.

(iv) Show that if $E \subseteq V$ is linearly dependent in V, then every superset of E is also linearly dependent.

(v) Show that if $E \subseteq V$ is linearly independent in V, then every subset of E is also linearly independent.

(vi) Show that if $\{u_1, \ldots, u_n\}$ is a linearly independent subset of V, and if Y is a subspace of V such that $\{u_1, \ldots, u_n\} \cap Y = \varnothing$, then every x in the span of $\{u_1, \ldots, u_n, Y\}$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$, $y \in Y$.

(vii) Show that if E_1 and E_2 are linearly independent subsets of V such that $(\operatorname{span} E_1 \cap (\operatorname{span} E_2) = \{0\}$, then $E_1 \cup E_2$ is linearly independent. $\Diamond$

Exercise 1.26 Let A be an $m \times n$ matrix of scalars with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$. Show the following:

(i) The equation $A\underline{x} = \underline{0}$ has a non-zero solution $\underline{x} \in \mathbb{F}^n$ if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ are linearly dependent.

(ii) For $\underline{y} \in \mathbb{F}^m$, the equation $A\underline{x} = \underline{y}$ has a solution $\underline{x} \in \mathbb{F}^n$ if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n, \underline{y}$ are linearly dependent, i.e., if and only if span $\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n, \underline{y}\} = \text{span}\,\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$, i.e., if and only if $\underline{y} \in \text{span}\,\{\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n\}$. $\Diamond$

Definition 1.11 (Basis) A subset E of a vector space V is said to be a *basis* of V if it is linearly independent and span $E = V$.

EXAMPLE 1.18 For each $j \in \{1, \ldots, n\}$, let $e_j \in \mathbb{F}^n$ be such that $e_j(i) = \delta_{ij}$, $i, j = 1, \ldots, n$. Then we have seen that $\{e_1, \ldots, e_n\}$ is linearly independent and its span is $\mathbb{F}^n$. Hence $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{F}^n$. $\Diamond$

EXAMPLE 1.19 For each $j \in \{1, \ldots, n\}$, let $\underline{e}_j \in \underline{\mathbb{F}}^n$ be such that $\underline{e}_j(i) = \delta_{ij}$, $i, j = 1, \ldots, n$. Then it is easily seen that $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is linearly independent and its span is $\underline{\mathbb{F}}^n$. Hence $\{\underline{e}_1, \ldots, \underline{e}_n\}$ is a basis of $\underline{\mathbb{F}}^n$. $\Diamond$

Definition 1.12 (Standard bases of $\mathbb{F}^n$ and $\underline{\mathbb{F}}^n$) The basis $\{e_1, \ldots, e_n\}$ of $\mathbb{F}^n$ is called the *standard basis* of $\mathbb{F}^n$, and the basis $\{\underline{e}_1, \ldots, \underline{e}_n\}$ of $\underline{\mathbb{F}}^n$ is called the *standard basis* of $\underline{\mathbb{F}}^n$.

EXAMPLE 1.20 Let $u_j = t^{j-1}$, $j \in \mathbb{N}$. Then $\{u_1, \ldots, u_{n+1}\}$ is a basis of $\mathcal{P}_n$, and $\{u_1, u_2, \ldots\}$ is a basis of $\mathcal{P}$. $\Diamond$

Exercise 1.27 Let $u_1 = 1$, and for $j = 2, 3, \ldots$, let $u_j = 1 + t + \ldots + t^{j-1}$. Show that $\{u_1, \ldots, u_{n+1}\}$ is a basis of $\mathcal{P}_n$, and $\{u_1, u_2, \ldots\}$ is a basis of $\mathcal{P}$. $\Diamond$

EXAMPLE 1.21 For $i = 1, \ldots, m$; $j = 1, \ldots, n$, let M_{ij} be the $m \times n$ matrix with its (i, j)-th entry as 1 and all other entries 0. Then

$$\{M_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$$

is a basis of $\mathbb{F}^{m \times n}$. $\Diamond$

Remark 1.3 A linearly independent subset of a subspace remains linearly independent in the whole space. $\Diamond$

Theorem 1.7 *Let V be a vector space and $E \subseteq V$. Then the following are equivalent.*

(i) E is a basis of V

(ii) E is a maximal linearly independent set in V, i.e., E is linearly independent, and a proper superset of E cannot be linearly independent.

(iii) E is a minimal spanning set of V, i.e., span of E is V, and a proper subset of E cannot span V.

Proof. $(i) \iff (ii)$: Suppose E is a basis of V. Suppose $\widetilde{E}$ is a proper superset of E. Let $x \in \widetilde{E} \setminus E$. Since E is a basis, $x \in \text{span}(E)$. This shows that $\widetilde{E}$ is linearly dependent, since $E \cup \{x\} \subseteq \widetilde{E}$.

Conversely, suppose E is a maximal linearly independent set. If E is not a basis, then there exists $x \notin \text{span}(E)$. Hence, it is seen that, $E \cup \{x\}$ is a linearly independent which is a proper superset of E – a contradiction to the maximality of E.

$(i) \iff (iii)$: Suppose E is a basis of V. Suppose F is a proper subset of E. Then, it is clear that there exists $x \in E \setminus F$ which is not in the span of F, since $F \cup \{x\} \subseteq E$. Hence, F does not span V.

Conversely, suppose E is a minimal spanning set of V. If E is not a basis, then E is linearly dependent, and hence there exists $x \in \text{span}(E \setminus \{x\})$. Since E spans V, it follows that $E \setminus \{x\}$, which is a proper subset of E, also spans V – a contradiction to the fact that E is a minimal spanning set of V. ∎

Exercise 1.28 For $\lambda \in [a, b]$, let $u_\lambda(t) = \exp(\lambda t)$, $t \in [a, b]$. Show that $\{u_\lambda : \lambda \in [a, b]\}$ is an uncountable linearly independent subset of $C[a, b]$. ♢

Exercise 1.29 If $\{u_1, \ldots, u_n\}$ is a basis of a vector space V, then show that every $x \in V$, can be expressed uniquely as

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n,$$

that is, for every $x \in V$, there exists a unique n-tuple $(\alpha_1, \ldots, \alpha_n)$ of scalars such that $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$. ♢

Exercise 1.30 Consider the system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2 \\ \ldots & + & \ldots & + & \ldots & + & \ldots & = & \ldots \\ a_{m1}x_1 & + & a_{m1}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m \end{array}$$

Show that the above system has at most one solution if and only if the vectors

$$w_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, \quad w_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix}, \ldots, \quad w_n := \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}$$

are linearly independent. ♢

Exercise 1.31 Let $u_1, \ldots, u_n$ be linearly independent vectors in a vector space V. Let $[a_{ij}]$ be an $m \times n$ matrix of scalar, and let

$$\begin{array}{lcccccccc} v_1 & := & a_{11}u_1 & + & a_{21}u_2 & + & \ldots & + & a_{m1}u_n \\ v_2 & := & a_{12}u_1 & + & a_{22}u_2 & + & \ldots & + & a_{m2}u_n \\ \ldots & & \ldots & + & \ldots & + & \ldots & + & \ldots \\ v_n & := & a_{1n}u_1 & + & a_{2n}u_2 & + & \ldots & + & a_{mn}u_n. \end{array}$$

Show that the $v_1, \ldots, v_m$ are linearly independent if and only if the vectors

$$w_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}, \quad w_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix}, \ldots, \quad w_n := \begin{bmatrix} a_{1n} \\ a_{m2} \\ \ldots \\ a_{mn} \end{bmatrix}$$

are linearly independent. ♢

Exercise 1.32 Let $p_1(t) = 1 + t + 3t^2$, $p_2(t) = 2 + 4t + t^2$, $p_3(t) = 2t + 5t^2$. Are the polynomials p_1, p_2, p_3 linearly independent? ♢

1.5.1 Dimension of a Vector Space

Definition 1.13 (Finite dimensional space) A vector space V is said to be a *finite dimensional space* if there is a finite basis for V.

Recall that the empty set is considered as a linearly independent set, and its span is the zero space.

Definition 1.14 (Infinite dimensional space) A vector space which is not a finite dimensional space is called an *infinite dimensional space.*

Theorem 1.8 *If a vector space has a finite spanning set, then it has a finite basis. In fact, if S is a finite spanning set of V, then there exists a basis $E \subseteq S$.*

Proof. Let V be a vector space and S be a finite subset of V such that $\operatorname{span} S = V$. If S itself is linearly independent, then we are through. Suppose S is not linearly independent. Then there exists $u_1 \in S$ such that $u_1 \in \operatorname{span}(S \setminus \{u_1\})$. Let $S_1 = S \setminus \{u_1\}$. Clearly,

$$\operatorname{span} S_1 = \operatorname{span} S = V.$$

If S_1 is linearly independent, then we are through. Otherwise, there exists $u_2 \in S_1$ such that $u_2 \in \operatorname{span}(S_1 \setminus \{u_2\})$. Let $S_2 = S \setminus \{u_1, u_2\}$. Then, we have

$$\operatorname{span} S_2 = \operatorname{span} S_1 = V.$$

If S_2 is linearly independent, then we are through. Otherwise, continue the above procedure. This procedure will stop after a finite number of steps, as the original set S is a finite set, and we end up with a subset S_k of S which is linearly independent and $\operatorname{span} S_k = V$. ∎

By definition, an infinite dimensional space cannot have a finite basis. Is it possible for a finite dimensional space to have an infinite basis, or an infinite linearly independent subset? The answer is, as expected, negative. In fact, we have the following result.

Theorem 1.9 *Let V be a finite dimensional vector space with a basis consisting of n elements. Then every subset of V with more than n elements is linearly dependent.*

Proof. Let $\{u_1, \ldots, u_n\}$ be a basis of V, and $\{x_1, \ldots, x_{n+1}\} \subset V$. We show that $\{x_1, \ldots, x_{n+1}\}$ is linearly dependent.

If $\{x_1, \ldots, x_n\}$ is linearly dependent, then $\{x_1, \ldots, x_{n+1}\}$ is linearly dependent. So, let us assume that $\{x_1, \ldots, x_n\}$ is linearly independent. Now, since $\{u_1, \ldots u_n\}$ is a basis of V, there exist scalars $\alpha_1, \ldots, \alpha_n$ such that

$$x_1 = \alpha_1 u_1 + \cdots + \alpha_n u_n.$$

Since $x_1 \neq 0$, one of $\alpha_1, \ldots, \alpha_n$ is nonzero. Without loss of generality, assume that $\alpha_1 \neq 0$. Then we have $u_1 \in \operatorname{span}\{x_1, u_2, \ldots, u_n\}$ so that

$$V = \operatorname{span}\{u_1, u_2, \ldots, u_n\} = \operatorname{span}\{x_1, u_2, \ldots, u_n\}.$$

Let $\alpha_1^{(2)}, \ldots, \alpha_n^{(2)}$ be scalars such that

$$x_2 = \alpha_1^{(2)} x_1 + \alpha_2^{(2)} u_2 + \cdots + \alpha_n^{(2)} u_n.$$

Since $\{x_1, x_2\}$ is linearly independent, at least one of $\alpha_2^{(2)}, \ldots, \alpha_n^{(2)}$ is nonzero. Without loss of generality, assume that $\alpha_2^{(2)} \neq 0$. Then we have $u_2 \in \operatorname{span}\{x_1, x_2, u_3, \ldots, u_n\}$ so that

$$V = \operatorname{span}\{x_1, u_2, \ldots, u_n\} = \operatorname{span}\{x_1, x_2, u_3, \ldots, u_n\}.$$

Now, let $1 \le k \le n-1$ be such that

$$V = \text{span}\,\{x_1, x_2, \ldots, x_k, u_{k+1}, \ldots, u_n\}.$$

Suppose $k < n-1$. Then there exist scalars $\alpha_1^{(k+1)}, \ldots, \alpha_n^{(k+1)}$ such that

$$x_{k+1} = \alpha_1^{(k+1)} x_1 + \cdots + \alpha_k^{(k+1)} x_k + \alpha_{k+1}^{(k+1)} u_{k+1} + \cdots + \alpha_n^{(k+1)} u_n.$$

Since $\{x_1, \ldots, x_{k+1}\}$ is linearly independent, at least one of the scalars $\alpha_{k+1}^{(k+1)}, \ldots, \alpha_n^{(k+1)}$ is nonzero. Without loss of generality, assume that $\alpha_{k+1}^{(k+1)} \neq 0$. Then we have $u_{k+1} \in \text{span}\,\{x_1, \ldots, x_{k+1}, u_{k+2}, \ldots, u_n\}$ so that

$$\begin{aligned} V &= \text{span}\,\{x_1, \ldots, x_k, u_{k+1}, \ldots, u_n\} \\ &= \text{span}\,\{x_1, \ldots, x_{k+1}, u_{k+2}, \ldots, u_n\}. \end{aligned}$$

Thus, the above procedure leads to $V = \text{span}\,\{x_1, \ldots, x_{n-1}, u_n\}$ so that there exist scalars $\alpha_1^{(n)}, \ldots, \alpha_n^{(n)}$ such that

$$x_n = \alpha_1^{(n)} x_1 + \cdots + \alpha_{n-1}^{(n)} x_{n-1} + \alpha_n^{(n)} u_n.$$

Since $\{x_1, \ldots, x_n\}$ is linearly independent, it follows that $\alpha_n^{(n)} \neq 0$. Hence,

$$u_n \in \text{span}\,\{x_1, \ldots, x_n\}.$$

Consequently,

$$V = \text{span}\,\{x_1, x_2, \ldots, x_{n-1}, u_n\} = \text{span}\,\{x_1, x_2, \ldots, x_{n-1}, x_n\}.$$

Thus, $x_{n+1} \in \text{span}\,\{x_1, \ldots, x_n\}$, showing that $\{x_1, \ldots, x_{n+1}\}$ is linearly dependent. ∎

The following three corollaries are easy consequences of Theorem 1.9. Their proofs are left as exercises for the reader.

Corollary 1.10 *If V is a finite dimensional vector space, then any two bases of V have the same number of elements.*

Corollary 1.11 *If a vector space contains an infinite linearly independent subset, then it is an infinite dimensional space.*

Corollary 1.12 *If (a_{ij}) is an $m \times n$ matrix with $a_{ij} \in \mathbb{F}$ and $n > m$, then there exists a nonzero $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ such that*

$$a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = 0, \qquad i = 1, \ldots, m.$$

Exercise 1.33 Assuming Corollary 1.12, give an alternate proof for Theorem 1.9. $\diamondsuit$

By Corollary 1.12, we see that if $A \in \underline{\mathbb{F}}^{m\times n}$, then there exists $\underline{\mathbb{F}}^n$ such that

$$A\underline{x} = \underline{0}.$$

Definition 1.15 (n-vector) An $n \times 1$ matrix is also called an *n-vector.*

In view of Corollary 1.10, the following definition makes sense.

Definition 1.16 (Dimension) Suppose V is a finite dimensional vector space. Then the *dimension* of V is the number of elements in a basis of V, and this number is denoted by $\dim V$. If V is infinite dimensional, then its dimension is defined to be infinity and we write $\dim X = \infty$.

EXAMPLE 1.22 The spaces $\mathbb{F}^n$ and $\mathcal{P}_{n-1}$ are of dimension n. $\diamondsuit$

EXAMPLE 1.23 It is seen that the set $\{e_1, e_2, \ldots, \} \subseteq \mathcal{F}(\mathbb{N}, \mathbb{F})$ with $e_j(i) = \delta_{ij}$ is a linearly independent subset of the spaces $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$. Hence, it follows that $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are infinite dimensional spaces. $\diamondsuit$

EXAMPLE 1.24 We see that $\{u_1, u_2, \ldots, \}$ with $u_j(t) = t^{j-1}$, $j \in \mathbb{N}$, is linearly independent in $C^k[a, b]$ for every $k \in \mathbb{N}$. Hence, the space $C^k[a, b]$ for each $k \in \mathbb{N}$ is infinite dimensional. $\diamondsuit$

EXAMPLE 1.25 Suppose S is a finite set consisting of n elements. Then $\mathcal{F}(S, \mathbb{F})$ is of dimension n. To see this, let $S = \{s_1, \ldots, s_n\}$, and for each $j \in \{1, \ldots, n\}$, define $f_j \in \mathcal{F}(S, \mathbb{F})$ by

$$f_j(s_i) = \delta_{ij}, \qquad i \in \{1, \ldots, n\}.$$

Then the set $\{f_1, \ldots, f_n\}$ is a basis of $\mathcal{F}(S, \mathbb{F})$: Clearly,

$$\sum_{j=1}^{n} \alpha_j f_j = 0 \implies \alpha_i = \sum_{j=1}^{n} \alpha_j f_j(s_i) = 0 \quad \forall\, i.$$

Thus, $\{f_1, \ldots, f_n\}$ is linearly independent. Also, note that

$$f = \sum_{j=1}^{n} f(s_j) f_j \quad \forall\, f \in \mathcal{F}(S, \mathbb{F}).$$

Thus span $\{f_1, \ldots, f_n\} = \mathcal{F}(S, \mathbb{F})$. ◊

1.5.2 Dimension of Sum of Subspaces

Theorem 1.13 *Suppose V_1 and V_2 are subspaces of a finite dimensional vector space V. If $V_1 \cap V_2 = \{0\}$, then*

$$\dim (V_1 + V_2) = \dim V_1 + \dim V_2.$$

Proof. Suppose $\{u_1, \ldots, u_k\}$ is a basis of V_1 and $\{v_1, \ldots, v_\ell\}$ is a basis of V_2. We show that $E := \{u_1, \ldots, u_k, v_1, \ldots, v_\ell\}$ is a basis of $V_1 + V_2$. Clearly (*Is it clear?*) span $E = V_1 + V_2$. So, it is enough to show that E is linearly independent. For this, suppose $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell$ are scalars such that $\alpha_1 u_1 + \ldots + \alpha_k u_k + \beta_1 v_1 + \ldots + \beta_\ell v_\ell = 0$. Then we have

$$x := \alpha_1 u_1 + \ldots + \alpha_k u_k = -(\beta_1 v_1 + \ldots + \beta_\ell v_\ell) \in V_1 \cap V_2 = \{0\}$$

so that $\alpha_1 u_1 + \ldots + \alpha_k u_k = 0$ and $\beta_1 v_1 + \ldots + \beta_\ell v_\ell = 0$. From this, by the linearly independence of u_i's and v_j's, it follows that $\alpha_i = 0$ for $i \in \{1, \ldots, k\}$ and $\beta_j = 0$ for all $j \in \{1, \ldots, \ell\}$. Hence, E is linearly independent. This completes the proof. ∎

In fact, the above theorem is a particular case of the following.

Theorem 1.14 *Suppose V_1 and V_2 are subspaces of a finite dimensional vector space V. Then*

$$\dim (V_1 + V_2) = \dim V_1 + \dim V_2 - \dim (V_1 \cap V_2).$$

For the proof of the above theorem we shall make use of the following result.

Proposition 1.15 *Let V be a finite dimensional vector space. If E_0 is a linearly independent subset of V, then there exists a basis E of V such that $E_0 \subseteq E$.*

Proof. Let $E_0 = \{u_1, \ldots, u_k\}$ be a linearly independent subset of V, and let $\{v_1, \ldots, v_n\}$ be a basis of V. Let

$$E_1 = \begin{cases} E_0 & \text{if } v_1 \in \text{span}\,(E_0), \\ E_0 \cup \{v_1\} & \text{if } v_1 \notin \text{span}\,(E_0). \end{cases}$$

Clearly, E_1 is linearly independent, and

$$E_0 \subseteq E_1, \qquad \{v_1\} \subseteq \text{span}\,(E_1).$$

Then define

$$E_2 = \begin{cases} E_1 & \text{if } v_2 \in \text{span}\,(E_1), \\ E_1 \cup \{v_2\} & \text{if } v_2 \notin \text{span}\,(E_1). \end{cases}$$

Again, it is clear that E_2 is linearly independent, and

$$E_1 \subseteq E_2, \qquad \{v_1, v_2\} \subseteq \text{span}\,(E_2).$$

Having defined $E_1, \ldots, E_j$, $j < n$, we define

$$E_{j+1} = \begin{cases} E_j & \text{if } v_{j+1} \in \text{span}\,(E_j), \\ E_j \cup \{v_{j+1}\} & \text{if } v_{j+1} \notin \text{span}\,(E_j). \end{cases}$$

Thus, we get linearly independent sets $E_1, E_2, \ldots, E_n$ such that

$$E_0 \subseteq E_1 \subseteq \ldots E_n, \qquad \{v_1, v_2, \ldots, v_n\} \subseteq \text{span}\,(E_n).$$

Since $\{v_1, \ldots, v_n\}$ is a basis of V, it follows that $E := E_n$ is a basis of V such that $E_0 \subseteq E_n = E$. ∎

Proof of Theorem 1.14. Let $\{u_1, \ldots, u_k\}$ be a basis of the subspace $V_1 \cap V_2$. By Proposition 1.15, there exists $v_1, \ldots, v_\ell$ in V_1 and $w_1, \ldots, w_m$ in V_2 such that $\{u_1, \ldots, u_k, v_1, \ldots, v_\ell\}$ is a basis of V_1, and $\{u_1, \ldots, u_k, w_1, \ldots, w_m\}$ is a basis of V_2. We show that $E := \{u_1, \ldots, u_k, v_1, \ldots, v_\ell, w_1, \ldots, w_m\}$ is a basis of $V_1 + V_2$.

Clearly, $V_1 + V_2 = \text{span}\,(E)$. Hence, it is enough to show that E is linearly independent. For this, let $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell, \gamma_1, \ldots, \gamma_m$ be scalars such that

$$\sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{\ell} \beta_i v_i + \sum_{i=1}^{m} \gamma_i w_i = 0. \qquad (*)$$

Then

$$x := \sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{\ell} \beta_i v_i = -\sum_{i=1}^{m} \gamma_i w_i \in V_1 \cap V_2.$$

Hence, there exists scalars $\delta_1, \ldots, \delta_k$ such that

$$\sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{\ell} \beta_i v_i = \sum_{i=1}^{k} \delta_i u_i, \quad i.e., \quad \sum_{i=1}^{k} (\alpha_i - \delta_i) u_i + \sum_{i=1}^{\ell} \beta_i v_i = 0$$

Since $\{u_1, \ldots, u_k, v_1, \ldots, v_\ell\}$ is a basis of V_1, it follows that $\alpha_i = \delta_i$ for all $i = 1, \ldots, k$, and $\beta_j = 0$ for $j = 1, \ldots, \ell$. Hence, from $(*)$,

$$\sum_{i=1}^{k} \alpha_i u_i + \sum_{i=1}^{m} \gamma_i w_i = 0.$$

Now, since $\{u_1, \ldots, u_k, w_1, \ldots, w_m\}$ is a basis of V_2, it follows that $\alpha_i = 0$ for all $i = 1, \ldots, k$, and $\gamma_j = 0$ for all $j = 1, \ldots, m$.

Thus, we have shown that $\{u_1, \ldots, u_k, v_1, \ldots, v_\ell, w_1, \ldots, w_m\}$ is a basis of $V_1 + V_2$. Since $\dim(V_1 + V_2) = k + \ell + m$, $\dim V_1 = k + \ell$, $\dim V_2 = k + m$ and $\dim(V_1 \cap V_2) = k$, we get

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

This completes the proof. ∎

Exercise 1.34 Prove that if V is a finite dimensional vector space and V_0 is a proper subspace of V, then $\dim(V_0) < \dim(V)$. ◊

2

Linear Transformations

2.1 Introduction

We may recall from the theory of matrices that if A is an $m \times n$ matrix, and if $\underline{x}$ is an n-vector, then $A\underline{x}$ is an m-vector. Moreover, for any two n-vectors $\underline{x}$ and $\underline{y}$, and for every scalar α,

$$A(\underline{x} + \underline{y}) = A\underline{x} + A\underline{y}, \qquad A(\alpha\underline{x}) = \alpha A\underline{x}).$$

Also, we recall from calculus that if f and g are real-valued differentiable functions (defined on an interval J), and α is a scalar, then

$$\frac{d}{dt}(f + g) = \frac{d}{dt}f + \frac{d}{dt}g, \qquad \frac{d}{dt}(\alpha f) = \alpha\frac{d}{dt}f.$$

Note also that, if f and g are continuous real-valued functions defied on an interval $[a, b]$, then

$$\int_a^b (f+g)(t)dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt, \qquad \int_a^b (\alpha f)(t) = \alpha \int_a^b f(t)\,dt,$$

and for every $s \in [a, b]$,

$$\int_a^s (f+g)(t)dt = \int_a^s f(t)\,dt + \int_a^s g(t)\,dt, \qquad \int_a^s (\alpha f)(t) = \alpha \int_a^s f(t)\,dt.$$

Abstracting the above operations between specific vector spaces, we define the notion of a linear transformation between general vector spaces.

2.2 What is a Linear Transformation?

Definition 2.1 (Linear transformation) Let V_1 and V_2 be vector spaces (over the same scalar field $\mathbb{F}$). A a function $T : V_1 \to V_2$ is said to be a *linear transformation* or a *linear operator* from V_1 to V_2 if

$$T(x+y) = T(x) + T(y), \qquad T(\alpha x) = \alpha T(x)$$

for every $x, y \in V_1$ and for every $\alpha \in \mathbb{F}$.

A linear transformation with codomain space as the scalar field is called a **linear functional**. $\Diamond$

- Linear functionals are usually denoted by small-scale letters.
- If $V_1 = V_2 = V$, and if if $T : V \to V$ is a linear transformation, then we say that T is a linear transformation on V.
- The linear transformation which maps each $x \in V$ onto itself is called the *identity transformation* on V, and is usually denoted by I. Thus, $I : V \to V$ is defined by

$$I(x) = x \qquad \forall\, x \in V.$$

- If $T : V_1 \to V_2$ is a linear transformation, then for $x \in V_1$, we shall the element $T(x) \in V_2$ also by Tx..

EXAMPLE 2.1 (Multiplication by a scalar) Let V be a vector space and λ be a scalar. Define $T : V \to V$ by $T(x) = \lambda x$, $x \in V$. Then we see that T is a linear transformation. $\Diamond$

EXAMPLE 2.2 (Matrix as linear transformation) Let $A = (a_{ij})$ be an $m \times n$-matrix of scalars. For $x \in \underline{\mathbb{F}}^n$, let $T(x) = Ax$ for every $x \in \underline{\mathbb{F}}^n$. Then, it can be easily seen that $T : \underline{\mathbb{F}}^n \to \underline{\mathbb{F}}^m$ is a linear transformation.

Note that

$$Tx = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}.$$

$\Diamond$

Another example similar the above:

EXAMPLE 2.3 Let $A = (a_{ij})$ be an $m \times n$-matrix of scalars. For $x = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{F}^n$, let

$$Tx = (\beta_1, \ldots, \beta_m), \qquad \beta_i = \sum_{j=1}^{n} a_{ij}\alpha_j, \quad i = 1, \ldots, m.$$

Then $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation. $\Diamond$

More generally, we have the following.

EXAMPLE 2.4 Let V_1 and V_2 be finite dimensional vector space with bases $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ respectively. Let $A = (a_{ij})$ be an $m \times n$-matrix of scalars. For $x = \sum_{j=1}^{n} \alpha_j u_j \in V$, let

$$Tx = \sum_{i=1}^{m} \beta_i v_i \quad \text{with} \quad \beta_i = \sum_{j=1}^{n} a_{ij}\alpha_j \quad \text{for} \quad i \in \{1, \ldots, m\}.$$

Then $T : V_1 \to V_2$ is a linear transformation. Thus,

$$T\Big(\sum_{j=1}^{n} \alpha_j u_j\Big) := \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij}\alpha_j\Big).$$

$\Diamond$

EXAMPLE 2.5 For each $j \in \{1, \ldots, n\}$, the function $f_j : \mathbb{F}^n \to \mathbb{F}$ defined by $f_j(x) = x_j$ for $x = (\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$, is a linear functional. $\Diamond$

More generally, we have the following example.

EXAMPLE 2.6 Let V be an n-dimensional space and let $E = \{u_1, \ldots, u_n\}$ be a basis of V. For $x = \sum_{j=1}^{n} \alpha_j u_j \in V$, and for each $j \in \{1, \ldots, n\}$, define $f_j : V \to \mathbb{F}$ by

$$f_j(x) = \alpha_j.$$

Then f_j is a linear functional. $\Diamond$

Definition 2.2 The linear functionals $f_1, \ldots, f_n$ defined as in Example 2.6 are called **coordinate functionals** on V with respect to the basis E of V. $\Diamond$

Remark 2.1 We observe that if $f_1, \ldots, f_n$ are the coordinate functionals on V with respect to the basis $E = \{u_1, \ldots, u_n\}$ of V, then

$$f_j(u_i) = \delta_{ij} \quad \forall\, i, j = 1, \ldots, n.$$

It is to be remarked that these linear functionals depend not only on the basis $E = \{u_1, \ldots, u_n\}$, but also on the order in which $u_1, \ldots, u_n$ appear in the representation of any $x \in V$. ◊

EXAMPLE 2.7 (Evaluation of functions) For a given point $\tau \in [a, b]$, let $f_\tau : C[a, b] \to \mathbb{F}$ be defined by

$$f_\tau(x) = x(\tau), \qquad x \in C[a, b].$$

Then f_τ is a linear functional. ◊

More generally, we have the following example.

EXAMPLE 2.8 Given points $\tau_1, \ldots, \tau_n$ in $[a, b]$, and $\omega_1, \ldots \omega_n$ in $\mathbb{F}$, let $T : C[a, b] \to \mathbb{F}$ be defined by

$$f(x) = \sum_{i=1}^{n} x(\tau_i)\omega_i, \qquad x \in C[a, b].$$

Then f is a linear functional. ◊

EXAMPLE 2.9 (Differentiation) Let $T : C^1[a, b] \to C[a, b]$ be defined by

$$Tx = x', \qquad x \in C^1[a, b],$$

where f' denotes the derivative of f. Then T is a linear transformation. ◊

EXAMPLE 2.10 For $\lambda, \mu \in \mathbb{F}$, the function $T : C^1[a, b] \to C[a, b]$ defined by

$$Tx = \lambda x + \mu x', \qquad x \in C^1[a, b],$$

is a linear transformation. ◊

More generally, we have the following example.

EXAMPLE 2.11 Let T_1 and T_2 be linear transformations from V_1 to V_2 and λ and μ be scalars. Then $T : V_1 \to V_2$ defined by

$$T(x) = \lambda T_1(x) + \mu T_2(x), \quad x \in V_1,$$

is a linear transformation.

◊

EXAMPLE 2.12 (Definite integration) Let $T : C[a,b] \to \mathbb{F}$ be defined by

$$f(x) = \int_a^b x(t)\,dt, \qquad x \in C[a,b].$$

Then f is a linear functional. $\Diamond$

EXAMPLE 2.13 (Indefinite integration) Let $T : C[a,b] \to C[a,b]$ be defined by

$$(Tx)(s) = \int_a^s x(t)\,dt, \qquad x \in C[a,b], \quad s \in [a,b].$$

Then T is a linear transformation. $\Diamond$

EXAMPLE 2.14 Let V be a finite dimensional vector space and $E_1 = \{u_1, \ldots, u_n\}$ be a basis of V. For $\underline{x} \in \underline{\mathbb{F}}^n$, let

$$T\underline{x} = \sum_{j=1}^{n} \underline{x}_j u_j.$$

Then $T : \underline{\mathbb{F}}^n \to V$ is a linear transformation. $\Diamond$

In the above example, $\underline{e_k} \in \underline{\mathbb{F}}^n$ such that $\underline{e_k}_j = \delta_{kj}$, then $T\underline{e_k} = u_k$ for $k \in \{1, \ldots, n\}$. More generally, we have the following.

EXAMPLE 2.15 Let V_1 and V_2 be vector spaces with $\dim V_1 = n$. Let $E_1 = \{u_1, \ldots, u_n\}$ be a basis of V_1 and $E_2 = \{v_1, \ldots, v_m\}$ be a subset of V_2. For $x = \sum_{j=1}^{n} \alpha_j u_j \in V_1$, define $T : V_1 \to V_2$ by

$$Tx = \sum_{i=1}^{m} \alpha_i v_i.$$

Then T is a linear transformation. $\Diamond$

Exercise 2.1 Show that the linear transformation T in Example 2.15 is

(a) one-one if and only if E_2 is linearly independent,

(b) onto if and only if $\operatorname{span}(E_2) = V_2$. $\Diamond$

Exercise 2.2 Let V_1 and V_2 be vector spaces, $E_1 = \{u_1, \ldots, u_n\}$ be a linearly independent subset of V_1 and $E_2 = \{v_1, \ldots, v_m\}$ be a subset of V_2.

(a) Show that there exists a linear transformation $T : V_1 \to V_2$ such that

$$Tu_j = v_j, \qquad j \in \{1, \ldots, n\}.$$

(a) Show that the transformation T in (a) is unique if and only if E_1 is a basis of V_1. ♢

Exercise 2.3 Let V_1 and V_2 be vector spaces, and V_0 be a subspace of V_1. Let $T_0 : V_0 \to V_2$ be a linear transformation. Show that there exists a linear transformation $T : V_1 \to V_2$ such that $T_{|V_0} = A_0$. ♢

Theorem 2.1 *Let $T : V_1 \to V_2$ be a linear transformation which is one to one and onto. Then $T^{-1} : V_2 \to V_1$ is also a linear transformation.*

Proof. For $y_1, y_2 \in V_2$, let $x_1, x_2 \in V_1$ be such that $Tx_1 = y_1$ and $Tx_2 = y_2$. Then for $\alpha, \beta \in \mathbb{F}$, by linearity of T, we have

$$\alpha y_1 + \beta y_2 = \alpha Tx_1 + \beta Tx_2 = T(\alpha x_1 + \beta x_2).$$

Hence,

$$T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2 = \alpha T^{-1}(x_1) + \beta T^{-1}(x_2).$$

This completes the proof. ∎

Definition 2.3 (Isomorphism of vector spaces) Vector spaces V_1 and V_2 are said to be *linearly isomorphic* if there exists a bijective linear transformation $T : V_1 \to V_2$, and in that case T is called a *linear isomorphism* from V_1 onto V_2. ♢

Theorem 2.2 *Any two finite dimensional vector spaces of the same dimension are linearly isomorphic.*

Proof. Let V_1 and V_2 be finite dimensional vector spaces of the same dimension, say n. Let $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_n\}$ be bases of V_1 and V_2, respectively. For $x = \sum_{j=1}^{n} \alpha_j u_j \in V_1$, define $T : V_1 \to V_2$ by

$$Tx = \sum_{i=1}^{m} \alpha_i v_i.$$

Clearly, T is a linear transformation. Note that T is bijective as well: For $x = \sum_{j=1}^{n} \alpha_j u_j \in V_1$,

$$Tx = 0 \Longrightarrow (\alpha_1, \ldots, \alpha_n) = 0 \Longrightarrow x = 0.$$

Also, for $y = \sum_{j=1}^{n} \beta_j v_j \in V_2$, the element $x = \sum_{j=1}^{n} \beta_j u_j \in V_1$, satisfies $Tx = y$. ∎

EXAMPLE 2.16 Let $\dim(V) = n$ and $E = \{u_1, \ldots, u_n\}$ be an ordered basis of V. Then for $x = \sum_{i=1}^{n} \alpha_i u_i \in V$, $T_1 : V \to \mathbb{F}^n$ and $T_2 : V \to \mathbb{F}^n$ defined by

$$T_1 x = (\alpha_1, \ldots, \alpha_n), \qquad T_2 x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

are a bijective linear isomorphisms.

The above isomorphisms are called *canonical isomorphisms* between the spaces involved w.r.t. the basis E. ♢

2.3 Space of Linear Transformations

Let $\mathcal{L}(V_1, V_2)$ denote the set of all linear transformations from V_1 to V_2. On $\mathcal{L}(V_1, V_2)$ we define addition and scalar multiplication *pointwise*, i.e., for T, T_1, T_2 in $\mathcal{L}(V_1, V_2)$ and $\alpha \in \mathbb{F}$, linear transformations $T_1 + T_2$ and αT are defined by

$$(T_1 + T_2)(x) = T_1 x + T_2 x,$$

$$(\alpha T)(x) = \alpha T x$$

for all $x \in V$. Then it is seen that $\mathcal{L}(V_1, V_2)$ is a vector space with its zero element as the zero operator $O : V_1 \to V_2$ defined by

$$Ox = 0 \quad \forall\, x \in V_1$$

and the additive inverse $-T$ of $T \in \mathcal{L}(V_1, V_2)$ is $-T : V_1 \to V_2$ defined by

$$(-T)(x) = -Tx \quad \forall\, x \in V_1.$$

Definition 2.4 The space $\mathcal{L}(V, \mathbb{F})$ of all linear functionals on V is called the **dual space** of V, and this space is also denoted by V'. ♢

Theorem 2.3 *Let V be a finite dimensional vector space, and let $E = \{u_1, \ldots, u_n\}$ be a basis of V. If $f_1, \ldots, f_n$ are the coordinate functionals on V with respect to E, then we have the following:*

(i) *Every $x \in V$ can be written as $x = \sum_{j=1}^{n} f_j(x) u_j$.*

(ii) *$\{f_1, \ldots, f_n\}$ is a basis of V'.*

Proof. Since $E = \{u_1, \ldots, u_n\}$ is a basis of V, for every $x \in V$, there exist unique scalars $\alpha_1, \ldots \alpha_n$ such that $x = \sum_{j=1}^n \alpha_j u_j$. Now, using the relation $f_i(u_j) = \delta_{ij}$, it follows that

$$f_i(x) = \sum_{j=1}^n \alpha_j f_i(u_j) = \alpha_i, \qquad i = 1, \ldots, n.$$

Therefore, the result in (i) follows.

To see (ii), first we observe that if $\sum_{i=1}^n \alpha_i f_i = 0$, then

$$\alpha_j = \sum_{i=1}^n \alpha_i f_i(u_j) = 0 \quad \forall\, j = 1, \ldots, n.$$

Hence, $\{f_1, \ldots, f_n\}$ is linearly independent in $\mathcal{L}(V, \mathbb{F})$. It remains to show that the span $\{f_1, \ldots, f_n\} = \mathcal{L}(V, \mathbb{F})$. For this, let $f \in \mathcal{L}(V, \mathbb{F})$ and $x \in V$. Then using the representation of x in (i), we have

$$f(x) = \sum_{j=1}^n f_j(x) f(u_j) = \left(\sum_{j=1}^n f(u_j) f_j\right)(x)$$

for all $x \in V$. Thus, $f = \sum_{j=1}^n f(u_j) f_j$ so that $f \in \text{span}\,\{f_1, \ldots, f_n\}$. This completes the proof. ∎

Definition 2.5 ((Dual basis)) Let V be a finite dimensional vector space and let $E = \{u_1, \ldots, u_n\}$ be a basis of V, and $f_1, \ldots, f_n$ be the associated coordinate functionals. The basis $F := \{f_1, \ldots, f_n\}$ of V' is called the *dual basis* of E, or *dual to the basis* E. ◊

2.4 Matrix Representations

Let V_1 and V_2 be finite dimensional vector spaces, and $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be ordered bases of V_1 and V_2, respectively. In Example 2.4 we have seen that an $m \times n$ matrix of scalars induces a linear transformation from V_1 to V_2. Now, we show the reverse.

Let $T : V_1 \to V_2$ be a linear transformation. Note that for every $x \in V_1$, there exists a unique $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ such that $x = \sum_{j=1}^n \alpha_j u_j$. Then, by the linearity of T, we have

$$T(x) = \sum_{j=1}^n \alpha_j T(u_j).$$

Since $T(u_j) \in V_2$ for each $j = 1, \ldots, n$ and $\{v_1, \ldots, v_m\}$ is a basis of V_2, Tu_j can be written as

$$T(u_j) = \sum_{i=1}^{m} a_{ij} v_i$$

for some scalars $a_{1j}, a_{2j}, \ldots, a_{mj}$. Thus,

$$T(x) = \sum_{j=1}^{n} \alpha_j T u_j = \sum_{j=1}^{n} \alpha_j \Big(\sum_{i=1}^{m} a_{ij} v_i \Big) = \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} a_{ij} \alpha_j \Big) v_i. \qquad (*)$$

For $x := \sum_{i=1}^{n} \alpha_i u_i \in V_1$, let $\vec{x} \in \mathbb{F}^n$ be the column vector $[\alpha_1, \ldots, \alpha_n]^T$. Then the relation $(*)$ connecting the linear transformation T and the matrix $A = (a_{ij})$ can be written as

$$Tx = \sum_{i=1}^{m} (A\vec{x})_i v_i.$$

In view of the above representation of T, we say that the $m \times n$ matrix $A := (a_{ij})$ is the **matrix representation** of T, with respect to the ordered bases E_1 and E_2 of V_1 and V_2 respectively. This fact is written as

$$[T]_{E_1,E_2} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

or simply $[T] = (a_{ij})$ when the bases are understood.

Definition 2.6 The matrix $[T]_{E_1,E_2}$ is called the *matrix representation* of T, with respect to $\{E_1, E_2\}$. $\Diamond$

Clearly, the above discussion also shows that for every $m \times n$ matrix $A = (a_{ij})$, there exists a unique linear transformation $T \in \mathcal{L}(V_1, V_2)$ such that $[T] = (a_{ij})$. Thus, there is a one-one correspondence between $\mathcal{L}(V_1, V_2)$ onto $\mathbb{F}^{m \times n}$, namely,

$$T \mapsto [T].$$

Suppose $J_1 : V_1 \to \underline{\mathbb{F}}^n$ and $J_2 : V_2 \to \underline{\mathbb{F}}^m$ be the canonical isomorphisms, that is, for $x = \sum_{i=1}^n \alpha_i u_i \in V_1$ and $y = \sum_{j=1}^n \beta_i v_i \in V_2$,

$$J_1(x) := \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}, \qquad J_2(x) := \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Then we see that $J_2 T J_1^{-1} : \underline{\mathbb{F}}^n \to \underline{\mathbb{F}}^m$ and $J_2^{-1}[T]J_1 : V_1 \to V_2$ are linear transformations such that

$$J_2 T J_1^{-1} \underline{x} = [T]\underline{x}, \qquad \underline{x} \in \underline{\mathbb{F}}^n,$$

$$J_2^{-1}[T]J_1 x = Tx, \qquad x \in V_1.$$

Exercise 2.4 Prove the last statement. ♢

Exercise 2.5 Let V be an n-dimensional vector space and $\{u_1, \ldots, u_n\}$ be an ordered basis of V. Let f be a linear functional on V. Prove the following:

(i) There exists a unique $(\beta_1, \ldots, \beta_n) \in \mathbb{F}^n$ such that

$$f(\alpha_1 u_1 + \ldots + \alpha_n u_n) = \alpha_1 \beta_1 + \ldots + \alpha_n \beta_n.$$

(ii) The matrix representations of $[f]_{E,\{1\}}$ is $[\beta_1 \cdots \beta_n]$. ♢

Exercise 2.6 Let V_1 and V_2 be finite dimensional vector spaces, and $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively. Show the following:

(a) If $\{g_1, \ldots, g_m\}$ is the dual of E_2, then for every $T \in \mathcal{L}(V_1, V_2)$,

$$[T]_{E_1,E_2} = \Big(g_i(Tu_j)\Big).$$

(b) If $T_1, T_2 \in \mathcal{L}(V_1, V_2)$ and $\alpha \in \mathbb{F}$, then

$$[T_1 + T_2]_{E_1,E_2} = [T_1]_{E_1,E_2} + [T_2]_{E_1,E_2}, \qquad [\alpha T]_{E_1,E_2} = \alpha [T]_{E_1,E_2}.$$

(c) Suppose $\{A_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$ is a basis of $\mathbb{F}^{m \times n}$. If $T_{ij} \in \mathcal{L}(V_1, V_2)$ is such that $[T_{ij}]_{E_1,E_2} = A_{ij}$, then

$$\{T_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$$

is a basis of $\mathcal{L}(V_1, V_2)$. (e.g., A_{ij} as in Example 1.21. ♢

Exercise 2.7 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_2 + x_3,\ x_3 + x_1,\ x_1 + x_2).$$

Find $[T]_{E_1,E_2}$ in each of the following cases.
(a) $\{(1,0,0),(0,1,0),(0,0,1)\},\ E_2 = \{(1,0,0),(1,1,0),(1,1,1)\}$
(b) $E_1 = \{(1,0,0),(1,1,0),(1,1,1)\},\ E_2 = \{(1,0,0),(0,1,0),(0,0,1)\}$
(c) $E_1 = \{(1,1,-1),(-1,1,1),(1,-1,1)\}$,
$E_2 = \{(-1,1,1),(1,-1,1),(1,1,-1)$ ♢

Exercise 2.8 Let $T : \mathcal{P}^3 \to \mathcal{P}^2$ be defined by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2.$$

Find Find $[T]_{E_1,E_2}$ in each of the following cases.
(a) $E_1 = \{1, t, t^2, t^3\}$, $E_2 = \{1 + t, 1 - t, t^2\}$
(b) $E_1 = \{1, 1 + t, 1 + t + t^2, t^3\}$, $E_2 = \{1, 1 + t, 1 + t + t^2\}$
(c) $E_1 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$, $E_2 = \{t^2, t, 1\}$
♢

Exercise 2.9 Let $T : \mathcal{P}^2 \to \mathcal{P}^3$ be defined by

$$T(a_0 + a_1 t + a_2 t^2) = a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3.$$

Find Find $[T]_{E_1,E_2}$ in each of the following cases.
(a) $E_1 = \{1 + t, 1 - t, t^2\}$, $E_2 = \{1, t, t^2, t^3\}$,
(b) $E_1 = \{1, 1 + t, 1 + t + t^2\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, t^3\}$,
(c) $E_1 = \{t^2, t, 1\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$, ♢

2.5 Rank and Nullity

Let V_1 and V_2 be vector spaces and $T : V_1 \to V_2$ be a linear transformation. Then it is easily seen that the sets

$$R(T) = \{Tx : x \in V_1\}, \qquad N(T) = \{x \in V_1 : Tx = 0\}$$

are subspaces of V_1 and V_2, respectively (Verify!).

Definition 2.7 (Range and null space) The subspaces $R(T)$ and $N(T)$ associated with a linear transformation $T : V_1 \to V_2$ are called the *range* of T and *null space* of T, respectively.

Definition 2.8 (Rank and nullity) The dimension of $R(T)$ is called the *rank* of T, denoted by $\operatorname{rank} T$, and the dimension of $N(T)$ is called the *nullity* of T, denoted by $\operatorname{null} T$.

Let $T : V_1 \to V_2$ be a linear transformation. We observe that

- T is onto or surjective if and only if $R(T) = V_2$,
- T is one-one or injective if and only if $N(T) = \{0\}$.

The proof of the following theorem is easy, and hence left as an exercise.

Theorem 2.4 *Let $T : V_1 \to V_2$ be a linear transformation. The we have the following.*

(a) *If $u_1, \ldots, u_k$ are linearly independent in V_1 and if T is one-one, then $Tu_1, \ldots, Tu_k$ are linearly independent in V_2.*

(b) *If $\{u_1, \ldots, u_k\} \subset V_1$ is such that $Tu_1, \ldots, Tu_k$ are linearly independent subset in V_2, then $u_1, \ldots u_k$ are linearly independent in V_2.*

From the above theorem we can deduce the following theorem.

Theorem 2.5 *Let V_1 and V_2 be finite dimensional vector spaces and $T : V_1 \to V_2$ be a linear transformation. Then T is one-one if and only if* $\operatorname{rank} T = \dim V_1$. *In particular, if* $\dim V_1 = \dim V_2$, *then*

T is one-one if and only if T is onto.

In fact, the above result is a particular case of the following theorem as well.

Theorem 2.6 (Rank-nullity theorem) *Let V_1 and V_2 be vector spaces and $T : V_1 \to V_2$ be a linear transformation. Then*

$$\operatorname{rank} T + \operatorname{null} T = \dim V_1.$$

Proof. First we observe that, if either $\operatorname{null} T = \infty$ or $\operatorname{rank} T = \infty$, then $\dim V_1 = \infty$(*Why?*). Therefore, assume that both

$$r := \operatorname{rank} T < \infty, \qquad k := \operatorname{null} T < \infty.$$

Suppose $E_0 = \{u_1, \ldots, u_k\}$ is a basis of $N(T)$ and $E = \{v_1, \ldots, v_r\}$ is a basis of $R(T)$. Let $E_1 = \{w_1, \ldots, w_r\} \subseteq V_1$ such that $Tw_j = v_j$,

$j = 1, \ldots, r$. We show that $E_0 \cup E_1$ is a basis of V_1. Note that $E_0 \cap E_1 = \varnothing$.

Let $x \in V_1$. Since $E = \{v_1, \ldots, v_r\}$ is a basis of $R(T)$, there exist scalars $\alpha_1, \ldots, \alpha_r$ such that

$$Tx = \sum_{i=1}^{r} \alpha_j v_j = \sum_{i=1}^{r} \alpha_j T w_j$$

Hence,

$$T\Big(x - \sum_{i=1}^{r} \alpha_j w_j\Big) = 0$$

so that $x - \sum_{i=1}^{r} \alpha_j w_j \in N(T)$. Since E_0 is a basis of $N(T)$, there exist scalars $\beta_1, \ldots, \beta_k$ such that

$$x - \sum_{i=1}^{r} \alpha_j w_j = \sum_{i=1}^{k} \beta_j u_j.$$

Thus, $x \in \operatorname{span}(E_0 \cup E_1)$. It remains to show that $E_0 \cup E_1$ is linearly independent. For this, suppose $a_1, \ldots, a_k$ and $b_1, \ldots, b_r$ are scalars such that

$$\sum_{i=1}^{k} a_j u_j + \sum_{i=1}^{r} b_j w_j = 0.$$

Applying T to the above equation, it follows that $\sum_{i=1}^{r} b_j v_j = 0$ so that, by the linear independence of E, $b_j = 0$ for all $j = 1, \ldots, r$. Therefore, we have $\sum_{i=1}^{k} a_j u_j = 0$. Now, by the linear independence of E_0, $a_j = 0$, for all $j = 1, \ldots, k$. This completes the proof. ∎

Exercise 2.10 Prove Theorem 2.4. ♢

Recall that for a square matrix A, $\det(A) = 0$ if and only if its columns are linearly dependent. Hence, in view of Theorem 2.4, we have the following:

Theorem 2.7 *Let $T : V_1 \to V_2$ be a linear transformation and let A be a matrix representation of A. Then T is one-one if and only if columns of A are linearly independent.*

Definition 2.9 (finite rank transformations) A linear transformation $T : V \to W$ is said to be of *finite rank* if $\operatorname{rank} T < \infty$.

Exercise 2.11 Let $T : V_1 \to V_2$ be a linear transformation between vector spaces V_1 and V_2. Show that T is of finite rank if and only if there exists $n \in \mathbb{N}$, $\{v_1, \ldots, v_n\} \subset V_2$ and $\{f_1, \ldots, f_n\} \subset \mathcal{L}(V_1, \mathbb{F})$ such that $Tx = \sum_{j=1}^{n} f_j(x) v_j$ for all $x \in V_1$. $\Diamond$

2.6 Composition of Linear of Transformations

Let V_1, V_2, V_3 be vector spaces, and let $T_1 \in \mathcal{L}(V_1, V_2)$, $T_2 \in \mathcal{L}(V_2, V_3)$. Then the composition of T_1 and T_2, namely, $T_2 \circ T_1 : V_1 \to V_3$ defined by

$$(T_2 \circ T_1)(x) = T_2(T_1 x), \qquad x \in V_1,$$

is a linear transformation and it is dented by $T_2 T_1$.

Note that if $V_1 = V_2 = V_3 = V$, then both $T_1 T_2, T_2 T_1$ are well-defined and belong to $\mathcal{L}(V)$. In particular, if $T \in \mathcal{L}(V)$, we can define *powers of* T, namely, T^n for any $n \in \mathbb{N}$ inductively: $T^1 := T$ and for $n > 1$,

$$T^n = T(T^{n-1}).$$

Using this, we can define **polynomials in** T as follows: .

For $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}_n$, say $p(t) = a_1 + a_1 t + \cdots + a_n t^n$, we define $p(T) : V \to V$ by

$$p(T) = a_1 I + a_1 T + \cdots + a_n T^n.$$

We shall also use the convention: $T^0 := I$.

EXAMPLE 2.17 Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$\begin{aligned} T_1(\alpha_1, \alpha_2, \alpha_2) &= (\alpha_1 + \alpha_2 + 2\alpha_3,\ 2\alpha_1 - \alpha_2 + \alpha_3), \\ T_2(\beta_1, \beta_2) &= (\beta_1 + \beta_2,\ \beta_1 - \beta_2). \end{aligned}$$

Then the product transformation $T_2 T_1$ is given by

$$\begin{aligned} (T_2 T_1)(\alpha_1, \alpha_2, \alpha_2) &= T_2(\alpha_1 + \alpha_2 + 2\alpha_3,\ 2\alpha_1 - \alpha_2 + \alpha_3) \\ &= (\beta_1 + \beta_2,\ \beta_1 - \beta_2), \end{aligned}$$

where $\beta_1 = \alpha_1 + \alpha_2 + 2\alpha_3$, $\beta_2 = 2\alpha_1 - \alpha_2 + \alpha_3$. Thus,

$$(T_2 T_1)(\alpha_1, \alpha_2, \alpha_2) = (3\alpha_1 + 3\alpha_3,\ -\alpha_1 + 2\alpha_2 + \alpha_3)\,.$$

Now, consider the standard bases on $\mathbb{R}^3$ and $\mathbb{R}^2$, that is, $E_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $E_2 = \{(1,0), (0,1)\}$, respectively. Then we see that

$$[T_1]_{E_1,E_2} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}, \qquad [T_2]_{E_2,E_2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$[T_2T_1]_{E_1,E_2} = \begin{bmatrix} 3 & 0 & 3 \\ -1 & 2 & 1 \end{bmatrix}.$$

Note that

$$[T_2T_1]_{E_1,E_2} = [T_2]_{E_2,E_2}[T_1]_{E_2,E_3}.$$

♢

Exercise 2.12 Let V_1, V_2, V_3 be finite dimensional vector spaces with based E_1, E_2, E_3, respectively. Prove that if $T_1 \in \mathcal{L}(V_1, V_2)$ and $T_2 \in \mathcal{L}(V_2, V_3)$, then $[T_2T_1]_{E_1,E_3} = [T_2]_{E_2,E_3}[T_1]_{E_1,E_2}$.

Verify the above relation for the operators in Example 2.17. ♢

Recall from set theory that, if S_1 and S_2 are nonempty sets, then a function $f : S_1 \to S_2$ is one-one if and only if there exists a unique $g : R(f) \to S_1$ such that

$$g(f(x)) = x, \qquad f(g(y)) = y$$

for all $x \in S_1$ and for all $y \in R(f)$.

In the case of a linear transformation we have the following.

Theorem 2.8 *Let $T \in \mathcal{L}(V_1, V_2)$. Then T is one-one if and only if there exists a linear transformation $\widetilde{T} : R(T) \to V_1$ such that*

$$\widetilde{T}(Tx) = x \quad \forall\, x \in V_1, \qquad T(\widetilde{T}y) = y \quad \forall\, y \in R(T),$$

and in that case, such operator $\widetilde{T}$ is unique.

Proof. The fact that T is one-one if and only if there exists a unique function $\widetilde{T} : R(T) \to V_1$ such that

$$\widetilde{T}(Tx) = x \quad \forall\, x \in V_1, \qquad T(\widetilde{T}y) = y \quad \forall\, y \in R(T)$$

follows as in set theory. Thus, it is enough to prove that $\widetilde{T}$ is linear. For this, let y_1, y_2 be in $R(T)$ and let x_1, x_2 in V_1 be such that $Tx_i = y_i$, $i = 1, 2$. Let $\alpha \in \mathbb{F}$. Then, by linearity of T, we have

$$y_1 + \alpha y_2 = T\widetilde{T}y_1 + \alpha T\widetilde{T}y_2 = T(\widetilde{T}y_1 + \alpha\widetilde{T}y_2)$$

so that

$$\widetilde{T}(y_1 + \alpha y_2) = \widetilde{T}T(\widetilde{T}y_1 + \alpha\widetilde{T}y_2) = \widetilde{T}y_1 + \alpha\widetilde{T}y_2.$$

Thus, $\widetilde{T}$ is linear. ∎

Definition 2.10 If $T : V_1 \to V_2$ is an injective linear operator, then the unique linear operator $\widetilde{T} : R(T) \to V_1$ defined as in Theorem 2.8 is called the *inverse* of T, and is denoted by $T^{-1} : R(T) \to V_1$. ♢

Clearly, if $T \in \mathcal{L}(V_1, V_2)$ is bijective, then its inverse is defined on all of V_2. Thus, $T \in \mathcal{L}(V_1, V_2)$ is bijective if and only if there exists a unique operator $M \in \mathcal{L}(V_1, V_2)$ such that

$$TM = I_Y, \qquad MT = I_X,$$

and in that case(*verify*), M is also bijective and

$$(MT)^{-1} = M^{-1}T^{-1}.$$

Definition 2.11 A linear operator $T : V_1 \to V_2$ is said to be **invertible** if it is bijective. ♢

Exercise 2.13 Prove the following.

(i) If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear transformations such that T_2T_1 is bijective, then T_2 one-one and T_1 is onto.

(ii) If $T \in \mathcal{L}(V_1, V_2)$ is invertible, then $\dim(V_1) = \dim(V_2)$, and the converse need not be true.

(iii) If $\dim(V) < \infty$ and $T \in \mathcal{L}(V)$, then T is invertible if and only if for every basis E of V, $\det[T]_{E,E} \neq 0$. ♢

2.7 Eigenvalues and Eigenvectors

2.7.1 Definition and examples

Let $T : V \to V$ be a linear operator on a vector space V.

Definition 2.12 A scalar λ is called an **eigenvalue** of T if there exists a nonzero vector $x \in V$ such that

$$Tx = \lambda\, x,$$

and in that case, x is called an **eigenvector** of T corresponding to the eigenvalue λ. ♢

Suppose V is a finite dimensional space, say $\dim(V) = n$, and $T \in \mathcal{L}(V)$. Let A be the matrix representation of T w.r.t. a basis $\{u_1, \ldots, u_n\}$ of V. Then by the discussion in Section 2.4, we have

$$x = \sum_{i=1}^{n} (\vec{x})_i u_i, \qquad Tx = \sum_{i=1}^{n} (A\vec{x})_i u_i,$$

Hence, $x \neq 0$ if and only if $\vec{x} \neq 0$, and for $\lambda \in \mathbb{F}$,

$$Tx = \lambda x \iff A\vec{x} = \lambda\vec{x}.$$

Note that $\lambda \in \mathbb{F}$ is an eigenvalue of T if and only if $T - \lambda I$ is not one-one, and in that case, the subspace $N(T - \lambda I)$ consists of all eigenvectors of T corresponding to the eigenvalue λ together with the zero vector.

Definition 2.13 If λ is an eigenvalue of T, then the space $N(T - \lambda I)$ is called the **eigenspace** of T corresponding to λ.

The set of all eigenvalues of T is called the **eigenspectrum** of T, and will be denoted by $\mathrm{Eig}(T)$. $\Diamond$

In view of Theorem 2.7, we have the following:

Theorem 2.9 *Let V be finite dimensional and $T \in \mathcal{L}(V)$. If A is a matrix representation of T with respect to a basis of V, then λ is an eigenvalue of T if and only if* $\det(A - \lambda I) = 0$.

EXAMPLE 2.18 The conclusions in (i)-(vi) below can be verified easily:

(i) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3).$$

Then $\mathrm{Eig}(T) = \{1\}$ and $N(A - I) = \mathrm{span}\,\{(0, 0, 1)\}$.

(ii) Let $T : \mathbb{F}^2 \to \mathbb{F}^2$ be defined by

$$T(\alpha_1, \alpha_2) = (\alpha_1 + \alpha_2, \alpha_2).$$

Then $\mathrm{Eig}(T) = \{1\}$ and $N(T - I) = \mathrm{span}\,\{(1, 0)\}$.

(iii) Let $T : \mathbb{F}^2 \to \mathbb{F}^2$ be defined by

$$T(\alpha_1, \alpha_2) = (\alpha_2, -\alpha_1).$$

If $\mathbb{F} = \mathbb{R}$, then A has no eigenvalues, i.e., $\text{Eig}(A) = \varnothing$.

(iv) Let T be as in (ii) above. If $\mathbb{F} = \mathbb{C}$, then $\text{Eig}(T) = \{i, -i\}$, $N(A - iI) = \text{span}\,\{(1, i)\}$ and $N(T + iI) = \text{span}\,\{(1, -i)\}$.

(v) Let $T : \mathcal{P} \to \mathcal{P}$ be defined by

$$(Tx)(t) = tx(t), \qquad x \in \mathcal{P}.$$

Then $\text{Eig}(T) = \varnothing$.

(vi) Let X be $\mathcal{P}[a, b]$ and $A : V \to X$ be defined by

$$(Tx)(t) = \frac{d}{dt}x(t), \qquad x \in \mathcal{P}.$$

Then $\text{Eig}(T) = \{0\}$ and $N(T) = \text{span}\,\{x_0\}$, where $x_0(t) = 1$ for all $t \in [a, b]$. ♢

Theorem 2.10 *Let $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Let $V_0 = \{0\}$ and for $j \in \mathbb{N}$, let $V_j := N((T - \lambda I)^j)$. Then the following hold.*

(i) $\{0\} \subseteq N(T - \lambda I) \subseteq N((T - \lambda I)^2) \subseteq N((T - \lambda I)^3) \subseteq \ldots.$

(ii) *If $N((T - \lambda I)^k) = N((T - \lambda I)^{k+1})$ for some $k \in \mathbb{N} \cup \{0\}$, then $N((T - \lambda I)^k) = N((T - \lambda I)^{k+j})$ for all $j \in \mathbb{N}$.*

Suppose V is finite dimensional. Then every inclusion in Theorem 2.10(i) cannot be proper. Thus, the following corollary is immediate from Theorem 2.10.

Corollary 2.11 *Let $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T. If V is a finite dimensional vector space, then there exists $\ell \in \{1, \ldots, n\}$ such that $N((T - \lambda I)^\ell) = N((T - \lambda I)^{\ell+j})$ for all $j \in \mathbb{N}$.*

Definition 2.14 Let $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T. The number $\dim N(T - \lambda I)$ is called the *geometric multiplicity* of λ. If there exists $\ell \in \mathbb{N}$ such that $N((T - \lambda I)^\ell) = N((T - \lambda I)^{\ell+j})$ for all $j \in \mathbb{N}$, then

(i) ℓ is called the *index* of λ,

(ii) $N((T - \lambda I)^\ell)$ is called the *generalized eigenspace* of T corresponding to λ,

(iii) $\dim N((T - \lambda I)^\ell)$ is called the *algebraic multiplicity* of λ. ♢

If V is finite dimensional, $T \in \mathcal{L}(V)$ and λ be an eigenvalue of T, then it is obvious that λ has (finite) index and algebraic multiplicity. If g_λ, ℓ_λ and m_λ are the geometric multiplicity, index and algebraic multiplicity, respectively, of λ, then from Theorem 2.10(i), we have

$$g_\lambda + \ell_\lambda - 1 \leq m_\ell \,.$$

It is also known[1] that

$$m \leq \ell g.$$

Thus, if $\ell = 1$, then $g = m$, that is, generalized eigenspace coincides with eigenspace.

2.7.2 Existence of an eigenvalue

From the above examples we observe that in those cases in which the eigenspectrum is empty, either the scalar field is $\mathbb{R}$ or the vector space is infinite dimensional. The next result shows that if the space is finite dimensional and if the scalar field is the set of all complex numbers, then the eigenspectrum is nonempty.

Theorem 2.12 *Let V be a finite dimensional vector space over $\mathbb{C}$. Then every linear operator on V has at least one eigenvalue.*

Proof. Let $\dim(V) = n$ and $T : V \to V$ be a linear operator. Let x be a nonzero element in V. Since $\dim(V) = n$, the set $\{x, Tx, T^2x, \ldots, T^nx\}$ is linearly dependent. Let $a_0, a_1 \ldots, a_n$ be scalars with at least one of them being nonzero such that

$$a_0x + a_1Tx + \cdots + a_nT^nx = 0.$$

Let $k = \max\{j : a_j \neq 0,\ j = 1, \ldots, n\}$. Then writing

$$p(t) = a_0 + a_1t + \cdots + a_kt^k, \quad p(T) = a_0I + a_1T + \cdots + a_kT^k,$$

we have

$$p(T)(x) = 0.$$

By fundamental theorem of algebra, there exist $\lambda_1, \ldots, \lambda_k$ in $\mathbb{C}$ such that

$$p(t) = a_k(t - \lambda_1)(t - \lambda_2) \ldots (t - \lambda_k).$$

[1] M.T. Nair: *Multiplicities of an eigenvalue: Some observations*, Resonance, Vol. 7 (2002) 31-41.

Thus, we have

$$(T - \lambda_1 I)(T - \lambda_2 I) \ldots (T - \lambda_k I)(x) = p(T)(x) = 0.$$

Hence, at least one of $T - \lambda_1 I, \ \ldots, \ T - \lambda_k I$ is not one-one so that at least one of $\lambda_1, \ldots, \lambda_k$ is an eigenvalue of A. ∎

Theorem 2.13 *Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of a linear operator $T : V \to V$ with corresponding eigenvectors $u_1, \ldots, u_k$, respectively. Then $\{u_1, \ldots, u_k\}$ is a linearly independent set.*

Proof. We prove this result by induction. Clearly $\{u_1\}$ is linearly independent. Now, assume that $\{u_1, \ldots, u_m\}$ is linearly independent, where $m < k$. We show that $\{u_1, \ldots, u_{m+1}\}$ is linearly independent. So, let $\alpha_1, \ldots, \alpha_{m+1}$ be scalars such that

$$\alpha_1 u_1 + \cdots + \alpha_m u_m + \alpha_{m+1} u_{m+1} = 0. \qquad (*)$$

Applying T and using the fact that $Tu_j = \lambda_j u_j$, we have

$$\alpha_1 \lambda_1 u_1 + \cdots + \alpha_m \lambda_k u_m + \alpha_{m+1} \lambda_{m+1} u_{m+1} = 0.$$

From $(*)$, multiplying by λ_{m+1}, we have

$$\alpha_1 \lambda_{m+1} u_1 + \cdots + \alpha_m \lambda_{m+1} u_m + \alpha_{m+1} \lambda_{m+1} u_{m+1} = 0.$$

Thus,

$$\alpha_1 (\lambda_1 - \lambda_{m+1}) u_1 + \cdots + \alpha_m (\lambda_{m+1} - \lambda_m) u_m = 0.$$

Now, using the fact that $\{u_1, \ldots, u_m\}$ is linearly independent in V, and $\lambda_1, \ldots, \lambda_m, \lambda_{m+1}$ are distinct, we obtain $\alpha_j = 0$ for $j = 1, \ldots, m$. Therefore, from $(*)$, $\alpha_{m+1} = 0$. This completes the proof. ∎

By the above theorem we can immediately infer that if V is finite dimensional, then the eigenspectrum of every linear operator on V is a finite set.

Definition 2.15 Let $T \in \mathcal{L}(V)$ and V_0 is a subspace of V. Then V_0 is is said to be *invariant under* T if $T(V_0) \subseteq V_0$, that is,

$$x \in V_0 \Longrightarrow Tx \in V_0.$$

If V_0 is invariant under T, then the *restriction of* T to the space V_0 is the operator $T_0 \in \mathcal{L}(V_0)$ defined by

$$T_0 x = Tx \qquad \forall\, x \in V_0.$$

◊

Exercise 2.14 Prove that if $T \in \mathcal{L}(V)$, $\lambda \in \mathrm{Eig}(T)$ and for $k \in \mathbb{N}$ if $V_k = N((T - \lambda I)^k)$ and $W_k = R((T - \lambda I)^k)$, then show that V_k and W_k are invariant under T. $\Diamond$

Exercise 2.15 Prove that if $T \in \mathcal{L}(V)$, V_0 is invariant under T, and $T_0 \in \mathcal{L}(V_0)$ is the restriction of T to V_0, then $\mathrm{Eig}(T_0) \subseteq \mathrm{Eig}(T)$. $\Diamond$

2.7.3 Diagonalizability

Definition 2.16 Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear operator. Then T is said to be **diagonalizable** if V has a basis E such that $[T]_{E,E}$ is a diagonal matrix. $\Diamond$

The proof of the following theorem is immediate (Write details!).

Theorem 2.14 *Suppose V is a finite dimensional vector space and $T : V \to V$ is a linear operator. Then T is diagonalizable if and only if V has a basis E consisting of eigenvectors of T.*

Hence, in view of Theorem 2.13, we have the following.

Theorem 2.15 *Suppose* $\dim(V) = n$ *and $T : V \to V$ is a linear operator having n distinct eigenvalues. Then T is diagonalizable.*

It is to be observed that, in general, a linear operator $T : V \to V$ need not be diagonalizable (See Example 2.18(ii)). However, we shall see in the next chapter that if V has some additional structure, namely that V is an *inner product space*, and if T satisfies an additional condition with respect to this new structure, namely, *self-adjointness*, then T is diagonalizable.

Exercise 2.16 Which of the following linear transformation T is diagonalizable? If it is diagonalizable, find the basis E and $[T]_{E,E}$.

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3,\ x_1 + x_2 - x_3,\ x_1 - x_2 + x_3).$$

(ii) $T : \mathcal{P}_3 \to \mathcal{P}_3$ such that

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2.$$

(iii) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$Te_1 = 0, \quad Te_2 = e_1, \quad Te_3 = e_2.$$

(iv) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$Te_1 = e_2, \quad Te_2 = e_3, \quad Te_3 = 0.$$

(iv) $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$Te_1 = e_3, \quad Te_2 = e_2, \quad Te_3 = e_1.$$

$\Diamond$

3

Inner Product Spaces

3.1 Motivation

In Chapter 1 we defined a vector space as an abstraction of the familiar Euclidian space. In doing so, we took into account only two aspects of the set of vectors in a plane, namely, the vector addition and scalar multiplication. Now, we consider the third aspect, namely the *angle* between vectors.

Recall from plane geometry that if $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are two *non-zero* vectors in the plane $\mathbb{R}^2$, then the angle $\theta_{x,y}$ between $\vec{x}$ and $\vec{y}$ is given by

$$\cos\theta_{x,y} := \frac{x_1y_1 + x_2y_2}{|\vec{x}|\,|\vec{y}|},$$

where for a vector $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$, $|\vec{u}|$ denotes the *absolute value* of the vector $\vec{u}$, i.e.,

$$|\vec{u}| := \sqrt{u_1^2 + u_2^2},$$

which is the distance of the point $(u_1, u_2) \in \mathbb{R}^2$ from the coordinate origin.

We may observe that the angle $\theta_{x,y}$ between the vectors $\vec{x}$ and $\vec{y}$ is completely determined by the quantity $x_1y_1 + x_2y_2$, which is the *dot product* of $\vec{x}$ and $\vec{y}$. Breaking the convention, let us denote this quantity, i.e., the *dot product* of $\vec{x}$ and $\vec{y}$, by $\langle\vec{x}, \vec{y}\rangle$, i.e.,

$$\langle\vec{x}, \vec{y}\rangle = x_1y_1 + x_2y_2.$$

A property of the function $(\vec{x}, \vec{y}) \mapsto \langle\vec{x}, \vec{y}\rangle$ that one notices immediately is that, for every fixed $\vec{y} \in \mathbb{R}^2$, the function

$$x \mapsto \langle\vec{x}, \vec{y}\rangle, \qquad \vec{x} \in \mathbb{R}^2,$$

is a linear transformation from $\mathbb{R}^2$ into $\mathbb{R}$, i.e.,

$$\langle \vec{x} + \vec{u}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{u}, \vec{y} \rangle, \qquad \langle \alpha\vec{x}, \vec{y} \rangle = \alpha \langle \vec{x}, \vec{y} \rangle \tag{3.1}$$

for all $\vec{x}, \vec{u}$ in $\mathbb{R}^2$. Also, we see that for all $\vec{x}$, $\vec{y}$ in $\mathbb{R}^2$,

$$\langle \vec{x}, \vec{x} \rangle) \geq 0, \tag{3.2}$$

$$\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}, \tag{3.3}$$

$$\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle. \tag{3.4}$$

If we take $\mathbb{C}^2$ instead of $\mathbb{R}^2$, and if we define $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2$, for $\vec{x}, \vec{y}$ in $\mathbb{C}^2$, then the above properties are not satisfied by all vectors in $\mathbb{C}^2$. In order to accommodate the complex situation, we define a *generalized dot product*, as follows: For $\vec{x}$, $\vec{y}$ in $\mathbb{F}^2$, let

$$\langle \vec{x}, \vec{y} \rangle_* = x_1 \bar{y}_1 + x_2 \bar{y}_2,$$

where for a complex number z, $\bar{z}$ denotes its complex conjugation. It is easily seen that $\langle \cdot, \cdot \rangle_*$ satisfies properties (3.1) – (3.4).

Now, we shall consider the abstraction of the above *modified dot product*.

3.2 Definition and Some Basic Properties

Definition 3.1 (Inner Product) An *inner product* on a vector space V is a map $(x, y) \mapsto \langle x, y \rangle$ which associates each pair (x, y) of vectors in V, a unique scalar $\langle x, y \rangle$ which satisfies the following axioms:

(a) $\langle x, x \rangle \geq 0 \quad \forall\, x \in V$,

(b) $\langle x, x \rangle = 0 \iff x = 0$,

(c) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \forall\, x, y, z \in V$,

(d) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle \quad \forall\, \alpha \in \mathbb{F}$ and $\forall\, x, y \in V$, and

(e) $\langle x, y \rangle = \overline{\langle y, x \rangle} \quad \forall\, x, y \in V$. ♢

Definition 3.2 (Inner Product Space) A vector space together with an inner product is called an *inner product space*. ♢

If an inner product $\langle \cdot, \cdot \rangle$ is defined on a vector space V, and if V_0 is a subspace of V, then the restriction of $\langle \cdot, \cdot \rangle$ to $V_0 \times V_0$, i.e., the map $(x, y) \mapsto \langle x, y \rangle$ for $(x, y) \in V_0 \times V_0$ is an inner product on V_0.

Before giving examples of inner product spaces, let us observe some properties of an inner product.

Proposition 3.1 *Let V be an inner product space. For a given $y \in V$, let $f : V \to \mathbb{F}$ be defined by*

$$f(x) = \langle x, y \rangle, \qquad x \in V.$$

Then f is a linear functional on V.

Proof. The result follows from axioms (c) and (d) in the definition of an inner product: Let $x, x' \in V$ and $\alpha \in \mathbb{F}$. Then, by axioms (c) and (d),

$$f(x + x') = \langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle = f(x) + fx'),$$

$$f(\alpha x) = \langle \alpha x, y \rangle = \alpha \langle x, y \rangle = \alpha f(x).$$

Hence, f is a linear transformation. ▮

Proposition 3.2 *Let V be an inner product space. Then for every x, y, u, v in V, and for every $\alpha \in \mathbb{F}$,*

$$\langle x, u + v \rangle = \langle x, u \rangle + \langle x, v \rangle, \qquad \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle.$$

Proof. The result follows from axioms (c),(d) and (e) in the definition of an inner product: Let x, y, u, v in V and $\alpha \in \mathbb{F}$.

$$\langle x, u+v \rangle = \overline{\langle u + v, x \}} = \overline{\langle u, x \rangle + \langle v, x \rangle} = \overline{\langle u, x \rangle} + \overline{\langle v, x \rangle} = \langle x, u \rangle + \langle x, v \rangle,$$

$$\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \overline{\alpha \langle y, x \rangle} = \bar{\alpha} \langle x, y \rangle.$$

This completes the proof. ▮

Exercise 3.1 Suppose V is an inner product space over $\mathbb{C}$. Prove that $\mathrm{Re}\langle ix, y \rangle = -\mathrm{Im}\langle x, y \rangle$ for all $x, y \in V$. ♢

3.3 Examples of Inner Product Spaces

EXAMPLE 3.1 For $x = (\alpha_1, \ldots, \alpha_n)$ and $y = (\beta_1, \ldots, \beta_n)$ in $\mathbb{F}^n$, define

$$\langle x, y \rangle = \sum_{j=1}^{n} \alpha_j \bar{\beta}_j.$$

It is seen that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{F}^n$.

The above inner product is called the **standard inner product** on $\mathbb{F}^n$. ♢

EXAMPLE 3.2 Suppose V is a finite dimensional vector space, say of dimension n, and $E := \{u_1, \ldots, u_n\}$ is an *ordered basis* of V. For $x = \sum_{i=1}^n \alpha_i u_i$, $y = \sum_{i=1}^n \beta_i u_i$ in V, let

$$\langle x, y\rangle_E := \sum_{i=1}^n \alpha_i \bar{\beta}_i.$$

Then it is easily seen that $\langle \cdot, \cdot \rangle_E$ is an inner product on V. ◊

EXAMPLE 3.3 For $f, g \in C[a, b]$, let

$$\langle f, g\rangle := \int_a^b f(t)\overline{g(t)}\,dt.$$

This defines an inner product on $C[a, b]$: Clearly,

$$\langle f, f\rangle = \int_a^b |f(t)|^2\,dt \geq 0 \quad \forall f \in C[a, b],$$

and by continuity of the function f,

$$\langle f, f\rangle := \int_a^b |f(t)|^2\,dt = 0 \iff f(t) = 0 \quad \forall\, t \in [a, b].$$

The other axioms can be verified easily. ◊

Exercise 3.2 Let $T : V \to \mathbb{F}^n$ be a linear isomorphism. Show that

$$\langle x, y\rangle_T := \langle Tx, Ty\rangle_{\mathbb{F}^n}, \qquad x, y \in V,$$

defines an inner product on V. Here, $\langle \cdot, \cdot\rangle_{\mathbb{F}^n}$ is the standard inner product on $\mathbb{F}^n$. ◊

Exercise 3.3 Let $\tau_1, \ldots, \tau_{n+1}$ be distinct real numbers. Show that

$$\langle p, q\rangle := \sum_{i=1}^{n+1} p(\tau_i)\overline{q(\tau_i)}, \qquad p, q \in \mathcal{P}_n,$$

defines an inner product on $\mathcal{P}_n$. ◊

3.4 Norm of a Vector

Recall that the absolute value of a vector $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$, is given by

$$|\vec{x}| = \sqrt{x_1^2 + x_2^2}$$

Denoting the standard inner product on $\mathbb{R}^2$ by $\langle x, x\rangle_2$, it follows that

$$|\vec{x}| = \sqrt{\langle \vec{x}, \vec{x}\rangle_2}.$$

As an abstraction of the above notion, we define the *norm* of a vector.

Definition 3.3 (Norm of a Vector) Let V be an inner product space. Then for $x \in V$, then *norm* of x is defined as the non-negative square root of $\langle x, x\rangle$, and it is denoted by $\|x\|$, i.e,

$$\|x\| := \sqrt{\langle x, x\rangle}, \qquad x \in V.$$

The map $x \mapsto \|x\|$ is also called a *norm* on V. $\Diamond$

Definition 3.4 A vector in an inner product space is said to be a *unit vector* if it is of norm 1. $\Diamond$

Exercise 3.4 If x is a non-zero vector, then show that $u := x/\|x\|$ is a vector of norm 1. $\Diamond$

Recall from elementary geometry that if a, b are the lengths of the adjacent sides of a parallelogram, and if c, d are the lengths of its diagonals, then $2(a^2 + b^2) = c^2 + d^2$. This is the well-known parallelogram law. This has a generalized version in the setting of inner product spaces.

Theorem 3.3 (Parallelogram law) *For vectors x, y in an inner product space V,*

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right).$$

Exercise 3.5 Verify the parallelogram law (Theorem 3.3). $\Diamond$

Exercise 3.6 Let V be an inner product space, and let $x, y \in V$. Then, show the following:

(a) $\|x\| \geq 0$.
(b) $\|x\| = 0$ iff $x = 0$.
(c) $\|\alpha x\| = |\alpha|\,\|x\|$ for all $\alpha \in \mathbb{F}$..

$\Diamond$

Exercise 3.7 Let V_1 and V_2 be an inner product spaces, using the same notations for inner products (respectively, norms) on both the spaces. Let $T : V_1 \to V_2$ be a linear transformation. Prove that, for all $(x, y) \in V_1 \times V_2$,

$$\langle Tx, Ty\rangle = \langle x, y\rangle \iff \|Tx\| = \|x\|.$$

[Hint: For the only if part, use $\langle T(x+y), T(x+y)\rangle = \langle x+y, x+y\rangle$ and use Exercise 3.1.] $\Diamond$

3.5 Orthogonality

Recall that the angle $\theta_{x,y}$ between vectors $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^2$ is given by

$$\cos\theta_{x,y} := \frac{\langle\vec{x}, \vec{y}\rangle_2}{|\vec{x}|\,|\vec{y}|}.$$

Hence, we can conclude that the vectors $\vec{x}$ and $\vec{y}$ are *orthogonal* if and only if $\langle\vec{x}, \vec{y}\rangle_2 = 0$. This observation motivates us to have the following definition.

Definition 3.5 (Orthogonal vectors) Vectors x and y in an inner product space V are said to be *orthogonal to each other* or x *is orthogonal to* y if $\langle x, y\rangle = 0$. In this case we write $x \perp y$, and read x perpendicular to y, or x perp y. $\Diamond$

Note that

- for x, y in V, $x \perp y \iff y \perp x$, and
- $0 \perp x$ for all $x \in V$.

3.5.1 Cauchy-Schwarz inequality

Recall from the geometry of $\mathbb{R}^2$ that if $\vec{x}$ and $\vec{y}$ are nonzero vectors in $\mathbb{R}^2$, then the projection vector $\vec{p}_{x,y}$ of $\vec{x}$ along $\vec{y}$ is given by

$$\vec{p}_{x,y} := \frac{\langle\vec{x}, \vec{y}\rangle_2}{\|\vec{y}\|^2}\vec{y}$$

and it has less length atmost $\|\vec{x}\|$, that is, $\|\vec{p}_{x,y}\| \le \|\vec{x}\|$. Thus, we have

$$|\langle\vec{x}, \vec{y}\rangle_2| \le \|\vec{x}\|\,\|\vec{y}\|.$$

Further, the vectors $\vec{p}_{x,y}$ and $\vec{q}_{x,y} := \vec{x} - \vec{p}_{x,y}$ are orthogonal, so that by *Pythagoras theorem*,

$$\|\vec{x}\|^2 = \|\vec{p}_{x,y}\|^2 + \|\vec{q}_{x,y}\|^2.$$

Now, let us prove these concepts in the context of a general inner product space.

First recall *Pythagoras theorem* from elementary geometry that if a, b c are lengths of sides of a right angled triangle with c being the hypotenuse, then $a^2 + b^2 = c^2$. Here is the generalized form of it in the setting of an inner product space.

Theorem 3.4 (Pythagoras theorem) *Suppose x and y are vectors in an inner product space which are orthogonal to each other. Then*

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Proof. Left as an exercise. (Follows by writing the norms in terms of inner products and simplifying expressions.) ∎

Exercise 3.8 (i) If the scalar field is $\mathbb{R}$, then show that the converse of the Pythagoras theorem holds, that is, if $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then $x \perp y$.

(ii) If the scalar field is $\mathbb{C}$, then show that the converse of Pythagoras theorem need not be true .

[Hint: Take $V = \mathbb{C}$ with standard inner product, and for nonzero real numbers $\alpha, \beta \in \mathbb{R}$, take $x = \alpha$, $y = i\,\beta$.] ◊

Theorem 3.5 (Cauchy-Schwarz inequality) *Let V be an inner product space, and $x, y \in V$. Then*

$$|\langle x, y\rangle| \le \|x\|\ \|y\|.$$

Equality holds in the above inequality if and only if x and y are linearly dependent.

Proof. Clearly, the result holds if $y = 0$. So, assume that $y \neq 0$, and let $u_0 = y/\|y\|$. Let us write $x = u + v$, where

$$u = \langle x, u_0\rangle u_0, \qquad v = x - \langle x, u_0\rangle u_0.$$

Note that $\langle u, v\rangle = 0$ so that by Pythagoras theorem,

$$\|x\|^2 = \|u\|^2 + \|v\|^2 = |\langle x, u_0\rangle|^2 + \|v\|^2.$$

Thus, $|\langle x, u_0\rangle| \le \|x\|$. Equality holds in this inequality if and only if $v := x - \langle x, u_0\rangle u_0 = 0$, i.e., if and only if x is a scalar multiple of y if and only if x and y are linearly dependent. ∎

As a corollary of the above theorem we have the following.

Corollary 3.6 (Triangle inequality) *Suppose V is an inner product space. Then for every x, y in V,*

$$\|x + y\| \le \|x\| + \|y\|.$$

Proof. Let $x, y \in V$. Then, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&= \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}\,\langle x, y\rangle \\
&\le \|x\|^2 + \|y\|^2 + 2\,|\langle x, y\rangle| \\
&\le \|x\|^2 + \|y\|^2 + 2\,\|x\|\,\|y\| \\
&= (\|x\| + \|y\|)^2.
\end{aligned}$$

Thus, $\|x + y\| \le \|x\| + \|y\|$ for every $x, y \in V$. ∎

Remark 3.1 For nonzero vectors x and y in an inner product space V, by Schwarz inequality, we have

$$\frac{|\langle x, y\rangle|}{\|x\|\|y\|} \le 1.$$

This relation motivates us to define the *angle* between any two nonzero vectors x and y in V as

$$\theta_{x,y} := \cos^{-1}\left(\frac{|\langle x, y\rangle|}{\|x\|\,\|y\|}\right).$$

Note that if $x = cy$ for some nonzero scalar c, then $\theta_{x,y} = 0$, and if $\langle x, y\rangle = 0$, then $\theta_{x,y} = \pi/2$.

3.5.2 Orthogonal and orthonormal sets

Theorem 3.7 *Let V be an inner product space, and $x \in V$. If $\langle x, y\rangle = 0$ for all $y \in V$, then $x = 0$.*

Proof. Clearly, if $\langle x, y\rangle = 0$ for all $y \in V$, then $\langle x, x\rangle = 0$ as well. Hence $x = 0$ ∎

As an immediate consequence of the above theorem, we have the following.

Corollary 3.8 *Let V be an inner product space, and $u_1, u_2, \ldots, u_n$ be linearly independent vectors in V. Let $x \in V$. Then*

$$\langle x, u_i\rangle = 0 \quad \forall\, i \in \{1, \ldots, n\} \iff \langle x, y\rangle = 0 \quad \forall\, y \in \operatorname{span}\{u_1, \ldots, u_n\}.$$

In particular, if $\{u_1, u_2, \ldots, u_n\}$ is a basis of V, and if $\langle x, u_i\rangle = 0$ for all $i \in \{1, \ldots, n\}$, then $x = 0$.

Exercise 3.9 If $\dim V \geq 2$, and if $0 \neq x \in V$, then find a non-zero vector which is orthogonal to x. ◊

Definition 3.6 (Orthogonal to a set) Let S be a subset of an inner product space V, and $x \in S$. Then x is said to be *orthogonal to* S if $\langle x, y\rangle = 0$ for all $y \in S$. In this case, we write $x \perp S$. The set of vectors orthogonal to S is denoted by $S^\perp$, i.e.,

$$S^\perp := \{x \in V : x \perp S\}.$$

◊

Exercise 3.10 Let V be an inner product space.

(a) Show that $V^\perp = \{0\}$.

(b) If S is a basis of V, then show that $S^\perp = \{0\}$. ◊

Definition 3.7 (Orthogonal and orthonormal sets) Let S be a subset of an inner product space V. Then

(a) S is said to be an *orthogonal set* if $\langle x, y\rangle = 0$ for all distinct x, $y \in S$, i.e., for every $x, y \in S$, $x \neq y$ implies $x \perp y$.

(b) S is said to be an *orthonormal set* if it is an orthogonal set and $\|x\| = 1$ for all $x \in S$.

◊

Theorem 3.9 *Let S be an orthogonal set in an inner product space V. If $0 \notin S$, then S is linearly independent.*

Proof. Suppose $0 \notin S$ and $\{u_1, \ldots, u_n\} \subseteq S$. If $\alpha_1, \ldots, \alpha_n$ are scalars such that $\alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_n u_n = 0$, then for every $j \in \{1, \ldots, n\}$, we have

$$0 = \left\langle \sum_{i=1}^{n} \alpha_i u_i, u_j \right\rangle = \sum_{i=1}^{n} \langle \alpha_i u_i, u_j \rangle = \sum_{i=1}^{n} \alpha_i \langle u_i, u_j \rangle = \alpha_j \langle u_j, u_j \rangle.$$

Hence, $\alpha_j = 0$ for all $j \in \{1, \ldots, n\}$. ∎

By Theorem 3.9, it follows that every orthonormal set is linearly independent. In particular, if V is an n-dimensional inner product space and E is an orthonormal set consisting of n vectors, then E is a basis of V.

Definition 3.8 (Orthonormal basis) Suppose V is a finite dimensional inner product space. An orthonormal set in V which is also a basis of V is called an orthonormal basis of V. ♢

EXAMPLE 3.4 The standard basis $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{F}^n$ w.r.t. the standard inner product on $\mathbb{F}^n$. ♢

EXAMPLE 3.5 Consider the the vector space $C[0, 2\pi]$ with inner product defined by

$$\langle f, g \rangle := \int_0^{2\pi} f(t)\overline{g(t)}\, dt$$

for $f, g \in C[0, 2\pi]$. For $n \in \mathbb{N}$, let

$$u_n(t) := \sin(nt), \quad v_n(t) = \cos(nt), \qquad 0 \le t \le 2\pi.$$

Since

$$\int_0^{2\pi} \cos(kt)\, dt = 0 = \int_0^{2\pi} \sin(kt)\, dt \quad \forall\, k \in \mathbb{Z},$$

we have for $n \neq m$,

$$\langle u_n, u_m \rangle = \langle v_n, v_m \rangle = \langle u_n, v_n \rangle = \langle u_n, v_m \rangle = 0.$$

Thus, $u_n : n \in \mathbb{N}\}$ is an orthonormal set in $C[0, 2\pi]$. ♢

Exercise 3.11 If $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{F}^n$, then for every $i \neq j$, $e_i + e_j \perp e_i - e_j$ w.r.t. the standard inner product. ♢

Exercise 3.12 Let V_1 and V_2 be inner product spaces and $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be orthonormal bases of V_1 and V_2, respectively. Let $T \in \mathcal{L}(V_1, V_2)$. Prove that

(a) $[T]_{E_1,E_2} = \langle Tu_j, v_i \rangle$.

(b) $\langle Tx, y \rangle \quad \forall\, (x, y) \in V_1 \times V_2 \iff \langle Tu_j, u_i \rangle = \langle u_i, Tu_j \rangle$ $\forall\, i = 1, \ldots, m; j = 1, \ldots, n.$ ♢

3.5.3 Fourier expansion and Bessel's inequality

Theorem 3.10 *Suppose V is an inner product space, and $\{u_1, \ldots, u_n\}$ is an orthonormal subset of V. Then, for every $x \in \operatorname{span}\{u_1, \ldots, u_n\}$,*

$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \qquad \|x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2.$$

Proof. Let $x \in \operatorname{span}\{u_1, \ldots, u_n\}$, Then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = \alpha_1 u_1 + \cdots + \alpha_n u_n.$$

Hence, for every $i \in \{1, \ldots, n\}$,

$$\langle x, u_i \rangle = \alpha_1 \langle u_1, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle = \alpha_i.$$

and

$$\begin{aligned} \|x\|^2 = \langle x, x \rangle &= \left\langle \sum_{i=1}^{n} \alpha_i u_i, \sum_{j=1}^{n} \alpha_j u_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \langle u_i, u_j \rangle \\ &= \sum_{i=1}^{n} |\alpha_i|^2 = \sum_{i=1}^{n} |\langle x, u_i \rangle|^2. \end{aligned}$$

This completes the proof. ∎

The proof of the following corollary is immediate from the above theorem.

Corollary 3.11 (Fourier expansion and Parseval's identity) *If $\{u_1, \ldots, u_n\}$ is an orthonormal basis of an inner product space V, then for every $x \in V$,*

$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j, \qquad \|x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2.$$

Another consequence of Theorem 3.10 is the following.

Corollary 3.12 (Bessel's inequality) *Suppose V is an inner product space, and $\{u_1, \ldots, u_n\}$ is an orthonormal subset of V. Then, for every $x \in V$,*

$$\sum_{j=1}^{n} |\langle x, u_j\rangle|^2 \leq \|x\|^2.$$

Proof. Let $x \in V$, and let $y = \sum_{i=1}^{n} \langle x, u_i\rangle u_i$. By Theorem 3.10,

$$\|y\|^2 = \sum_{i=1}^{n} |\langle y, u_i\rangle|^2.$$

Note that $\langle y, u_i\rangle = \langle x, u_i\rangle$ for all $i \in \{1, \ldots, n\}$, i.e., $\langle x-y, u_i\rangle = 0$ for all $i \in \{1, \ldots, n\}$. Hence, $\langle x - y, y\rangle = 0$. Therefore, by Pythagoras theorem,

$$\|x\|^2 = \|y\|^2 + \|x - y\|^2 \geq \|y\|^2 = \sum_{i=1}^{n} |\langle x, u_i\rangle|^2.$$

This completes the proof. ▮

EXAMPLE 3.6 Let $V = C[0, 2\pi]$ with inner product $\langle x, y\rangle := \int_0^{2\pi} x(t)\overline{y(t)}dt$ for x, y in $C[0, 2\pi]$. For $n \in \mathbb{Z}$, let u_n be defined by

$$u_n(t) = \frac{e^{i\,nt}}{\sqrt{2\pi}}, \qquad t \in [0, 2\pi].$$

Then it is seen that

$$\langle u_n, u_m\rangle = \int_0^{2\pi} e^{i\,(n-m)t}dt = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

Hence, $\{u_n : n \in \mathbb{Z}\}$ is an orthonormal set in $C[0, 2\pi]$. By Theorem 3.10, if $x \in \text{span}\,\{u_j : j = -N, -N+1, \ldots, 0, i, \ldots, N\}$,

$$x = \sum_{j=-N}^{N} a_n\, e^{i\,nt} \quad \text{with} \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} x(t)e^{-i\,nt}dt.$$

◊

Theorem 3.13 (Riesz representation theorem) *Let V be a finite dimensional inner product space. Then for very linear functional $f : V \to \mathbb{F}$, there exists a unique $y \in V$ such that*

$$f(x) = \langle x, y\rangle \quad \forall\, x \in V.$$

Proof. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of V. Then, by Corollary 3.11, every $x \in V$ can be written as $x = \sum_{j=1}^{n} \langle x, u_j\rangle u_j$. Thus, if $f : V \to \mathbb{F}$ is a linear functional, then

$$f(x) = \sum_{j=1}^{n} \langle x, u_j\rangle f(u_j) = \sum_{j=1}^{n} \langle x, \overline{f(u_j)} u_j\rangle = \left\langle x, \sum_{j=1}^{n} \overline{f(u_j)} u_j \right\rangle.$$

Thus, $y := \sum_{j=1}^{n} \overline{f(u_j)} u_j$ satisfies the requirements. To see the uniqueness, let y_1 and y_2 be in V such that

$$f(x) = \langle x, y_1\rangle, \quad f(x) = \langle x, y_2\rangle \quad \forall\, x \in V.$$

Then

$$\langle x, y_1 - y_2\rangle = 0 \quad \forall\, x \in V$$

so that by Theorem 3.7, $y_1 - y_2 = 0$, i.e., $y_1 = y_2$. ∎

3.6 Gram-Schmidt Orthogonalization

A question that naturally arises is: Does every finite dimensional inner product space has an orthonormal basis? We shall answer this question affirmatively.

Theorem 3.14 (Gram-Schmidt orthogonalization) *Let V be an inner product space and $u_1, u_2, \ldots, u_n$ are linearly independent vectors in V. Then there exist orthogonal vectors $v_1, v_2, \ldots, v_n$ in V such such that*

$$\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\} \quad \forall\, k \in \{1, \ldots, n\}.$$

In fact, the vectors $v_1, v_2, \ldots, v_n$ defined by

$$\begin{aligned} v_1 &:= u_1 \\ v_{k+1} &:= u_{k+1} - \sum_{j=1}^{k} \frac{\langle u_{k+1}, v_j\rangle}{\langle v_j, v_j\rangle} v_j, \quad k = 1, 2, \ldots, n-1, \end{aligned}$$

satisfy the requirements.

Proof. We construct orthogonal vectors $v_1, v_2, \ldots, v_n$ in V such such that $\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$ for all $k \in \{1, \ldots, n\}$.

Let $v_1 = u_1$. Let us write u_2 as

$$u_2 = \alpha u_1 + v_2,$$

where α is chosen in such a way that $v_2 := u_2 - \alpha u_1$ is orthogonal to v_1, i.e., $\langle u_2 - \alpha u_1, v_1 \rangle = 0$, i.e.,

$$\alpha = \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle}.$$

Thus, the vector

$$v_2 := u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

is orthogonal to v_1. Moreover, using the linearly independence of u_1, u_2, it follows that $v_2 \neq 0$, and $\operatorname{span}\{u_1, u_2\} = \operatorname{span}\{v_1, v_2\}$. Next, we write

$$u_3 = (\alpha_1 v_1 + \alpha_2 v_2) + v_3,$$

where α_1, α_2 are chosen in such a way that $v_3 := u_3 - (\alpha_1 v_1 + \alpha_2 v_2)$ is orthogonal to v_1 and v_2, i.e.,

$$\langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2), v_1 \rangle = 0, \qquad \langle u_3 - (\alpha_1 v_1 + \alpha_2 v_2), v_2 \rangle = 0.$$

That is, we take

$$\alpha_1 = \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle}, \qquad \alpha_2 = \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle}.$$

Thus, the vector

$$v_3 := u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

is orthogonal to v_1 and v_2. Moreover, using the linearly independence of u_1, u_2, u_3, it follows that $v_3 \neq 0$, and

$$\operatorname{span}\{u_1, u_2, u_3\} = \operatorname{span}\{v_1, v_2, v_3\}.$$

Continuing this procedure, we obtain orthogonal vectors $v_1, v_2, \ldots, v_n$ defined by

$$v_{k+1} := u_{k+1} - \frac{\langle u_{k+1}, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_{k+1}, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \ldots - \frac{\langle u_{k+1}, v_k \rangle}{\langle v_k, v_k \rangle} v_k$$

which satisfy

$$\operatorname{span}\{u_1, \ldots, u_k\} = \operatorname{span}\{v_1, \ldots, v_k\}$$

for each $k \in \{1, 2, \ldots, k-1\}$. ∎

Exercise 3.13 Let V be an inner product space, and let $u_1, u_2, \ldots, u_n$ be orthonormal vectors. Define $w_1, w_2, \ldots, w_n$ iteratively as follows:

$$v_1 := u_1 \quad \text{and} \quad w_1 = \frac{v_1}{\|v_1\|}$$

and for each $k \in \{1, 2, \ldots, n-1\}$, let

$$v_{k+1} := u_{k+1} - \sum_{i=1}^{k} \langle u_{k+1}, w_i \rangle w_i \quad \text{and} \quad w_{k+1} = \frac{v_{k+1}}{\|v_{k+1}\|}.$$

Show that $\{w_1, w_2, \ldots, w_n\}$ is an orthonormal set, and

$$\operatorname{span}\{w_1, \ldots, w_k\} = \operatorname{span}\{u_1, \ldots, u_k\}, \qquad k = 1, 2, \ldots, n.$$

◊

From Theorem 3.14, we obtain the following two theorem.

Theorem 3.15 *Every finite dimensional inner product space has an orthonormal basis.*

From Theorem 3.15 we deduce the following.

Theorem 3.16 (Projection theorem) *Suppose V is a finite dimensional inner product space and V_0 is a subspace of V. Then there exists a subspace W of V such that*

$$V = V_0 + W \quad \textit{and} \quad V_0 \perp W.$$

Proof. If $V_0 = V$, then $W = \{0\}$. Now, assume that $\dim(V) = n$ and $\dim(V_0) = k < n$. Let $E_0 = \{u_1, \ldots, u_k\}$ be an orthonormal basis of V_0. Now, extend E_0 to a basis E of V. Now, orthonormalization of E by Gram-Schmidt orthogonalization process will give an orthonormal basis $\widetilde{E} = \{u_1, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ of V. Then $W = \operatorname{span}\{u_{k+1}, \ldots, u_n\}$ will satisfy the requirements. ∎

3.6.1 Examples

EXAMPLE 3.7 Let $V = \mathbb{F}^3$ with standard inner product. Consider the vectors $u_1 = (1,0,0)$, $u_2 = (1,1,0)$, $u_3 = (1,1,1)$. Clearly, u_1, u_2, u_3 are linearly independent in $\mathbb{F}^3$. Let us orthogonalize these vectors according to the Gram-Schmidth orthogonalizaion procedure:

Take $v_1 = u_1$, and

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1.$$

Note that $\langle v_1, v_1\rangle = 1$ and $\langle u_2, v_1\rangle = 1$. Hence, $v_2 = u_2 - v_1 = (0,1,0)$. Next, let

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2.$$

Note that $\langle v_2, v_2\rangle = 1$, $\langle u_3, v_1\rangle = 1$ and $\langle u_3, v_2\rangle = 1$ Hence, $v_3 = u_2 - v_1 - v_2 = (0,0,1)$. Thus,

$$\{(1,0,0), (0,1,0), (0,0,1)\}$$

is the Gram-Schmidt orthogonalization of $\{u_1, u_2, u_3\}$. ◊

EXAMPLE 3.8 Again let $V = \mathbb{F}^3$ with standard inner product. Consider the vectors $u_1 = (1,1,0)$, $u_2 = (0,1,1)$, $u_3 = (1,0,1)$. Clearly, u_1, u_2, u_3 are linearly independent in $\mathbb{F}^3$. Let us orthogonalize these vectors according to the Gram-Schmidth orthogonalizaion procedure:

Take $v_1 = u_1$, and

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1.$$

Note that $\langle v_1, v_1\rangle = 2$ and $\langle u_2, v_1\rangle = 1$. Hence,

$$v_2 = (0,1,1) - \frac{1}{2}(1,1,0) = (-1/2, 1/2, 1).$$

Next, let

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2.$$

Note that $\langle v_2, v_2\rangle = 3/2$, $\langle u_3, v_1\rangle = 1$ and $\langle u_3, v_2\rangle = 1/2$ Hence,

$$v_3 = (1,0,1) - \frac{1}{2}(1,1,0) - \frac{1}{3}(-1/2, 1/2, 1) = (-2/3, 2/3, -2/3).$$

Thus,

$$\{(1,1,0),(-1/2,1/2,1),(-2/3,2/3,-2/3)\}$$

is the Gram-Schmidt orthogonalization of $\{u_1, u_2, u_3\}$. ◊

EXAMPLE 3.9 Let $V = \mathcal{P}$ be with the the inner product

$$\langle p, q\rangle = \int_{-1}^{1} p(t)\overline{q(t)}\,dt, \qquad p, q \in V.$$

Let $u_j(t) = t^{j-1}$ for $j = 1, 2, 3$ and consider the linearly independent set $\{u_1, u_2, u_3\}$ in V. Now let $v_1(t) = u_1(t) = 1$ for all $t \in [-1, 1]$, and let

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1.$$

Note that

$$\langle v_1, v_1\rangle = \int_{-1}^{1} v_1(t)\overline{v_1(t)}\,dt = \int_{-1}^{1} dt = 2,$$

$$\langle u_2, v_1\rangle = \int_{-1}^{1} u_2(t)\overline{v_1(t)}\,dt = \int_{-1}^{1} t\,dt = 0.$$

Hence, we have $v_2(t) = u_2(t) = t$ for all $t \in [-1, 1]$. Next, let

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2.$$

Here,

$$\langle u_3, v_1\rangle = \int_{-1}^{1} u_3(t)\overline{v_1(t)}\,dt = \int_{-1}^{1} t^2\,dt = \frac{2}{3},$$

$$\langle u_3, v_2\rangle = \int_{-1}^{1} u_3(t)\overline{v_2(t)}\,dt = \int_{-1}^{1} t^3\,dt = 0.$$

Hence, we have $v_3(t) = t^2 - \frac{1}{3}$ for all $t \in [-1, 1]$. Thus,

$$\left\{1, t, t^2 - \frac{1}{3}\right\}$$

is an orthogonal set of polynomials. ◊

Definition 3.9 (Legendre polynomials) The polynomials

$$p_o(t), p_1(t), p_2(t) \dots$$

obtained by orthogonalizing $1, t, t^2, \dots$ using the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(t)\overline{q(t)}\, dt, \qquad p, q \in \mathcal{P},$$

are called *Legendre polynomials.* ♢

It is clear that the n-th Legendre polynomial $p_n(t)$ is of degree n. We have seen in Example 3.9 that

$$p_0(t) = 1, \quad p_1(t) = t, \quad p_2(t) = t^2 - \frac{1}{3}.$$

Remark 3.2 For nonzero vectors x and y in an inner product space V, by Schwarz inequality, we have

$$\frac{|\langle x, y \rangle|}{\|x\| \|y\|} \le 1.$$

This relation motivates us to define the *angle* between any two nonzero vectors x and y in V as

$$\theta_{x,y} := \cos^{-1}\left(\frac{|\langle x, y \rangle|}{\|x\| \; \|y\|} \right).$$

Note that if $x = cy$ for some nonzero scalar c, then $\theta_{x,y} = 0$, and if $\langle x, y \rangle = 0$, then $\theta_{x,y} = \pi/2$.

Exercise 3.14 Let V_1 and V_2 be finite dimensional inner product spaces, and E_1 and E_2 be ordered orthonormal bases of V_1 and V_2 respectively. Let $A : \underline{\mathbb{F}}^n \to \underline{\mathbb{F}}^m$ be the linear transformation corresponding to the matrix $[T]_{E_1,E_2}$. Prove that

$$\langle Tx, y \rangle = \langle A\underline{x}, \underline{y} \rangle \qquad \forall\, (x, y) \in V_1 \times V_2.$$

Deduce that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $(x, y) \in V_1 \times V_2$ if and only $[T]_{E_1,E_2}$ is hermitian. ♢

3.7 Diagonalization

In this section we show the diagonalizability of certain type operators as promised at the end of Section 2.7.3.

3.7.1 Self-adjoint operators and their eigenvalues

Recall from the theory of matrices that a square matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

with complex entries is said to be *hermitian* if its conjugate transpose is itself, that is,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \cdots & \overline{a}_{n1} \\ \overline{a}_{12} & \overline{a}_{22} & \cdots & \overline{a}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \overline{a}_{1n} & \overline{a}_{2n} & \cdots & \overline{a}_{nn} \end{bmatrix}.$$

The real analogue of hermitian matrices are the so called *symmetric matrices*, that is, matrices whose transpose is itself.

In the context of a general inner product space, there is a analogue for the above concepts too.

Definition 3.10 A linear transformation $T : V \to V$ on an inner product space V is said to be a *self-adjoint* if

$$\langle Tx, y\rangle = \langle x, Ty\rangle \qquad \forall\, x, y \in V.$$

$\diamondsuit$

Suppose V is finite dimensional, $E := \{u_1, \ldots, u_n\}$ is an (ordered) orthonormal basis of V, and $T : V \to V$ is a self-adjoint linear transformation. Then we have the following:

- $\langle Tu_j, u_i\rangle = \langle u_j, Tu_i\rangle$ for all $i, j = 1, \ldots, n$,
- $\mathbb{F} = \mathbb{C}$ and $V = \underline{\mathbb{C}}^n$ implies $[T]_{E,E}$ is a hermitian matrix,
- $\mathbb{F} = \mathbb{R}$ and $V = \underline{\mathbb{R}}^n$ implies $[T]_{E,E}$ is a symmetric matrix.

Observe that if $T : V \to V$ is a self-adjoint operator, then

$$\langle Tx, x\rangle \in \mathbb{R} \qquad \forall\, x \in V.$$

Using this fact we prove the following.

Theorem 3.17 *Eigenvalues of a self-adjoint operator are real.*

Proof. Let $T : V \to V$ be a self-adjoint operator and $\lambda \in \mathbb{F}$ be an eigenvalue of T. Let x be a corresponding eigenvector. Then $x \neq 0$ and $Tx = \lambda x$. Hence,

$$\langle Tx, x\rangle = \langle \lambda x, x\rangle = \lambda\langle x, x\rangle.$$

Since $\langle Tx, x\rangle$ and $\langle x, x\rangle$ is nonzero real, λ is also real. ∎

Corollary 3.18 *Eigenvalues of a Hermitian matrix are real.*

We also observe the following.

Theorem 3.19 *Eigenvectors associated with distinct eigenvalues of a self-adjoint operator are orthogonal.*

Proof. Let $T : V \to V$ be a self-adjoint operator and λ and μ be distinct eigenvalues of T. Let x and y be eigenvectors corresponding to λ and μ, respectively. Then, we have

$$\lambda\langle x, y\rangle = \langle \lambda x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \mu\langle x, y\rangle.$$

Hence, $\langle x, y\rangle = 0$. ∎

Next we have another important property of self-adjoint operators.

Theorem 3.20 *Every self-adjoint operator on a finite dimensional inner product space has an eigenvalue.*

Proof. We already know that if $\mathbb{F} = \mathbb{C}$, then every linear operator on a finite dimensional linear space has an eigenvalue. Hence, assume that V be an inner product space over $\mathbb{R}$ and $T : V \to V$ is self-adjoint.

Let $\dim(V) = n$ and let $A = (a_{ij})$ be a matrix representation of T with respect to an orthonormal basis $\{u_1, \ldots, u_n\}$. Then A as an operator on $\underline{\mathbb{C}}^n$ has an eigenvalue, say λ. Since $A : \underline{\mathbb{C}}^n \to \underline{\mathbb{C}}^n$ is self-adjoint, $\lambda \in \mathbb{R}$. Let $\underline{x} \in \mathbb{C}^n$ be an eigenvector corresponding to λ, that is, $\underline{x} \neq 0$ and $A\underline{x} = \lambda\underline{x}$. Let $\underline{u}$ and $\underline{v}$ be real and imaginary parts of $\underline{x}$. Then we have

$$A(\underline{u} + i\underline{v}) = \lambda(\underline{u} + i\underline{v}).$$

Therefore,

$$A\underline{u} = \lambda\,\underline{u}, \qquad A\underline{v} = \lambda\,\underline{v}.$$

Since $\underline{x}$ is nonzero, at least one of $\underline{u} \neq 0$ and $\underline{v} \neq 0$ is nonzero. Without loss of generality, assume that $\underline{u} \neq 0$. Thus, If $\alpha_1, \ldots, \alpha_n$ are the coordinates of $\underline{u}$, then $u := \sum_{j=1}^{n} \alpha_j u_j$ satisfies the equation $Tu = \lambda u$. ▮

In the Appendix (Section 3.11) we have given another proof for Theorem 3.20 which does not depend on the matrix representation of T.

We end this subsection with another property.

Theorem 3.21 *Let $T : V \to V$ be a self-adjoint operator on a finite dimensional inner product space V and V_0 be a subspace of V. Then*

$$T(V_0) \subseteq V_0 \Longrightarrow T(V_0^{\perp}) \subseteq V_0^{\perp}.$$

Proof. Suppose V_0 is a subspace of V such that $T(V_0) \subseteq V_0$. Now, let $x \in V_0^{\perp}$ and $y \in V_0$. Since $Ty \in V_0$, we have

$$\langle Tx, y \rangle = \langle x, Ty \rangle = 0.$$

Thus, $Ty \in V_0^{\perp}$ for every $y \in V_0^{\perp}$. ▮

3.7.2 Diagonalization of self-adjoint operators

First we prove the diagonalization theorem in the general context. Then state it in the setting of matrices.

Theorem 3.22 (Diagonalization theorem) *Let $T : V \to V$ be a self-adjoint operator on a finite dimensional inner product space V. Then there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T and let

$$V_0 = N(A - \lambda_1 I) + \ldots + N(A - \lambda_k I).$$

Then the union of orthonormal bases of $N(A - \lambda_1 I), \ldots, N(A - \lambda_k I)$ will be an orthonormal basis of V_0. Thus, if $V_0 = V$, then we are through.

Suppose $V_0 \neq V$. Then $V_0^{\perp} \neq 0$ (See Theorem 3.16). Now, it can be easily seen that $T(V_0) \subseteq V_0$. Hence, by Theorem 3.21, $T(W) \subseteq W$. Also, the operator $T_1 : V_0^{\perp} \to V_0^{\perp}$ defined by $T_1 x = Tx$ for every $x \in V_0^{\perp}$ is self-adjoint. Hence, T_1 has an eigenvalue, say λ. If $x \in V_0^{\perp}$ is a corresponding eigenvector, then we see also have $Tx = \lambda x$, so that $\lambda = \lambda_j$ for some $j \in \{1, \ldots, k\}$. Therefore, $x \in N(T - \lambda_j I) \subseteq V_0$. Thus, we obtain

$$0 \neq x \in V_0 \cap V_0^{\perp} = \{0\},$$

which is clearly a contradiction. ∎

We observe that, the method of the proof of the above theorem shows that, if $T : V \to V$ is a self-adjoint operator on a finite dimensional inner product space V and $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of T and if $\{u_{j1}, \ldots, u_{jm_j}\}$ is an ornthonromal basis of the eigenspace $N(T - \lambda_j I)$ for $j = 1, \ldots, k$, then the matrix representation of T with respect to the ordered basis

$$\{u_{11}, \ldots, u_{1m_j}, u_{21}, \ldots, u_{2m_j}, \ldots, u_{k1}, \ldots, u_{km_j}\}$$

is a diagonal matrix with diagonal entries

$$\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_k$$

with each λ_j appearing m_j times where

$$m_1 + \ldots + m_k = \dim(V), \qquad m_j = \dim N(T - \lambda_j I), \quad j = 1, \ldots, k.$$

If A is a square matrix, then let us denote the conjugate transpose of A by A^*. For the next result we introduce the following definition.

Definition 3.11 A square matrix A with entries from $\mathbb{F}$ is said to be a

1. *self-adjoint matrix* if $A^* = A$, and
2. *unitary matrix* if $A^* A = I = AA^*$.

◊

We note the following (Verify!):

- If A is unitary then it is invertible and $A^* = A^{-1}$ and columns of A form an orthonormal basis for $\underline{\mathbb{F}}^n$.

In view of Theorem 3.22 and the above observations, we have the following.

Theorem 3.23 *Let A be a self-adjoint matrix. Then there exists a unitary matrix U such that $U^{-1}AU$ is a diagonal matrix.*

3.8 Best Approximation

In applications one may come across functions which are too complicated to handle for computational purposes. In such cases, one would like to replace them by functions of "simpler forms" which are easy to handle. This is often done by approximating the given function by certain functions belonging to a finite dimensional space spanned by functions of simple forms. For instance, one may want to approximate a continuous function f defined on certain interval $[a, b]$ by a polynomial, say a polynomial p in $\mathcal{P}_n$ for some specified n. It is desirable to find that polynomial p such that

$$\|f - p\| \leq \|f - q\| \qquad \forall\, q \in \mathcal{P}_n.$$

Here, $\|.\|$ is a norm on $C[a, b]$. Now the question is whether such a polynomial exists, and if exists, then is it unique; and if there is a unique such polynomial, then how can we find it. These are the issues that we discuss in this section, in an abstract frame work of inner product spaces.

Definition 3.12 Let V be an inner product spaceand V_0 be a subspace of V. Let $x \in V$. A vector $x_0 \in V_0$ is a called a **best approximation** of x from V_0 if

$$\|x - x_0\| \leq \|x - v\| \qquad \forall\, v \in V_0.$$

$\diamondsuit$

Proposition 3.24 *Let V be an inner product space , V_0 be a subspace of V, and $x \in V$. If $x_0 \in V_0$ is such that $x - x_0 \perp V_0$, then x_0 is a best approximation of x, and it is the unique best approximation of x from V_0.*

Conversely, if $x_0 \in V_0$ is a best approximation of x, then $x - x_0 \perp V_0$.

Proof. Suppose $x_0 \in V_0$ is such that $x - x_0 \perp V_0$. Then, for every $u \in V_0$,

$$\begin{aligned} \|x - u\|^2 &= \|(x - x_0) + (x_0 - u)\|^2 \\ &= \|x - x_0\|^2 + \|x_0 - u\|^2. \end{aligned}$$

Hence

$$\|x - x_0\| \leq \|x - v\| \qquad \forall\, v \in V_0,$$

showing that x_0 is a best approximation.

To see the uniqueness, suppose that $v_0 \in V_0$ is another best approximation of x. Then, we have

$$\|x - x_0\| \le \|x - v_0\| \quad \text{and} \quad \|x - v_0\| \le \|x - x_0\|,$$

so that $\|x - x_0\| = \|x - v_0\|$. Therefore, using the fact that $\langle x - x_0, x_0 - v_0\rangle = 0$, we have

$$\|x - v_0\|^2 = \|x - x_0\|^2 + \|x_0 - v_0\|^2.$$

Hence, it follows that $\|x_0 - v_0\| = 0$. Thus $v_0 = x_0$.

Conversely, suppose that $x_0 \in V_0$ is a best approximation of x. Then $\|x - x_0\| \le \|x - u\|$ for all $u \in V_0$. In particular, if $v \in V_0$,

$$\|x - x_0\| \le \|x - (x_0 + \alpha v\| \quad \forall \alpha \in \mathbb{F}.$$

Hence, for every $\alpha \in \mathbb{F}$,

$$\begin{aligned} \|x - x_0\|^2 &\le \|x - (x_0 + \alpha v\|^2 \\ &= \langle (x - x_0) + \alpha v, (x - x_0) + \alpha v\rangle \\ &= \|x - x_0\|^2 - 2\mathrm{Re}\langle x - x_0, \alpha v\rangle + |\alpha|^2\|v\|^2. \end{aligned}$$

Taking $\alpha = \langle x - x_0, v\rangle/\|v\|^2$, we have

$$\langle x - x_0, \alpha v\rangle = \frac{|\langle x - x_0, v\rangle|^2}{\|v\|^2} = |\alpha|^2\|v\|^2$$

so that

$$\begin{aligned} \|x - x_0\|^2 &\le \|x - x_0\|^2 - 2\mathrm{Re}\langle x - x_0, \alpha v\rangle + |\alpha|^2\|v\|^2 \\ &= \|x - x_0\|^2 - \frac{|\langle x - x_0, v\rangle|}{\|v\|^2}. \end{aligned}$$

Hence, $\langle x - x_0, v\rangle = 0$. ∎

By the above proposition, in order to find a best approximation of $x \in V$ from V_0, it is enough to find a vector $x_0 \in V_0$ such that $x - x_0 \perp V_0$; and we know that such vector x_0 is unique.

Theorem 3.25 *Let V be an inner product space, V_0 be a finite dimensional subspace of V, and $x \in V$. Let $\{u_1, \ldots, u_n\}$ be an orthonormal basis of V_0. Then for $x \in V$, the vector*

$$x_0 := \sum_{i=1}^{n} \langle x, u_i\rangle u_i$$

is the unique best approximation of x from V_0.

Proof. Clearly, $x_0 := \sum_{i=1}^{n} \langle x, u_i \rangle u_i$ satisfies the hypothesis of Proposition 3.24. ∎

The above theorem shows how to find a best approximation from a finite dimensional subspace V_0, provided we know an orthonormal basis of V_0. Suppose we know only a basis of V_0. Then, we can find an orthonormal basis by Gram-Schmidt procedure. Another way to find a best approximation is to use Proposition 3.24:

Suppose $\{v_1, \ldots, v_n\}$ is a basis of V_0. By Proposition 3.24, the vector x_0 that we are looking for should satisfy $\langle x - x_0, v_i \rangle$ for every $i = 1, \ldots, n$. Thus, we have to find scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\left\langle x - \sum_{j=1}^{n} \alpha_j v_j,\ v_i \right\rangle = 0 \qquad \forall\, i = 1, \ldots, n.$$

That is to find $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{j=1}^{n} \langle v_j,\ v_i \rangle \alpha_j = \langle x, v_i \rangle \qquad \forall\, i = 1, \ldots, n.$$

The above system of equations is uniquely solvable (*Why?*) to get $\alpha_1, \ldots, \alpha_n$. Note that if the basis $\{v_1, \ldots, v_n\}$ is an orthonormal basis basis of V_0, then $\alpha_j = \langle x, v_j \rangle$ for $j = 1, \ldots, n$.

Exercise 3.15 Show that, if $\{v_1, \ldots, v_n\}$ is a linearly independent subset of an inner product space V, then the columns of the matrix $M := (a_{ij})$ with $a_{ij} = \langle v_j, v_i \rangle$, are linearly independent. Deduce that, the matrix is invertible. ◊

EXAMPLE 3.10 Let $V = \mathbb{R}^2$ with usual inner product, and $V_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. Let us find the best approximation of $x = (0, 1)$ from V_0.

We have to find a vector of the form $x_0 = (\alpha, \alpha)$ such that $x - x_0 = (0, 1) - (\alpha, \alpha) = (-\alpha, 1 - \alpha)$ is orthogonal to V_0. Since V_0 is spanned by the single vector $(1, 1)$, the requirement is to find α such that $(-\alpha, 1 - \alpha)$ is orthogonal to $(1, 1)$, i.e., α has to satisfy the equation $-\alpha + (1 - \alpha = 0$, i.e., $\alpha = 1/2$. Thus the best approximation of $x = (0, 1)$ from V_0 is the vector $x_0 = (1/2, 1/2)$. ◊

EXAMPLE 3.11 Let V be the vector space $C[0, 1]$ over $\mathbb{R}$ with the inner product: $\langle x, u \rangle = \int_0^1 x(t)u(t)dt$, and let $V_0 = \mathcal{P}_1$. Let us find the best approximation of x define by $x(t) = t^2$ from space V_0.

We have to find a vector x_0 of the form $x_0(t) = a_0 + a_1 t$ such that the function $x - x_0$ defined by $(x - x_0)(t) = t^2 - a_0 - a_1 t$ is orthogonal to V_0. Since V_0 is spanned by u_1, u_2 where $u_1(t) = 1$ and $u_2(t) = t$, the requirement is to find a_0, a_1 such that

$$\langle x - x_0, u_1 \rangle = \int_0^1 (t^2 - a_0 - a_1 t)dt = 0,$$

$$\langle x - x_0, u_2 \rangle = \int_0^1 (t^3 - a_0 t - a_1 t^2)dt = 0.$$

That is

$$\int_0^1 (t^2 - a_0 - a_1 t)dt = [t^3/3 - a_0 t - a_1 t^2/2]_0^1 = 1/3 - a_0 - a_1/2 = 0,$$

$$\int_0^1 (t^3 - a_0 t - a_1 t^2)dt = [t^4/4 - a_0 t^2/2 - a_1 t^3/3]_0^1 = 1/4 - a_0/2 - a_1/3 = 0.$$

Hence, $a_0 = -1/6$ and $a_1 = 1$, so that the best approximation x_0 of t^2 from $\mathcal{P}_1$ is given by $x_0(t) := -1/63 + t$. $\diamondsuit$

Exercise 3.16 Let V be an inner product space and V_0 be a finite dimensional subspace of V. Show that for every $x \in V$, there exists a unique pair of vectors u, v with $u \in V_0$ and $v \in V_0^{\perp}$ satisfying $x = u + v$. In fact,

$$V = V_0 + V_0^{\perp}. \quad \blacklozenge$$

$\diamondsuit$

Exercise 3.17 Let $V = C[0,1]$ over $\mathbb{R}$ with inner product $\langle x, u \rangle = \int_0^1 x(t)u(t)dt$. Let $V_0 = \mathcal{P}_3$. Find best approximation for x from V_0, where $x(t)$ is given by

(i) e^t, (ii) $\sin t$, (iii) $\cos t$, (iv) t^4. $\diamondsuit$

3.9 Best Approximate Solution

In this section we shall make use of the results from the previous section to define and find a *best approximate solution* for an equation $Ax = y$ where $A : V_1 \to V_2$ is a linear transformation between vector spaces V_1 and V_2 with V_2 being an inner product space.

Definition 3.13 Let V_1 and V_2 be vector spaces with V_2 being an inner product space, and let $A : V_1 \to V_2$ be a linear transformation. Let $y \in V_2$. Then a vector $x_0 \in V_1$ is called a **best approximate solution** or **a least-square solution** of the equation $Ax = y$ if

$$\|Ax_0 - y\| \leq \|Au - y\| \qquad \forall\, u \in V_1.$$

$\diamondsuit$

It is obvious that $x_0 \in V_1$ is a best approximate solution of $Ax = y$ if and only if $y_0 := Ax_0$ is a best approximation of y from the range space $R(A)$. Thus, from Proposition 3.24, we can conclude the following.

Theorem 3.26 *Let V_1 and V_2 be vector spaces with V_2 being an inner product space, and let $A : V_1 \to V_2$ be a linear transformation. If $R(A)$ is a finite dimensional subspace of V_2, then the equation $Ax = y$ has a best approximate solution. Moreover, a vector $x_0 \in V_1$ is a best approximate solution if and only if $Ax_0 - y$ is orthogonal to $R(A)$.*

Clearly, a best approximate solution is unique if and only if A is injective.

Next suppose that $A \in \mathbb{R}^{m\times n}$, i.e., A is an $m \times n$ matrix of real entries. Then we know that range space of A, viewing it as a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, is the space spanned by the columns of A. Suppose $u_1, \ldots, u_n$ be the columns of A. Then, given $y \in \mathbb{R}^m$, a vector $x_0 \in \mathbb{R}^n$ is a best approximate solution of $Ax = y$ if and only if $Ax_0 - y$ is orthogonal to u_i for $i = 1, \ldots, n$, i.e., if and only if $u_i^T(Ax_0 - y) = 0$ for $i = 1, \ldots, n$, i.e., if and only if $A^T(Ax_0 - y) = 0$, i.e., if and only if

$$A^T Ax_0 = A^T y.$$

EXAMPLE 3.12 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and let $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly, the equation $Ax = y$ has no solution. It can be seen that $x_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution of the equation $A^T Ax = A^T y$. Thus, x_0 is a best approximate solution of $Ax = y$.

$\diamondsuit$

3.10 QR-Factorization and Best Approximate Solution

Suppose that $A \in \mathbb{R}^{m\times n}$, i.e., A is an $m \times n$ matrix of real entries with $n \le m$. Assume that the columns of A are linearly independent. Then we know that, if the equation $Ax = y$ has a solution, then the solution is unique. Now, let $u_1, \ldots, u_n$ be the columns of A, and let $v_1, \ldots, v_n$ are orthonormal vectors obtained by orthonormalizing $u_1, \ldots, u_n$. Hence, we know that for each $k \in \{1, \ldots, n\}$,

$$\text{span}\,\{u_1, \ldots, u_k\} = \text{span}\,\{v_1, \ldots, v_k\}.$$

Hence, there exists an upper triangular $n \times n$ matrix $R := (a_{ij})$ such that $u_j = a_{1j}v_1 + a_{2j}v_2 + \ldots + a_{nj}v_j, \qquad j = 1, \ldots, n$. Thus,

$$[u_1,\, u_2,\, \ldots,\, u_n] = [v_1,\, v_2,\, \ldots,\, v_n]R.$$

Note that $A = [u_1,\, u_2,\, \ldots,\, u_n]$, and the matrix $Q := [v_1,\, v_2,\, \ldots,\, v_n]$ satisfies the relation

$$Q^TQ = I.$$

Definition 3.14 The factorization $A = QR$ with columns of Q being orthonormal and R being an upper triangular matrix is called a **QR-factorization** of A. ◊

We have see that if columns of $A \in \mathbb{R}^{m\times n}$ are linearly independent, then A has a QR-factorization.

Now, suppose that $A \in \mathbb{R}^{m\times n}$ with columns of A are linearly independent, and $A = QR$ is the QR-factorization of A. Let $y \in \mathbb{R}^m$. Since columns of A are linearly independent, the equation $Ax = y$ has a unique best approximate solution, say x_0. Then we know that

$$A^TAx_0 = A^Ty.$$

Using the QR-factorization $A = QR$ of A, we have

$$R^TQ^TQRx_0 = R^TQ^Ty.$$

Now, $Q^TQ = I$, and R^T is injective, so that it follows that

$$Rx_0 = Q^Ty.$$

Thus, if $A = QR$ is the QR-factorization of A, then the best approximate solution of $Ax = y$ is obtained by solving the equation

$$Rx = Q^T y.$$

For more details on best approximate solution one may see
http://mat.iitm.ac.in/∼mtnair/LRN-Talk.pdf

3.11 Appendix

Another proof for Theorem 3.20. Assuming $\mathbb{F} = \mathbb{R}$, another way of proving Theorem 3.20 is as follows: Consider a new vector space $\widetilde{V} := \{u + iv : u, v \in V\}$ over $\mathbb{C}$. For $u + iv$, $u_1 + iv_1$, $u_2 + iv_2$ in $\widetilde{V}$ and $\alpha + i\beta) \in \mathbb{C}$ with $(\alpha, \beta) \in \mathbb{R}^2$ the addition and scalar multiplication are defined as

$$(u_1 + iv_1) + (u_2 + iv_2) := (u_1 + u_2) + i(v_1 + v_2),$$

$$(\alpha + i\beta)(u + iv) := (\alpha u - \beta v) + i(\alpha v + \beta u).$$

The inner product on $\widetilde{V}$ is defined as

$$\langle u_1 + iv_1,\, u_2 + iv_2\rangle := \langle u_1, u_2\rangle + \langle v_1, v_2\rangle.$$

Define $\widetilde{T} : \widetilde{V} \to \widetilde{V}$ by

$$\widetilde{T}(u + iv) := Tu + iTv.$$

Then, self-adjointness of T implies that $\widetilde{T}$ is also self-adjoint. Indeed,

$$\begin{aligned}
\langle \widetilde{T}(u_1 + iv_1),\, u_2 + iv_2\rangle &= \langle Tu_1 + iTv_1),\, u_2 + iv_2\rangle \\
&= \langle Tu_1, u_2\rangle + \langle Tv_1, v_2\rangle \\
&= \langle u_1, Tu_2\rangle + \langle v_1, Tv_2\rangle \\
&= \langle u_1 + iv_1,\, Tu_2 + iTv_2\rangle \\
&= \langle u_1 + iv_1,\, \widetilde{T}(u_2 + iv_2)\rangle.
\end{aligned}$$

Now, let $\lambda \in \mathbb{R}$ be an eigenvalue of $\widetilde{T}$ with a corresponding eigenvector $\tilde{x} := u + iv$. Since

$$\widetilde{T}\tilde{x} = \lambda\tilde{x} \iff Tu + iTv = \lambda u + i\lambda v \iff Tu = \lambda u \,\&Tv = \lambda v$$

and since one of u and v is nonzero, it follows that λ is an eigenvalue of T as well. ∎

4

Error Bounds and Stability of Linear Systems

4.1 Norms of Vectors and Matrices

Recall that a norm $\|\cdot\|$ on a vector space V is a function which associates each $x \in V$ a unique non-negative real number $\|x\|$ such that the following hold:

(a) For $x \in V$, $\|x\| = 0 \iff x = 0$

(b) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$,

(c) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in \mathbb{F}, \, x \in V$.

We have already seen that if V is an inner product space, then the function $x \mapsto \|x\| := \langle x, x\rangle^{1/2}$ is a norm on V. It can be easily sen that for $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$,

$$\|x\|_1 := \sum_{j=1}^{k} |x_j|, \qquad \|x\|_\infty := \max_{1 \leq i \leq k} |x_i|$$

define norms on $\mathbb{R}^k$. The norm induced by the standard inner product on $\mathbb{R}^k$ is denoted by $\|\cdot\|_2$, i.e.,

$$\|x\|_2 := \Big(\sum_{j=1}^{k} |x(t)|^2 \Big)^{1/2}.$$

Exercise 4.1 Show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ for every $x \in \mathbb{R}^k$. Compute $\|x\|_\infty$, $\|x\|_2$, $\|x\|_1$ for $x = (1, 1, 1) \in \mathbb{R}^3$.

We know that on $C[a, b]$,

$$\|x\|_2 := \langle x, x\rangle^{1/2} = \Big(\int_a^b |x(t)|^2 dt \Big)^{1/2}$$

defines a norm. It is easy to show that

$$\|x\|_1 := \int_a^b |x(t)|dt \qquad \|x\|_\infty := \max_{a\le b} |x(t)|$$

also define norms on $C[a,b]$.

Exercise 4.2 Show that there exists no constant $c>0$ such that $\|x\|_\infty \le c\,\|x\|_1$ for all $x \in C[a,b]$.

Next we consider norms of matrices. Considering an $n\times n$ matrix as an element of $\mathbb{R}^{n^2}$, we can obtain norms of matrices. Thus, analogues to the norms $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$ on $\mathbb{R}^n$, for $A=(a_{ij})\in\mathbb{R}^{n\times n}$, the quantities

$$\sum_{i=1}^n\sum_{j=1}^n |a_{ij}|, \qquad \Big(\sum_{i=1}^n\sum_{j=1}^n |a_{ij}|^2\Big)^{1/2}, \qquad \max_{1\le i,j\le n} |a_{ij}|$$

define norms on $\mathbb{R}^{n\times n}$.

Given a vector norm $\|\cdot\|$ on $\mathbb{R}^n$, it can be seen that

$$\|A\| := \sup_{\|x\|\le 1} \|Ax\|, \qquad A\in\mathbb{R}^{n\times n},$$

defines a norm on the space $\mathbb{R}^{n\times n}$. Since this norm is associated with the norm of the space $\mathbb{R}^n$, and since a matrix can be considered as a linear operator on $\mathbb{R}^n$, the above norm on $\mathbb{R}^{n\times n}$ is called a *matrix norm associated with a vector norm.*

The above norm has certain important properties that other norms may not have. For example, it can be seen that

- $\|Ax\| \le \|A\|\,\|x\| \quad \forall x\in\mathbb{R}^n$,
- $\|Ax\| \le c\,\|x\| \quad \forall x\in\mathbb{R}^n \implies \|A\|\le c,.$

Moreover, if $A, B\in\mathbb{R}^{n\times n}$ and if I is the identity matrix, then

- $\|AB\| \le \|A\|\,\|B\|, \qquad \|I\| = 1.$

Exercise 4.3 Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and and $A\in\mathbb{R}^{n\times n}$. Suppose $c>0$ is such that $\|Ax\|\le c\,\|x\|$ for all $x\in\mathbb{R}^n$, and there exists $x_0\ne 0$ in $\mathbb{R}^n$ such that $\|Ax_0\| = c\,\|x_0\|$. Then show that $\|A\| = c$.

In certain cases operator norm can be computed from the knowledge of the entries of the matrix. Let us denote the matrix norm associated with $\|\cdot\|_1$ and $\|\cdot\|_\infty$ by the same notation, i.e., for $p \in \{1, \infty\}$,

$$\|A\|_p := \sup_{\|x\|_p \le 1} \|Ax\|_p, \qquad A \in \mathbb{R}^{n\times n}.$$

Theorem 4.1 *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, then*

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|, \qquad \|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.$$

Proof. Note that for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned}\|Ax\|_1 &= \sum_{i=1}^n \Big|\sum_{j=1}^n a_{ij}x_j\Big| \le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}|\,|x_j| \\ &= \sum_{j=1}^n \Big(\sum_{i=1}^n |a_{ij}|\Big)|x_j| \le \Big(\max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|\Big) \sum_{j=1}^n |x_j|.\end{aligned}$$

Thus, $\|A\|_1 \le \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|$. Also, note that $\|Ae_j\|_1 = \sum_{i=1}^n |a_{ij}|$ for every $j \in \{1, \ldots, n\}$ so that $\sum_{i=1}^n |a_{ij}| \le \|A\|_1$ for every $j \in \{1, \ldots, n\}$. Hence, $\max_{1\le j\le n} \sum_{i=1}^n |a_{ij}| \le \|A\|_1$. Thus, we have shown that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{ij}|.$$

Next, consider the norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$. In this case, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$\|Ax\|_\infty = \max_{1\le i\le n} \Big|\sum_{j=1}^n a_{ij}x_j\Big|.$$

Since

$$\Big|\sum_{j=1}^n a_{ij}x_j\Big| \le \sum_{j=1}^n |a_{ij}|\,|x_j| \le \|x\|_\infty \sum_{j=1}^n |a_{ij}|,$$

it follows that

$$\|Ax\|_\infty \le \Big(\max_{1`i\le n} \sum_{j=1}^n |a_{ij}|\Big)\|x\|_\infty.$$

From this we have $\|A\|_\infty \le \max_{1`i\le n} \sum_{j=1}^n |a_{ij}|$. Now, let $i_0 \in \{1, \ldots, n\}$ be such that $\max_{1\le i\le n} \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{i_0 j}|$, and let $x_0 = (\alpha_1, \ldots, \alpha_n)$ be such that $\alpha_j = \begin{cases} |a_{i_0 j}|/a_{i_0 j} & \text{if } a_{i_0 j} \neq 0, \\ 0 & \text{if } a_{i_0 j} \neq 0. \end{cases}$ Then $\|x_0\|_\infty = 1$ and

$$\sum_{j=1}^n |a_{i_0 j}| = \Big|\sum_{j=1}^n a_{i_0 j}\alpha_j\Big| = |(Ax_0)_{i_o}| \le \|Ax_0\|_\infty \le \|A\|_\infty.$$

Thus, $\max_{1\le i\le n} \sum_{j=1}^n |a_{ij}| = \sum_{j=1}^n |a_{i_0 j}| \le \|A\|_\infty$. Thus we have proved that

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|.$$

This completes the proof of the theorem. ∎

What about the matrix norm

$$\|A\|_2 := \max_{\|x\|\le 1} \|Ax\|_2, \quad A \in \mathbb{R}^{n\times n},$$

induced by $\|\cdot\|_2$ on $\mathbb{R}^n$? In fact, there is no simple representation for this in terms of the entries of the matrix. However, we have the following.

Theorem 4.2 *Suppose $A = (a_{ij}) \in \mathbb{R}^{n\times n}$. Then*

$$\|A\|_2 \le \Big(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2\Big)^{1/2}.$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (non-negative) eigenvalues of the matrix $A^T A$, then

$$\|A\|_2 = \max_{1\le \mathrm{J}\le n} \sqrt{\lambda_j}.$$

Proof. Using the Cauchy-Schwarz inequality on $\mathbb{R}^n$, we have, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} \|Ax\|_2^2 &= \sum_{i=1}^{n} \Big| \sum_{j=1}^{n} a_{ij} x_j \Big|^2 \\ &\le \sum_{i=1}^{n} \Big[\Big(\sum_{j=1}^{n} |a_{ij}|^2 \Big) \Big(\sum_{j=1}^{n} |x_j|^2 \Big) \Big] \\ &\le \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \Big) \|x\|_2^2. \end{aligned}$$

Thus, $\|A\|_2 \le \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \Big)^{1/2}$.

Since $A^T A$ is a symmetric matrix, it has n real eigenvalues (may be some of the are repeated) with corresponding orthonormal eigenvectors $u_1, u_n, \ldots, u_n$. Note that, for every $j \in \{1, 2, \ldots, n\}$,

$$\lambda_j = \lambda_j \langle u_j, u_j \rangle = \langle \lambda_j u_j, u_j \rangle = \langle A^T A u_j, u_j \rangle = \langle A u_j, A u_j \rangle = \|A u_j\|^2$$

so that λ_j' s are non-negative, and $|\lambda_j| \le \|A\|$ for all j. Thus,

$$\max_{1 \le j \le n} \sqrt{\lambda_j} \le \|A\|.$$

To see the reverse inequality, first we observe that $u_1, u_n, \ldots, u_n$ form an orthonormal basis of $\mathbb{R}^n$. Hence, every $x \in \mathbb{R}^n$ can be written as $x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j$, so that

$$A^T A x = \sum_{j=1}^{n} \langle x, u_j \rangle A^T A u_j = \sum_{j=1}^{n} \langle x, u_j \rangle \lambda_j u_j.$$

Thus, we have $\|Ax\|^2 = \langle Ax, Ax \rangle = \langle ATAx, x \rangle$ so that

$$\begin{aligned} \|Ax\|^2 &= \Big\langle \sum_{j=1}^{n} \langle x, u_j \rangle \lambda_j u_j, \sum_{i=1}^{n} \langle x, u_i \rangle u_i \Big\rangle \\ &= \sum_{j=1}^{n} |\langle x, u_j \rangle|^2 \lambda_j \\ &\le \Big(\max_{1 \le j \le n} \lambda_j \Big) \|x\|^2. \end{aligned}$$

Hence, $\|A\|_2 \le \max_{1 \le j \le n} \sqrt{\lambda_j}$. This completes the proof. ∎

Exercise 4.4 Find $\|A\|_1$, $\|A\|_\infty$, for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

4.2 Error Bounds for System of Equations

Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$, consider the equation

$$Ax = b.$$

Suppose the data b is not known exactly, but a perturbed data $\tilde{b}$ is known. Let $\tilde{x} \in \mathbb{R}^n$ be the corresponding solution, i.e.,

$$A\tilde{x} = \tilde{b}.$$

Then, we have $x - \tilde{x} = A^{-1}(b - \tilde{b})$ so that

$$\|x-\tilde{x}\| \le \|A^{-1}\|b-\tilde{b}\| = \|A^{-1}\|b-\tilde{b}\|\frac{\|Ax\|}{\|b\|} \le \|A\|\,\|A^{-1}\|\frac{\|b-\tilde{b}\|}{\|b\|}\|x\|,$$

$$\|b-\tilde{b}\| \le \|A\|\,\|x-\tilde{x}\| = \|A\|\,\|x-\tilde{x}\|\frac{\|A^{-1}b\|}{\|x\|} \le \|A\|\,\|A^{-1}\|\frac{\|x-\tilde{x}\|}{\|x\|}\|b\|.$$

Thus, denoting the quantity $\|A\|\,\|A^{-1}\|$ by $\kappa(A)$,

$$\frac{1}{\kappa(A)}\frac{\|b-\tilde{b}\|}{\|b\|} \le \frac{\|x-\tilde{x}\|}{\|x\|} \le \kappa(A)\frac{\|b-\tilde{b}\|}{\|b\|}. \tag{4.1}$$

From the above inequalities, it can be inferred that if $\kappa(A)$ is large, then it can happen that for small relative error $\|b-\tilde{b}\|/\|b\|$ in the data, the relative error $\|x-\tilde{x}\|/\|x\|$ in the solution may be large. In fact, there do exist $b, \tilde{b}$ such that

$$\frac{\|x-\tilde{x}\|}{\|x\|} = \kappa(A)\frac{\|b-\tilde{b}\|}{\|b\|},$$

where $x, \tilde{x}$ are such that $Ax = b$ and $A\tilde{x} = \tilde{b}$. To see this, let x_0 and u be vectors such that

$$\|Ax_0\| = \|A\|\,\|x_0\|, \qquad \|A^{-1}u\| = \|A^{-1}\|\,\|u\|,$$

and let

$$b := Ax_0, \quad \tilde{b} := b + u, \quad \tilde{x} := x_0 + A^{-1}u.$$

Then it follows that $A\tilde{x} = \tilde{b}$ and

$$\frac{\|x_0 - \tilde{x}\|}{\|x_0\|} = \frac{\|A^{-1}u\|}{\|x_0\|} = \frac{\|A^{-1}\|\,\|u\|}{\|x_0\|} = \frac{\|A\|\,\|A^{-1}\|\,\|u\|}{\|Ax_0\|} = \kappa(A)\frac{\|b - \tilde{b}\|}{\|b\|}.$$

The quantity $\kappa(A) := \|A\|\,\|A^{-1}\|$ is called the **condition number** of the matrix A. To illustrate the observation in the preceding paragraph, let us consider

$$A = \begin{bmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

It can be seen that

$$A^{-1} = \frac{1}{\varepsilon^2}\begin{bmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{bmatrix} \quad \text{so that} \quad x = A^{-1}b = -\frac{1}{\varepsilon}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

From this, it is clear that, if ε is small, then for small $\|b\|$, $\|x\|$ can be very large. In this case, it can be seen that

$$\|A\|_\infty = 2+\varepsilon, \quad \|A^{-1}\|_\infty = \frac{1}{\varepsilon^2}(2 = \varepsilon), \quad \kappa(A) = \Big(\frac{2+\varepsilon}{\varepsilon}\Big)^2 > \frac{4}{\varepsilon^2}.$$

In practice, while solving $Ax = b$ by numerically, we obtain an approximate solution $\tilde{x}$ in place of the actual solution. One would like to know how much error incurred by this procedure. We can have inference on this from (4.1), by taking $\tilde{b} := A\tilde{x}$.

Exercise 4.5 Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Then there exist vectors x, u such that $\|Ax_0\| = \|A\|\,\|x_0\|$ and $\|A^{-1}u\| = \|A^{-1}\|\,\|u\|$ – Justify.

Exercise 4.6

1. Suppose A, B in $\mathbb{R}^{n\times n}$ are invertible matrices, and $b, \tilde{b}$ are in $\mathbb{R}^n$. Let $x, \tilde{x}$ are in $\mathbb{R}^n$ be such that $Ax = b$ and $B\tilde{x} = \tilde{b}$. Show that

   $$\frac{\|x - \tilde{x}\|}{\|x\|} \le \|A\|\,\|B^{-1}\|\Big(\frac{\|A - B\|}{\|A\|} + \frac{\|b - \tilde{b}\|}{\|b\|}\Big).$$

   [Hint: Use the fact that $B(x - \tilde{x}) = (B - A)x + (b - \tilde{b})$, and use the fact that $\|(B - A)x\| \le \|B - A\|\,\|x\|$, and $\|b - \tilde{b}\| = \|b - \tilde{b}\|\|Ax\|/\|b\| \le \|b - \tilde{b}\|\|A\|\,\|x\|/\|b\|$.]

2. Let $B \in \mathbb{R}^{n\times n}$. If $\|B\| < 1$, then show that $I - B$ is invertible, and $\|(I - B)^{-1}\| \le 1/(1 - \|B\|)$.

   [Hint: Show that $I - B$ is injective, by showing that for every x, $\|(I - B)x\| \ge (1 - \|B\|)\|x\|$, and then deduce the result.]

3. Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A - B\| < 1/\|A^{-1}\|$. Then, show that, B is invertible, and

$$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A - B\|\,\|A^{-1}\|}.$$

[Hint: Observe that $B = A - (A - B) = [I - (A - B)A^{-1}]A$, and use the previous exercise.]

4. Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A - B\| < 1/2\|A^{-1}\|$. Let $b, \tilde{b}, x, \tilde{x}$ be as in Exercise 1. Then, show that, B is invertible, and

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le 2\kappa(A)\Big(\frac{\|A - B\|}{\|A\|} + \frac{\|b - \tilde{b}\|}{\|b\|}\Big).$$

[Hint: Apply conclusion in Exercise 3 to that in Exercise 1.]

5

Fixed Point Iterations for Solving Equations

Suppose S is a non-empty set and $f : S \to S$ is a function. Our concern in this chapter is to find an $x \in S$ such that $x = f(x)$.

Definition 5.1 A point $x \in S$ is called a **fixed point** of $f : S \to S$ if $x = f(x)$.

It is to be mentioned that a problem of finding zeros of a function can be converted into a problem of finding fixed points of an appropriate function. A simplest case is the following:

Suppose S is a subset of a vector space V and $g : S \to V$. Then for $x \in S$,

$$g(x) = 0 \quad \text{if and only if} \quad x = f(x),$$

where $f(x) = x - g(x)$. Thus, if $x - g(x) \in S$ for every $x \in S$, then the problem of solving $g(x) = 0$ is same as finding a fixed point of $f : S \to S$.

It is to be remarked that a function may not have a fixed point or may have more than one fixed point. For example

- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ has no fixed point,
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$ has exactly one fixed point,
- $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ has exactly two fixed point,
- $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_2, x_1)$ has infinitely many fixed points
- $f : C[0,1] \to C[0,1]$ defined by $f(x)(t) = \int_0^t x(s)ds$ has no fixed point.

Now, suppose that S is a subset of a normed vector space V with a norm $\|\cdot\|$. For finding a fixed point of $f : S \to S$, one may consider the following iterative procedure to construct a sequence (x_n) in S:

> Start with some $x_0 \in S$, then define iteratively
>
> $$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots.$$
>
> One may enquire whether (x_n) converges to a fixed point of f.

Suppose the above iterations converge to some $x \in V$, i.e., suppose there exists an $x \in V$ such that $\|x - x_n\| \to 0$ as $n \to \infty$. Then the question is whether $x \in S$ and $f(x) = x$.

We require the following definition.

Definition 5.2 Let S be a subset of a normed vector space V, and $f : S \to S$.

(a) The set S is said to be a **closed set** if it has the property that $x \in V$ and $x = \lim_{n\to\infty} x_n$ for some sequence (x_n) in S implies $x \in S$.

(b) The function f is said to be **continuous** (on S) if for every sequence (x_n) in S which converges to a point $x \in S$, the sequence $(f(x_n))$ converges to $f(x)$.

Using the above definition, the proof of the following proposition is obvious.

Proposition 5.1 *Suppose S is a subset of a normed vector space V, and $f : S \to S$. Let $x_0 \in S$, and (x_n) is defined iteratively by $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$. Suppose that (x_n) converges to an $x \in V$. If f is continuous and S is closed then x is a fixed point of f.*

In the above proposition the assumption that the sequence (x_n) converges is a strong one. Sometimes it is easy to show that a sequence is a *Cauchy sequence*.

Definition 5.3 Let V be a normed vector space V. A sequence (x_n) is V is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a positive integer N such that $\|x_n - x_m\| < \varepsilon$ for all $n, m \geq N$.

A normed space in which every Cauchy sequence converges is called a *Banach space*

Examples of Banach spaces are

- $\mathbb{R}^k$ with $\|\cdot\|_1$ or $\|\cdot\|_2$ or $\|\cdot\|_\infty$,
- $C[a,b]$ with $\|x\|_\infty := \max\{|x(t)| : a \le t \le b\}$.

It is known that every finite dimensional vector space with any norm is a Banach space, whereas every infinite dimensional space need not be a Banach space with respect to certain norms. For instance, it can be shown easily that, $C[a,b]$ with the norm $\|x\|_1 := \int_a^b |x(t)|dt$ is not a Banach space.

Theorem 5.2 *Suppose S is a closed subset of a Banach vector space V, and $f : S \to S$ satisfies*

$$\|f(x) - f(y)\| \le \rho\|x - y\| \qquad \forall x, y \in S,$$

for some constant ρ satisfying $0 < \rho < 1$. Then f has a unique fixed point. In fact, for any $x_0 \in S$, if we define

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots,$$

iteratively, then (x_n) converges to a unique fixed point $x \in S$ of f, and

$$\|x_{n+1} - x_n\| \le \rho\|x_n - x_{n-1}\| \le \rho^n\|x_1 - x_0\| \quad \forall n \in \mathbb{N},$$

$$\|x_n - x_m\| \le \frac{\rho^m}{1-\rho}\|x_1 - x_0\| \quad \forall n > m,$$

$$\|x - x_n\| \le \frac{\rho^n}{1-\rho}\|x_1 - x_0\| \quad \forall n \in \mathbb{N}.$$

Proof. Let $x_0 \in S$, and define

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots.$$

Then

$$\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1}\| \le \rho\|x_n - x_{n-1}\| \quad \forall n \in \mathbb{N},$$

so that

$$\|x_{n+1} - x_n\| \le \rho^n\|x_1 - x_0\| \quad \forall n \in \mathbb{N}.$$

Now, let $n > m$. Then

$$\begin{aligned}\|x_n - x_m\| &\le \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \ldots + \|x_{m+1} - x_m\| \\ &\le (\rho^{n-1} + \rho^{n-2} + \ldots + \rho^m)\|x_1 - x_0\| \\ &\le \frac{\rho^m}{1-\rho}\|x_1 - x_0\|.\end{aligned}$$

Since $\rho^m \to 0$ as $m \to \infty$, (x_n) is a Cauchy sequence. Since V is a Banach space (x_n) converges to some $x \in V$, and since S is a closed set, $x \in S$. It also follows that, for all $m \in \mathbb{N}$,

$$\|x - x_m\| = \lim_{n\to\infty} \|x_n - x_m\| \le \frac{\rho^m}{1-\rho}\|x_1 - x_0\|.$$

Observe that

$$\begin{aligned}\|x - f(x)\| &\le \|(x - x_m) + (x_m - f(x)\| \\ &\le \|x - x_m\| + \|x_m - f(x)\| \\ &\le \|x - x_m\| + \rho\|x_{m-1} - x\| \\ &\le \frac{\rho^m}{1-\rho}\|x_1 - x_0\| + \rho\frac{\rho^{m-1}}{1-\rho}\|x_1 - x_0\| \\ &\le \frac{2\rho^m}{1-\rho}\|x_1 - x_0\|.\end{aligned}$$

Since $\rho^m \to 0$ as $m \to \infty$, it follows that $\|x - f(x)\| = 0$, i.e., $x = f(x)$, i.e., x is a fixed point of f. Now, to show that there is only one fixed point of f, suppose u and v are fixed points of f, i.e., $u = f(u)$ and $v = f(v)$. Then we have

$$\|u - v\| = \|f(u) - f(v)\| \le \rho\|u - v\|$$

so that $(1-\rho)\|u - v\| \le 0$. Since $1 - \rho > 0$ and $\|u - v\| \le 0$, we see that $\|u - v\| = 0$, i.e., $u = v$. ∎

Remark 5.1 For certain functions $f : S \to V$, sometimes one may be able to show that $\|f(x) - f(y)\| \le \rho\|x - y\|$ for all $x, y \in S$ for some $\rho > 0$, but the condition "$f(x) \in S$ for all $x \in S$" may not be satisfied. In this case the above theorem cannot be applied. For example, suppose $f : [1,2] \to \mathbb{R}$ be defined by $f(x) = x/2$. Then, we have $|f(x) - f(y)| = \rho|x - y|$ with $\rho = 1/2$, but $f(3/2) = 3/4 \notin [1,2]$.

Now, suppose that $\|f(x) - f(y)\| \le \rho\|x - y\|$ for all $x, y \in S$ and if we also know that

(i) f has a fixed point x^*, and

(ii) S contains $D_r := \{x \in V : \|x - x^*\| \le r\}$ for some $r > 0$. Then it follows that $f(x) \in D_r$ for all $x \in D_r$. Indeed, for $x \in D_r$,

$$\|f(x) - x^*\| = \|f(x) - f(x^*)\| \le \rho \|x - x^*\| < \|x - x^*\| \le r.$$

Thus, under the additional assumptions (i) and (ii), we can generate the iterations with any $x_0 \in D_r$.

Remark 5.2 In order to have certain accuracy of the approximation, say for the error $\|x - x_n\|$ to at most $\varepsilon > 0$, we have to take n large enough so that

$$\frac{\rho^n}{1-\rho}\|x_1 - x_0\| < \varepsilon,$$

that is, error $\|x - x_n\| \le \varepsilon > 0$ for all n satisfying

$$n \ge \frac{\log\Big(\|x_1 - x_0\|/\varepsilon(1-\rho)\Big)}{\log(1/\rho)}.$$

5.1 Iterative Methods for Solving $Ax = b$

Suppose $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. We would like to convert the problem of solving $Ax = b$ into that of finding a fixed point of certain other system. In this regard, the following result is of great use.

Theorem 5.3 *Let $C \in \mathbb{R}^{n\times n}$ and $d \in \mathbb{R}^n$. Let $x^{(0)} \in \mathbb{R}^n$ be given and $x^{(k)} \in \mathbb{R}^n$ be defined iteratively as*

$$x^{(k)} = Cx^{(k-1)} + d, \quad n = 1, 2, \ldots.$$

If $\|C\| < 1$, then $(x^{(k)})$ converges to a (unique) fixed point of the system $x = Cx + d$ and

$$\|x - x^{(k)}\| \le \frac{\rho^k}{1-\rho}\|x^{(1)} - x^{(0)}\|, \quad \rho := \|C\|.$$

Proof. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$F(x) = Cx + d, \quad x \in \mathbb{R}^n.$$

Then we have

$$\|F(x) - F(y)\| = \|C(x - y)\| \le \|C\|\,\|x - y\|.$$

Hence, the result follows from Theorem 5.2. ∎

Now let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Note that for $x \in \mathbb{R}^n$,

$$Ax = b \iff x = (I - A)x + b.$$

Hence, form Theorem 5.3, the iterations

$$x^{(k)} = (I - A)x^{(k-)} + b, \quad n = 1, 2, \ldots$$

converges to a unique solution of $Ax = b$, provided $\|I - A\| < 1$.

EXAMPLE 5.1 Consider the system $Ax = b$ with

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{2} & \frac{2}{9} \\ \frac{1}{6} & \frac{1}{3} & \frac{4}{5} \end{bmatrix} \quad \text{so that} \quad A - I = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{2}{9} \\ \frac{1}{6} & \frac{1}{3} & -\frac{1}{5} \end{bmatrix}$$

Thus, it follows that $\|I - A\|_\infty = 35/36$ and $\|I - A\|_1 = 31/30$. Thus, the error estimates for the above described iterative procedure for this example is valid if we take $\|\cdot\|_\infty$ on $\mathbb{R}^3$, but not with $\|\cdot\|_1$.

The idea of resorting to an iterative procedure for finding approximate solution of $Ax = b$ is when it is not easy to solve it exactly. Suppose we can write $A = A_1 + A_2$, where the system $A_1x = v$ can be solved easily. Then, we may write $Ax = b$ as $A_1x = b - A_2x$, so that the system $Ax = b$ is equivalent to the system

$$x = A_1^{-1}b - A_1^{-1}A_2x.$$

Suppose, for a given $x^{(0)}$, we define $x^{(k)}$ by

$$A_1x^{(k)} = b - A_2x^{(k-1)}, \quad k = 1, 2, \ldots.$$

Then, by Theorem 5.3, if $\|A_1^{-1}A_2\| < 1$, then $(x^{(k)})$ converges to a unique solution of $x = A_1^{-1}b - A_1^{-1}A_2x$ which is same as $Ax = b$.

5.1.1 Jacobi Method

Let $A = (a_{ij})$ be an $n \times n$ matrix. In Jacobi method, we assume that $a_{ii} \neq 0$ for all $i = 1, \ldots, n$, and define the Jacobi iterations by

$$x_i^{(k)} = \frac{1}{a_{ii}}\Big[b_i - \sum_{j\neq i} a_{ij}x_j^{(k-1)}\Big], \quad k = 1, 2, \ldots$$

for $i = 1, \ldots, n$. This is equivalent to *splitting* of A as $A = A_1 + A_2$ with A_1 being the diagonal matrix consisting of the diagonal entries of A. Thus, convergence of the Jacobi iterations to the unique solution of $Ax = b$ is ensured if $\|A_1^{-1}A_2\| < 1$. If we take the norm $\|\cdot\|_\infty$ on $\mathbb{R}^n$, then we have

$$\|A_1^{-1}A_2\|_\infty = \max_i \frac{1}{|a_{ii}|} \sum_{j \neq i} |a_{ij}|.$$

Hence, required condition is

$$\sum_{j \neq i} |a_{ij}| < |a_{ii}| \quad \forall i = 1, \ldots, n.$$

EXAMPLE 5.2 Consider the system $Ax = b$ with

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix}.$$

For applying Jacobi method, we take

$$A_1 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 11 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 3 \\ 3 & 4 & 0 \end{bmatrix}.$$

We see that

$$A_1^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{11} \end{bmatrix} \quad \text{so that} \quad A_1^{-1}A_2 = \begin{bmatrix} 0 & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{5} & 0 & \frac{3}{10} \\ \frac{3}{11} & \frac{4}{11} & 0 \end{bmatrix}.$$

Thus, it follows that $\|A_1^{-1}A_2\|_\infty = 7/11$ and $\|A_1^{-1}A_2\|_1 = 47/99$. Thus, the error estimates for the above described iterative procedure for this example is valid if we take either $\|\cdot\|_\infty$ or $\|\cdot\|_1$ on $\mathbb{R}^3$. For instance, taking $\|\cdot\|_1$ on $\mathbb{R}^3$ and $\rho = 47/99$, we have

$$\|x - x^{(k)}\|_1 \leq \frac{\rho^k}{1-\rho}\|x^{(1)} - x^{(0)}\| = \frac{99}{52}\Big(\frac{47}{99}\Big)^k \|x^{(1)} - x^{(0)}\|.$$

5.1.2 Gauss-Siedel Method

Let $A = (a_{ij})$ be an $n \times n$ matrix. In this method also, we assume that $a_{ii} \neq 0$ for all $i = 1, \ldots, n$. In this case we view the system $Ax = b$ as

$$A_1 x = b - A_2 x,$$

where A_1 is the lower triangular part of A including the diagonal, and $A_2 = A - A_1$, i.e.,

$$A_1 = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot & \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot & \\ 0 & 0 & 0 & \dots & a_{nn} \end{bmatrix}$$

Thus, the Gauss-Siedel iterations are defined by

$$x_i^{(1)} = \frac{1}{a_{11}}\Big[b_i - \sum_{j>i} a_{ij} x_j^{(k-1)}\Big],$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\Big[b_i - \sum_{j<i} a_{ij} x_j^{(k)} - \sum_{j>i} a_{ij} x_j^{(k-1)}\Big], \qquad k = 2, \dots$$

for $i = 1, \dots, n$. The convergence of the Gauss-Siedel iterations to the unique solution of $Ax = b$ is ensured if $\|A_1^{-1}A_2\| < 1$.

EXAMPLE 5.3 Again consider the system $Ax = b$ with

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix}.$$

For applying Gauss-Siedel method, we take

$$A_1 = \begin{bmatrix} 9 & 0 & 0 \\ 2 & 10 & 0 \\ 3 & 4 & 11 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

We see that

$$A_1^{-1} = \begin{bmatrix} \frac{1}{9} & 0 & 0 \\ -\frac{1}{45} & \frac{1}{10} & 0 \\ -\frac{1}{45} & -\frac{2}{55} & \frac{1}{11} \end{bmatrix} \quad \text{so that} \quad A_1^{-1}A_2 = \begin{bmatrix} 0 & \frac{1}{9} & \frac{1}{9} \\ 0 & -\frac{1}{45} & -\frac{5}{18} \\ 0 & -\frac{1}{45} & -\frac{13}{99} \end{bmatrix}.$$

Thus, it follows that $\|A_1^{-1}A_2\|_\infty = 3/10$ and $\|A_1^{-1}A_2\|_1 = 103/198 > 3/10$. Thus, the error estimates for the above described iterative procedure for this example is valid if we take either $\|\cdot\|_\infty$ or $\|\cdot\|_1$ on $\mathbb{R}^3$. For instance, taking $\|\cdot\|_\infty$ on $\mathbb{R}^3$ and $\rho = 3/10$, we have

$$\|x - x^{(k)}\|_1 \le \frac{\rho^k}{1-\rho}\|x^{(1)} - x^{(0)}\| = \frac{10}{7}\Big(\frac{3}{10}\Big)^k \|x^{(1)} - x^{(0)}\|.$$

5.2 Newton's Method for Solving $f(x) = 0$

Suppose $f : [a, b] \to \mathbb{R}$ is a function having a zero $x^* \in [a, b]$, i.e., $fx^*) = 0$. In practice, the exact location of x^* may not be known, but an (initial) approximation of x^*, say x_0 may be known. The idea of Newton's method is to find better approximations for x^* in an iterative manner. For this first we assume that

- f is differentiable at every $x \in [a, b]$, and $f'(x) \neq 0$ for every $x \in [a, b]$.

The idea is to choose an initial point $x_0 \in [a, b]$, and find a point x_1 as the point of intersection of the tangent at x_0 with the x-axis. Thus x_1 has to satisfy

$$f'(x_0) = \frac{f(x_0) - 0}{x_0 - x_1} = \frac{f(x_0)}{x_0 - x_1},$$

i.e., x_1 is defined by

$$x_1 := x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Now, repeat the above procedure with x_1 in place of x_0 to get a new point

$$x_2 := x_1 - \frac{f(x_1)}{f'(x_1)}.$$

In general, we define

$$x_n := x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n = 1, 2, \ldots.$$

There arises some questions:

- Does each x_n belong to $[a, b]$?
- Does the sequence (x_n) converge to x^*?

In order to answer the above questions we define a new function $g : [a, b] \to \mathbb{R}$ by

$$g(x) := x - \frac{f(x)}{f'(x)}, \quad x \in [a, b].$$

Theorem 5.4 *Suppose that $f : [a, b] \to \mathbb{R}$ is twice continuously differentiable at every $x \in [a, b]$, and that there exists $x^* \in [a, b]$ such that $f(x^*) = 0$. Then there exists a closed interval $J_0 \subseteq [a, b]$ such that $g : J_0 \to J_0$ is a contraction.*

Proof. Note that, under the above assumption, the function g is continuously differentiable at every $x \in [a, b]$, and

$$g'(x) := \frac{f(x)f''(x)}{[f'(x)]^2}, \quad x \in [a, b].$$

Now, by mean value theorem, for every $x, y \in [a, b]$, there exists $\xi_{x,y}$ in the interval whose end points are x and y, such that

$$g(x) - g(y) = g'(\xi_{x,y})(x - y).$$

Hence, g is a contraction in an interval J_0 if there exists ρ such that $0 < \rho < 1$ and $|g'(\xi_{x,y})| \le \rho$ for all $x, y \in J_0$. Note that the function g' is continuous in $[a, b]$ and $g'(x^*) = 0$. Hence, for every ρ with $0 < \rho < 1$, there exists a closed interval $J_\rho \subseteq [a, b]$ such that $|g'(x)| \le \rho$ for all $x, y \in J_0$. ∎

Assume that $f : [a, b] \to \mathbb{R}$ is twice continuously differentiable at every $x \in [a, b]$, and let J_0 be as in the above theorem. Then, taking $x_0 \in J_0$, the sequence (x_n) defined earlier converges to x^*, and

$$|x^* - x_n| \le \rho|x^* - x_{n-1}| \le \frac{\rho^n}{1 - \rho}|x_1 - x_0| \quad \forall n \in \mathbb{N}.$$

5.2.1 Error Estimates

Suppose $f : [a, b] \to \mathbb{R}$ is a twice continuously differentiable function, and there exits $x^* \in [a, b]$ such that $f(x^*) = 0$. We have already seen that for any given $\rho \in (0, 1)$, there exists a closed interval J_0 centered at x^* such that $x_0 \in J_0$ implies $x_n \in J_0$ for all $n \in \mathbb{N}$, and

$$|x^* - x_n| \le \rho|x^* - x_{n-1}| \le \frac{\rho^n}{1 - \rho}|x_1 - x_0| \quad \forall n \in \mathbb{N}.$$

Now we see that a better convergence estimate is possible.

Assume that $f'(x) \ne 0$ for all $x \in [a, b]$. Let us assume for the time being that the sequence (x_n) given iteratively as follows is well-defined:

$$x_{n+1} := x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots.$$

Then, by mean value theorem, we have

$$0 = f(x^*) = f(x_n) + (x^* - x_n)f'(x_n) + (x^* - x_n)^2 \frac{f''(\xi_n)}{2}$$

so that

$$0 = \frac{f(x_n)}{f'(x_n)} + (x^* - x_n) + (x^* - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)}$$

Now, by the definition of x_{n+1},

$$0 = (x_n - x_{n+1} + (x^* - x_n) + (x^* - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)}$$

so that

$$x^* - x_{n+1} = (x^* - x_n)^2 \frac{f''(\xi_n)}{2f'(x_n)}.$$

From the above relation, it is clear that if J_0 is as in Theorem 5.4, and if we know that there exists a constant $\kappa > 0$ such that $|f''(x)/2f'(y)| \leq \kappa$ for all $x, y \in J_0$, then

$$|x^* - x_{n+1}| \leq \kappa |x^* - x_n|^2 \quad \forall n.$$

Another way of looking at the issue of error estimates is the following: Since $f'(x^*) \neq 0$, there exists $\delta > 0$ such that $J_1 := [x^* - \delta, x^* + \delta] \subseteq [a, b]$ and $|f'(x)| \geq |f'(x^*)|/2$ for all $x \in J_1$. Let $M > 0$ be such that $|f''(x)| \leq M$ for all $x \in J_1$. Hence,

$$|f''(x)/2f'(y)| \leq \kappa_0 := M/|f'(x^*)| \quad \forall x \in J_1.$$

Then we see that

$$\kappa_0 |x^* - x_{n+1}| \leq \Big(\kappa_0 |x^* - x_n|\Big)^2 \leq \Big(\kappa_0 |x^* - x_0|\Big)^{2^n} \quad \forall n.$$

Thus, it is seen that if $x_0 \in J_2 := J_1 \cap \{x : |x^* - x| < 1/\kappa_0\}$, then $x_n \in J_2$ for all $n \in \mathbb{N}$ (x_n) converges to x^* as $n \to \infty$. Moreover, we have the error estimate

$$|x^* - x_{n+1}| \leq \kappa_0 |x^* - x_n|^2 \leq \kappa_0^{2^n - 1} |x^* - x_0|^{2^n} \quad \forall n.$$

Exercise 5.1

1. Consider the equation $f(x) := x^6 - x - 1 = 0$. Apply Newton's method for this equation and find x_n, $f(x_n)$ and $x_n - x_{n-1}$ for $n = 1, 2, 3, 4$ with initial guesses (i) $x_0 = 1.0$, (ii) $x_0 = 1.5$, (iii) $x_0 = 2.0$. Compare the results.

2. Using Newton's iterations, find approximations for the roots of the following equations with an error tolerance, $|x_n - x_{n-1}| \leq 10^{-6}$:

   (i) $x^3 - x^2 - x - 1 = 0$, (ii) $x = 1 + 0.3\cos(x)$, (iii) $x = e^{-x}$.

3. Write Newton's iterations for the problem of finding $1/b$ for a number $b > 0$ with $x_0 > 0$.

4. Show that the Newton's iterations for finding approximations for $\sqrt{a}$ for $a > 0$ has the error formula:

   $$\sqrt{a} - x_{n+1} = -\tfrac{1}{2x_n}(\sqrt{a} - x_n)^2.$$

5. Using Newton's method find approximations for m-th root of 2 for six significant digits, for $m = 2, 3, 4, 5$.

6

Interpolation and Numerical Integration

6.1 Interpolation

The idea of interpolation is to find a function φ which takes certain prescribed values $\beta_1, \beta_2, \ldots, \beta_n$ at a given set of points $t_1, t_2, \ldots, t_n$. In application the values $\beta_1, \beta_2, \ldots, \beta_n$ may be values of certain unknown function f at $t_1, t_2, \ldots, t_n$ respectively. The function φ is to be of some simple form for computational purposes. Thus, the interpolation problem is to find a function φ such that $\varphi(t_i) = \beta_i$, $i = 1, \ldots, n$.

Usually, one looks for φ in the span of certain known functions $u_1, \ldots, u_n$. Thus, the interpolation problem is to find scalars $\alpha_1, \ldots, \alpha_n$ such that the function $\varphi := \sum_{j=1}^{n} \alpha_j u_j$ satisfies $\varphi(t_i) = \beta_i$ for $i = 1, \ldots, n$, i.e., to find $\alpha_1, \ldots, \alpha_n$ such that

$$\sum_{j=1}^{n} \alpha_j u_j(t_i) = \beta_i, \quad i = 1, \ldots, n.$$

Obviously, the above problem has a unique solution if and only if the matrix $[u_j(t_i)]$ is invertible. Thus we have the following theorem.

Theorem 6.1 *Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$. Then there exists a unique $\varphi \in$ span $\{u_1, \ldots, u_n\}$ satisfying $\varphi(t_i) = \beta_i$ for $i = 1, \ldots, n$ if and only if the matrix $[u_j(t_i)]$ is invertible.*

Exercise 6.1 Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$. Show that, if the matrix $[u_j(t_i)]$ is invertible, then $u_1, \ldots, u_n$ are linearly independent.

Hint: A square matrix is invertible if and only if its columns are linearly independent.

Exercise 6.2 Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$ such that the matrix $[u_j(t_i)]$ is invertible. If $v_1, \ldots, v_n$ are linearly independent functions in span $\{u_1, \ldots, u_n\}$, then show that the matrix $[v_j(t_i)]$ is also invertible.

Hint: Let $X_0 := \text{span}\,\{u_1, \ldots, u_n\}$. Then observe that, if the matrix $[u_j(t_i)]$ is invertible, then the function $J : X_0 \to \mathbb{R}^n$ defined by $J(x) = [x(t_1), \ldots, x(t_n]^T$ is bijective.

Exercise 6.3 Let $t_1, \ldots, t_n$ be distinct points in $\mathbb{R}$, and for each $j \in \{1, 2, \ldots, n\}$, let

$$\ell_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i}.$$

Then show that $\{\ell_1, \ldots, \ell_n\}$ is a basis of $\mathcal{P}_{n-1}$, and it satisfies $\ell_j(t_i) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Deduce from the previous exercise that the matrix $[t_j^{i-1}]$ is invertible.

In general, if $t_1, \ldots, t_n$ are distinct points in $[a, b]$, and if $u_1, \ldots, u_n$ are functions which satisfy $u_j(t_i) = \delta_{ij}$, then the function $\varphi(t) := \sum_{j=1}^n \beta_j u_j(t)$ satisfies $\varphi(t_i) = \beta_i$. Thus, if $t_1, \ldots, t_n$ be distinct points in $[a, b]$, and if $u_1, \ldots, u_n$ are functions which satisfy $u_j(t_i) = \delta_{ij}$, the interpolation function of $f : [a, b] \to \mathbb{R}$, associated with the *nodes* $t_1, \ldots, t_n$ and the basis functions u_j's is

$$\varphi(t) := \sum_{j=1}^{n} f(t_j) u_j(t), \quad a \leq t \leq b.$$

EXAMPLE 6.1 Let $t_1, \ldots, t_n$ be distinct points in $[a, b]$. Define $u_1, \ldots, u_n$ as follows:

$$u_1(t) = \begin{cases} 1 & \text{if } a \leq t \leq t_1, \\ \frac{t - t_2}{t_1 - t_2} & \text{if } t_1 \leq t \leq t_2, \\ 0, & \text{elsewhere}, \end{cases}$$

$$u_n(t) = \begin{cases} 1 & \text{if } t_n \leq t \leq b, \\ \frac{t - t_{n-1}}{t_n - t_{n-1}} & \text{if } t_{n-1} \leq t \leq t_n, \\ 0, & \text{elsewhere}, \end{cases}$$

and for $2 \leq j \leq n - 1$,

$$u_j(t) = \begin{cases} \frac{t - t_{j-1}}{t_j - t_{j-1}} & \text{if } t_{j-1} \leq t \leq t_j, \\ \frac{t - t_{j+1}}{t_j - t_{j+1}} & \text{if } t_j \leq t \leq t_{j+1}, \\ 0, & \text{elsewhere}, \end{cases}$$

Because of their shapes $u_1, \ldots, u_n$ are called *hat functions*. Note that $u_j(t_i) = \delta_{ij}$. In this case the interpolation function φ is the polygonal line passing through the points $(t_1, f(t_1)), \ldots, (t_n, f(t_n))$. In this case it is also true that $\sum_{j=1}^n u_j(t) = 1$ for all $j \in [a, b]$. Hence,

$$f(t) - \varphi(t) = \sum_{j=1}^{n} [f(t) - \varphi(t_j)] u_j(t).$$

Note that, for $t_{i-1} \leq t \leq t \leq t_i$,

$$f(t) - \varphi(t) = \sum_{j=i-1}^{i} [f(t) - \varphi(t_j)] u_j(t).$$

6.1.1 Lagrange Interpolation

By Exercise 6.3, $\{\ell_1, \ldots, \ell_n\}$ is a basis of $\mathcal{P}_{n-1}$, and it satisfies $\ell_j(t_i) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Hence, by Theorem 6.1, it is clear that, given distinct points $t_1, \ldots, t_n$ in $[a, b]$, and numbers $\beta_1, \ldots, \beta_n$, there exists a unique polynomial $L_n(t) \in \mathcal{P}_{n-1}$ such that $L_n(t_i) = \beta_i$ for $i = 1, \ldots, n$, and it is given by

$$L_n(t) := \sum_{j=1}^{n} \beta_j \ell_j(t), \quad a \leq t \leq b.$$

The above polynomial is called the *Lagrange interpolating polynomial*, and the functions $\ell_1, \ldots, \ell_n$ are called *Lagrange basis polynomials*.

It can be seen that the Lagrange basis polynomials $\ell_i(t)$ also satisfies $\sum_{j=1}^n \ell_j(t) = 1$. Hence, if φ is the Lagrange interpolating polynomial of a function f associated with nodes $t_1, \ldots, t_n$, i.e., $L_n(t) := \sum_{j=1}^n f(t_j)\ell_j(t)$, then

$$f(t) - L_n(t) = \sum_{j=1}^{n} [f(t) - \varphi(t_j)] \ell_j(t).$$

The following theorem can be seen in any standard text book on Numerical Analysis.

Theorem 6.2 *If f is continuously differentiable $n+1$ times on the interval $[a, b]$, and if φ is the Lagrange interpolating polynomial of a* *function f associated with nodes $t_1, \ldots, t_n$, then there exists $\xi \in (a, b)$ such that*

$$f(t) - L_n(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - t_1)(t - t_2) \ldots (t - t_n).$$

By the above theorem,

$$|f(t) - L_n(t)| \leq \frac{|f^{(n+1)}(\xi)|}{n+1} \frac{(b-a)^n}{(n)!}.$$

Note that $\frac{(b-a)^n}{(n)!} \to 0$ as $n \to \infty$.Thus, if f is sufficiently smooth, then we can expect that L_n is close to φ for sufficiently large n.

Although we have a nice result in the above theorem, it is not at all clear whether L_n is close to f whenever the maximum width of the subintervals is close to zero. In fact it is known that

• if, for each $n \in \mathbb{N}$, $t_1^{(n)}, \ldots, t_1^{(n)}$ are points in $[a, b]$ then there exists $f \in C[a, b]$ such that $\|f - L_n\|_\infty \not\to 0$ as $n \to \infty$, where $L_n(t) := \sum_{j=1}^{n} f(t_j)\ell_j^{(n)}(t)$.

6.1.2 Piecewise Lagrange Interpolation

The disadvantage of Lagrange interpolation polynomial is that for large n, more computations are involved in obtaining the coefficients of the polynomial, and even for large n, it is not guaranteed that $L_n(t)$ is close to f for less smooth functions $f(t)$. One way to surmount this problem is to divide the interval $[a, b]$ into n equal parts, say by a partition $a = a_0 < a_1 < a_2 < \ldots < a_n = b$. In each subinterval $I_i = [a_{i-1}, a_i]$ we consider the Lagrange interpolation of the function. For this we consider points $\tau_{i1}, \ldots, \tau_{ik}$ in I_i, and for $t \in I_i$, $i = 1, \ldots, n$, define

$$p_{i,n}(t) := \sum_{j=1}^{k} f(\tau_{ij})\ell_{ij}^{(n)}(t), \quad \ell_{ij}(t) = \prod_{m \neq j} \frac{t - \tau_{im}}{\tau_{ij} - \tau_{im}},$$

and $p_n(t) = p_{i,n}(t)$ whenever $t \in I_i$. Note that p_n is a function on $[a, b]$ such that for each $i \in \{1, \ldots, n\}$, $p_n|_{I_i}$ is a polynomial in $\mathcal{P}_{k-1}$. Such a function p_n is called a **spline**. If $\tau_{i1} = a_{i-1}$ and $\tau_{ik} = a_i$, then we see that p_n is a continuous function on $[a, b]$.

Instead of taking arbitrary points $\tau_{i1}, \ldots, \tau_{ik}$ in I_i one may choose them by mapping a fixed number of points $\tau_1, \ldots, \tau_k$ in $[-1, 1]$ to each subinterval I_i by using functions $g_i : [-1, 1] \to I_i$ so as to obtain $\tau_{im} = g_i(\tau_m)$ for $m = 1, \ldots, k$. Points $\tau_1, \ldots, \tau_k$ in $[-1, 1]$ chosen as zeros of certain orthogonal polynomial of degree k has some advantages over other type of points.

If f is k times continuously differentiable, then by Theorem 6.2, for each $i \in \{1, \ldots, n\}$, there exists $\xi_i \in I_i$ such that for $t \in I_i$,

$$f(t) - p_n(t) = \frac{f^{(k+1)}(\xi_i)}{(k+1)!} \prod_{m=1}^{k} (t - \tau_{im}).$$

Hence, for every $t \in [a, b]$,

$$|f(t) - p_n(t)| \le \frac{\|f^{(k+1)}\|_\infty}{(k+1)!} \sup_{a_{i-1} \le t \le a_i} \prod_{m=1}^{k} (t - \tau_{im}).$$

In particular, if $a_i - a_{i-1} = h_n := (b-a)/n$ for all $i \in \{1, \ldots, n\}$, then

$$\|f - p_n\|_\infty \le \frac{\|f^{(k+1)}\|_\infty}{(k+1)!} h_n^k.$$

Note that $h_n \to 0$ as $n \to \infty$. Thus, for large enough n, p_n is an approximation of f.

6.2 Numerical Integration

The idea involved in numerical integration of a (Riemann integrable) function $f : [a, b] \to \mathbb{R}$ is to replace the integral $\int_a^b f(t)dt$ by another integral $\int_a^b \varphi(t)dt$, where φ is an interpolation of f based on certain points in $[a, b]$. Thus, numerical integration formulas are of the form

$$\sum_{j=1}^{n} f(t_j) w_j,$$

where $t_1, \ldots, t_n$ are called the *nodes* and $w_1, \ldots, w_n$ are called the *weights* of the formula. Numerical integration formulas are also called *quadrature rules.*

Suppose $u_1, \ldots, u_n$ are functions such that $u_j(t_i) = \delta_{ij}$ for $i, j = 1, \ldots, n$, and let φ be the interpolation of f based on $t_1, \ldots, t_n$ and $u_1, \ldots, u_n$,i.e., $\varphi(t) = \sum_{j=1}^{n} f(t_j) u_j(t)$. Then

$$\int_a^b \varphi(t)dt = \sum_{j=1}^{n} f(t_j) w_j \quad \text{with} \quad w_j = \int_a^b u_j(t)dt.$$

The above quadrature rule is called an **interpolatory quadrature rule**. Here are some special cases of interpolatory quadrature rules.

6.2.1 Trapezoidal rule

Suppose we approximate f by the interpolation polynomial φ of degree atmost 1 based on the points a and b, i.e., we approximate the graph of f by the straight line joining $(a, f(a))$ and $(b, f(b))$. Then we see that

$$\int_a^b \varphi(t)dt = \frac{b-a}{2}[f(a) + f(b)].$$

This quadrature rule is called the *trapezoidal rule.*

EXAMPLE 6.2 Consider $f(t) = 1/(1+t)$ for $0 \le t \le 1$. Then

$$\int_0^1 \varphi(t)dt = \frac{1}{2}[f(0) + f(1)] = \frac{1}{2}[1 + \frac{1}{2}] = \frac{3}{4} = 0.75.$$

We know that $\int_0^1 \frac{dt}{1+t} = \ln(2) \simeq 0.693147$.

$$\text{Error} = \log(2) - \int_0^1 \varphi(t)dt \simeq -0.056852819$$

6.2.2 Composite Trapezoidal rule

Let $a = a_0 < a_1 < \ldots < a_n = b$ be a partition of $[a, b]$. Suppose we approximate f by the piecewise interpolation polynomial φ which is of degree atmost 1 in each subinterval $[a_{i-1}, a_i]$ for $i = 1, \ldots, n$, based on the points a_{i-1} and a_i, i.e., we approximate the graph of f by a polygonal line line passing through the poits $(a_i, f(a_i))$ for $i = 0, 1, 2, \ldots, n$. Then we see that

$$\int_a^b \varphi(t)dt = \sum_{i=1}^{n} \int_{a_{i-1}}^{a_i} \varphi(t)dt = \sum_{i=1}^{n} \frac{a_i - a_{i-1}}{2}[f(a_{i-1}) + f(a_i)].$$

This quadrature rule is called the *composite trapezoidal rule.* In particular, if $h_n := a_i - a_{i-1} = (b-a)/n$, then

$$\int_a^b \varphi(t)dt = h_n\Big[\frac{f(a_0)}{2} + f(a_1) + \ldots + f(a_{n-1}) + \frac{f(a_n)}{2}\Big].$$

EXAMPLE 6.3 Consider $f(t) = 1/(1+t)$ for $0 \le t \le 1$. Taking $n = 2$, $h_n = 1/2$ and

$$\begin{aligned}\int_0^1 \varphi(t)dt &= \frac{1}{2}\Big[\frac{f(0)}{2} + f(1/2) + \frac{f(1)}{2}\Big] \\ &= \frac{1}{2}\Big[\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\Big] = \frac{17}{24} = 0.70833.\end{aligned}$$

$$\text{Error} = \ln(2) - \int_0^1 \varphi(t)dt \simeq -0.01518$$

6.2.3 Simpson's rule

Suppose we approximate f by the interpolation polynomial φ of degree atmost 2 based on the points a, $c := (a+b)/2$ and b, i.e., we approximate the graph of f by a quadratic polynomial φ. We know that

$$\varphi(t) = \frac{(t-c)(t-b)}{(a-c)(a-b)}f(a) + \frac{(t-a)(t-b)}{(c-a)(c-b)}f(c) + \frac{(t-a)(t-c)}{(b-a)(b-c)}f(b).$$

If we take $h = (b-a)/2$, then we see that

$$\int_a^b \frac{(t-c)(t-b)}{(a-c)(a-b)}dt = \frac{h}{3},$$

$$\int_a^b \frac{(t-a)(t-b)}{(c-a)(c-b)}dt = \frac{4h}{3},$$

$$\int_a^b \frac{(t-a)(t-c)}{(b-a)(b-c)}dt = \frac{h}{3}.$$

Hence,

$$\int_a^b \varphi(t)dt = \frac{h}{3}\Big[f(a) + 4f(c) + f(b)\Big].$$

This quadrature rule is called the *Simpson's rule*.

EXAMPLE 6.4 Consider $f(t) = 1/(1+t)$ for $0 \le t \le 1$. Taking $n = 2$, $h_n = 1/2$ and

$$\begin{aligned}\int_0^1 \varphi(t)dt &= \frac{h}{3}\Big[f(a) + 4f(c) + f(b)\Big]\\ &= \frac{1}{6}\Big[f(0) + 4f(1/2) + f(1)\Big] = \frac{1}{6}\Big[\frac{1}{2} + 4\Big(\frac{2}{3}\Big) + \frac{1}{2}\Big]\\ &= \frac{25}{36} \simeq 0.69444.\end{aligned}$$

$$\text{Error} = \ln(2) - \int_0^1 \varphi(t)dt \simeq -0.001293$$

6.2.4 Composite Simpson's rule

Consider the partition $a = a_0 < a_1 < \ldots < a_n = b$ of of $[a, b]$ n is an even number, and $h_n := a_i - a_{i-1} = (b-a)/n$ for every $i = 1, 2, \ldots, n$. Suppose we approximate f by the piecewise interpolation polynomial φ which is of degree atmost 2 in each subinterval $[a_{2i}, a_{2i+2}]$ based on the points a_{2i}, a_{2i+1} and a_{2i+2} for $i = 0, 1, 2, \ldots, k$ where $2k = n$. Then we see that

$$\int_a^b \varphi(t)dt = \sum_{i=0}^{k-1} \int_{a_{2i}}^{a_{2i+2}} \varphi(t)dt = \sum_{i=0}^{k-1} \frac{h_n}{3}\Big[f(a_{2i}) + 4f(a_{2i+1}) + f(a_{2i+2})\Big]$$

Thus,

$$\begin{aligned}
\int_a^b \varphi(t)dt &= \frac{h_n}{3}\Big[f(a_0) + 4f(a_1) + 2f(a_2) + 4f(a_3) + 2f(a_4) + \\
&\quad \ldots + 2f(a_{n-2}) + 4f(a_{n-1}) + f(a_n)\Big] \\
&= \frac{h_n}{3}\Big[f(a_0) + \sum_{i=1}^{k} 4f(a_{2i-1}) + \sum_{i=1}^{k-1} 2f(a_{2i}) + f(a_n)\Big]
\end{aligned}$$

This quadrature rule is called the *composite Simpson's rule.*

EXAMPLE 6.5 Consider $f(t) = 1/(1+t)$ for $0 \le t \le 1$. Taking $n = 2k = 4$, $h_n = 1/4$ and

$$\begin{aligned}
\int_0^1 \varphi(t)dt &= \frac{h_n}{3}\Big[f(a_0) + 4f(a_1) + 2f(a_2) + 4f(a_3) + f(a_4)\Big] \\
&= \frac{(1/4)}{3}\Big[f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1)\Big] \\
&= \frac{1}{12}\Big[1 + \frac{16}{5} + \frac{4}{3} + \frac{16}{7} + \frac{1}{2}\Big] \\
&= \frac{1}{12} \times \frac{1747}{210} \simeq 0.693253968
\end{aligned}$$

$$\text{Error} = \ln(2) - \int_0^1 \varphi(t)dt \simeq -0.000106787$$

Exercise 6.4 Apply trapezoidal rule, composite trapezoidal rule, Simpson's rule, and composite Simpson's rule for approximating the the following integrals:

$$\int_0^1 e^{-t^2} dt, \qquad \int_0^4 \frac{dt}{1+t^2} dt, \qquad \int_0^{2\pi} \frac{dt}{2+\cos(t)} dt.$$

7

Additional Exercises

In the following V denotes a vector space over $\mathbb{F}$ which is $\mathbb{R}$ or $\mathbb{C}$.

1. Let V be a vector space. For $x, y \in V$, show that $x + y = x$ implies $y = \theta$.

2. Suppose that $x \in V$ is a nonzero vector. Then show that $\alpha x \neq \beta x$ for every $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$.

3. Let $\mathcal{R}[a, b]$ be the set of all real valued Riemann integrable functions on $[a, b]$. Show that $\mathcal{R}[a, b]$ is a vector space over $\mathbb{R}$.

4. Let V be the set of all polynomials of degree 3. Is it a vector space with respect to the usual addition and scalar multiplication?

5. Let S be a nonempty set, $s_0 \in S$. Show that the set V of all functions $f : S \to \mathbb{R}$ such that $f(s_0) = 0$ is a vector space with respect to the usual addition and scalar multiplication of functions.

6. Find a bijective linear transformation between $\mathbb{F}^n$ and $\mathcal{P}_{n-1}$.

7. Let V be the set of real sequences with only a finite number of nonzero entries. Show that V is a vector space over $\mathbb{R}$ and find a bijective map $T : V \to \mathcal{P}$ which also satisfies $T(x + \alpha y) = T(x) + \alpha T(y)$ for all $x, y \in V$ and $\alpha \in \mathbb{R}$.

8. In each of the following, a set V is given and some operations are defined. Check whether V is a vector space with these operations:

   (a) Let $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0\}$ with addition and scalar multiplication as in $\mathbb{R}^2$.

(b) Let $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : 2x_1 + 3x_2 = 0\}$ with addition and scalar multiplication as in $\mathbb{R}^2$.

(c) Let $V = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ with addition and scalar multiplication as for $\mathbb{R}^2$.

(d) Let $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$, let $x + y := (x_1 + y_1, x_2 + y_2)$ and for all $\alpha \in \mathbb{R}$,

$$\alpha x := \begin{cases} (0, 0) & \alpha = 0, \\ (\alpha x_1, x_2/\alpha), & \alpha \neq 0. \end{cases}$$

(e) Let $V = \mathbb{C}^2$, $\mathbb{F} = \mathbb{C}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$, let

$$x + y := (x_1 + 2y_1, x_2 + 3y_2) \quad \text{and} \quad \alpha x := (\alpha x_1, \alpha x_2) \quad \forall \alpha \in \mathbb{C}.$$

(f) Let $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$, let

$$x + y := (x_1 + y_1, x_2 + y_2) \quad \text{and} \quad \alpha x := (x_1, 0) \quad \forall \alpha \in \mathbb{R}.$$

9. Let $A \in \mathbb{R}^{n \times n}$, O is the zero in $\mathbb{R}^{n \times 1}$. Show that the set V_0 of of all $n \times 1$ matrices X such that $AX = O$, is a subspace of $\mathbb{R}^{n \times 1}$.

10. Let V be the space of all sequences of real numbers, and let $\ell^1(\mathbb{N})$ be the set of all absolutely convergent real sequences. Show that $\ell^1(\mathbb{N})$ is a subspace of V.

11. Let V be the space of all sequences of real numbers, and let $\ell^\infty(\mathbb{N})$ be the set of all bounded sequences of real numbers. Show that $\ell^\infty(\mathbb{N})$ is a subspace of the space of V.

12. For a nonempty set S, V be the set of all functions from S to $\mathbb{R}$, and let let $B(S)$ be the set of all bounded functions on S. Show that $B(S)$ is a subspace of V

13. Suppose V_0 is a subspace of a vector space V, and V_1 is a subspace of V_0. Then show that V_1 is a subspace of V.

14. Give an example to show that union of two subspaces need not be a subspace.

15. Let S be a subset of a vector space V. Show that S is a subspace if and only if $S = \operatorname{span} S$.

16. Let V be a vector space. Show that the the following hold.

    (i) Let S be a subset of V. Then $\operatorname{span} S$ is the intersection of all subspaces of V containing S.

    (ii) Suppose V_0 is a subspace of V and $x_0 \in V \setminus V_0$. Then for every $x \in \operatorname{span}\{x_0; X_0\}$, there exist a unique $\alpha \in \mathbb{F}$, $y \in V_0$ such that $x = \alpha x_0 + y$.

17. Show that

    (a) $\mathcal{P}_n$ is a subspace of $\mathcal{P}_m$ for $n \leq m$,

    (b) $C[a,b]$ is a subspace of $\mathcal{R}[a,b]$,

    (c) $C^k[a,b]$ is a subspace of $C[a,b]$.

18. For each λ in the open interval $(0,1)$, let $u_\lambda = (1, \lambda, \lambda^2, \ldots)$. Show that $u_\lambda \in \ell^1$ for each $\in (0,1)$, and $\{u_\lambda : 0 < \lambda < 1\}$ is a linearly independent subset of ℓ^1.

19. Let A be an $m \times n$ matrix, and $\mathbf{b}$ be a column m-vector. Show that the system $A\mathbf{x} = \mathbf{b}$ has a solution n-vector if and only if $\mathbf{b}$ is in the span of columns of A.

20. Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$. What is the span of $\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$?

21. What is the span of $S = \{t^n : n = 0, 2, 4, \ldots\}$ in $\mathcal{P}$?

22. Let S be a subset of a vector space V. Show that S is a subspace if and only if $S = \operatorname{span} S$.

23. Let V be a vector space. Show that the the following hold.

    (a) Let S be a subset of V. Then

    $$\operatorname{span} S = \bigcap \{Y : Y \text{ is a subspace of } V \text{ containing } S\}.$$

    (b) Suppose V_0 is a subspace of V and $x_0 \in V \setminus V_0$. Then for every $x \in \operatorname{span}\{x_0; X_0\}$, there exist a unique $\alpha \in \mathbb{F}$, $y \in V_0$ such that $x = \alpha x_0 + y$.

24. Consider the system of equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \dots & + & \dots & + & \dots & + & \dots & = & \dots \\ a_{m1}x_1 & + & a_{m1}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Let

$$u_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix}, u_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix}, \dots, u_n := \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}.$$

(a) Show that the above system has a solution vector $x = [x_1, \dots, x_n]^T$ if and only if $b = [b_1, \dots, b_n]^T \in span(\{u_1, \dots, u_n\}$.

(b) Show that the above system has atmost one solution vector $x = [x_1, \dots, x_n]^T$ if and only if $\{u_1, \dots, u_n\}$ is linearly independent.

25. Show that every superset of a linearly dependent set is linearly dependent, and every subset of a linearly independent set is linearly independent.

26. Give an example to justify the following: E is a subset of vector space such that there exists an vector $u \in E$ which is not a linear combination of other members of E, but E is linearly dependent.

27. Is union (resp., intersection) of two linearly independent sets a linearly independent? Why?

28. Is union (resp., intersection) of two linearly dependent sets a linearly dependent? Why?

29. Show that vectors $u = (a, c)$, $v = (b, d)$ are linearly independent in $\mathbb{R}^2$ iff $ad - bc \neq 0$. Can you think of a generalization to n vectors in $\mathbb{R}^n$.

30. Show that $V_0 := \{x = (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$. Find a basis for V_0.

31. Show that $E := \{1 + t^n, t + t^n, t^2 + t^n, \dots, t^{n-1} + t^n, t^n\}$ is a basis of $\mathcal{P}_n$.

32. Let $u_1, \ldots, u_n$ are linearly independent vectors in a vector space V. Let $[a_{ij}]$ be an $m \times n$ matrix of scalar, and let

$$\begin{array}{lllllllll} v_1 & := & a_{11}u_1 & + & a_{21}u_2 & + & \ldots & + & a_{m1}u_n \\ v_2 & := & a_{12}u_1 & + & a_{22}u_2 & + & \ldots & + & a_{m2}u_n \\ \ldots & & \ldots & + & \ldots & + & \ldots & + & \ldots \\ v_n & := & a_{1n}u_1 & + & a_{2n}u_2 & + & \ldots & + & a_{mn}u_n. \end{array}$$

Show that the $v_1, \ldots, v_m$ are linearly independent if and only if the vectors

$$w_1 := \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix}, \quad w_2 := \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix}, \ldots, \quad w_n := \begin{bmatrix} a_{1n} \\ a_{m2} \\ \ldots \\ a_{mn} \end{bmatrix}$$

are linearly independent.

33. Let $u_1(t) = 1$, and for $j = 2, 3, \ldots$, let $u_j(t) = 1 + t + \ldots + t^j$. Show that span of $\{u_1, \ldots, u_n\}$ is $\mathcal{P}_n$, and span of $\{u_1, u_2, \ldots\}$ is $\mathcal{P}$.

34. Let $p_1(t) = 1 + t + 3t^2$, $p_2(t) = 2 + 4t + t^2$, $p_3(t) = 2t + 5t^2$. Are the polynomials p_1, p_2, p_3 linearly independent?

35. Show that a basis of a vector space is a minimal spanning set, and maximal linearly independent set.

36. Suppose V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$. Show that every $x \in V_1 + V_2$ can be written *uniquely* as $x = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$.

37. Suppose V_1 and V_2 are subspaces of a vector space V. Show that $V_1 + V_2 = V_1$ if and only if $V_2 \subseteq V_1$.

38. Let V be a vector space.

    (a) Show that a subset $\{u_1, \ldots, u_n\}$ of V is linearly independent if and only if the function $(\alpha_1, \ldots, \alpha_n) \mapsto \alpha_1 u_1 + \cdots + \alpha_n u_n$ from $\mathbb{F}^n$ into V is injective.

    (b) Show that if $E \subseteq V$ is linearly dependent in V, then every superset of E is also linearly dependent.

(c) Show that if $E \subseteq V$ is linearly independent in V, then every subset of E is also linearly independent.

(d) Show that if $\{u_1, \ldots, u_n\}$ is a linearly independent subset of V, and if Y is a subspace of V such that $(\text{span}\,\{u_1, \ldots, u_n\}) \cap Y = \{0\}$, then every V in the span of $\{u_1, \ldots, u_n, Y\}$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$, $y \in Y$.

(e) Show that if E_1 and E_2 are linearly independent subsets of V such that $(\text{span}\, E_1) \cap (\text{span}\, E_2) = \{0\}$, then $E_1 \cup E_2$ is linearly independent.

39. For each $k \in \mathbb{N}$, let $\underline{\mathbb{F}}^k$ denotes the set of all column k-vectors, i.e., the set of all $k \times 1$ matrices. Let A be an $m \times n$ matrix of scalars with columns $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$. Show the following:

    (a) The equation $A\underline{x} = \underline{0}$ has a non-zero solution if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n$ are linearly dependent.

    (b) For $\underline{y} \in \underline{\mathbb{F}}^m$, the equation $A\underline{x} = \underline{y}$ has a solution if and only if $\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n, \underline{y}$ are linearly dependent, i.e., if and only if $\underline{y}$ is in the span of columns of A.

40. For $i = 1, \ldots, m$; $j = 1, \ldots, n$, let E_{ij} be the $m \times n$ matrix with its (i, j)-th entry as 1 and all other entries 0. Show that

$$\{E_{ij} : i = 1 \ldots, m;\ j = 1, \ldots, n\}$$

    is a basis of $\mathbb{F}^{m \times n}$.

41. If $\{u_1, \ldots, u_n\}$ is a basis of a vector space V, then show that every $x \in V$, can be expressed *uniquely* as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$; i.e., for every $x \in V$, there exists a unique n-tuple $(\alpha_1, \ldots, \alpha_n)$ of scalars such that $x = \alpha_1 u_1 + \cdots + \alpha_n u_n$.

42. Suppose S is a set consisting of n elements and V is the set of all real valued functions defined on S. Show that V is a vector space of dimension n.

43. Given real numbers $a_0, a_1, \ldots, a_k$, let X be the set of all solutions $x \in C^k[a, b]$ of the differential equation

$$a_0 \frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + \cdots + a_k x = 0.$$

Show that X is a linear space over $\mathbb{R}$. What is the dimension of X?

44. Let $t_0, t_1, \ldots, t_n$ be in $[a, b]$ such that $a = t_0 < t_1 < \ldots < t_n = b$. For each $j \in \{1, \ldots, n\}$, let u_j be in $C([a, b], \mathbb{R})$ such that

$$u_j(t_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and the restriction of u_j to each interval $[t_{j-1}, t_j]$ is a polynomial of degree atmost 1. Show that the span of $\{u_1, \ldots, u_n\}$ is the space of all continuous functions whose restrictions to each subinterval $[t_{i-1}, t_i]$ is a polynomial of degree atmost 1.

45. State with reason whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ in each of the following is a linear transformation:

(a) $T(x_1, x_2) = (1, x_2)$, (b) $T(x_1, x_2) = (x_1, x_2^2)$
(c) $T(x_1, x_2) = (\sin(x_1), x_2)$ (d) $T(x_1, x_2) = (x_1, 2 + x_2)$

46. Check whether the functions T in the following are linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + y, x + y^2)$.
(ii) $T : C^1[0, 1] \to \mathbb{R}$ defined by $T(u) = \int_0^1 [u(t)]^2 dt$.

(b) $T : C^1[-1, 1] \to \mathbb{R}^2$ defined by $T(u) = \left(\int_{-1}^1 u(t)dt, u'(0) \right)$.

(c) $T : C^1[0, 1] \to \mathbb{R}$ defined by $T(u) = \int_0^1 u'(t)dt$.

47. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be linear transformations. Show that the function $T : V_1 \to V_3$ defined by $Tx = T_2(T_1 x)$, $x \in V_1$, is a linear transformation.

[The above transformation T is called the *composition* of T_2 and T_1, and is usually denoted by $T_2 T_1$.]

48. If $T_1 : C^1[0, 1] \to C[0, 1]$ is defined by $T_1(u) = u'$, and $T_2 : C[0, 1] \to \mathbb{R}$ is defined by $T_2(v) = \int_0^1 v(t)dt$, then find $T_2 T_1$.

49. Let V_1, V_2, V_3 be finite dimensional vector spaces, and let E_1, E_2, E_3 be bases of V_1, V_2, V_3 respectively. If $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ are linear transformations. Show that $[T_2 T_1]_{E_1, E_3} = [T_2]_{E_2, E_3} [T_1]_{E_1, E_2}$.

50. If $T_1 : \mathcal{P}_n[0,1] \to \mathcal{P}_n[0,1]$ is defined by $T_1(u) = u'$, and $T_2 : \mathcal{P}_n[0,1] \to \mathbb{R}$ is defined by $T_2(v) = \int_0^1 v(t)dt$, then find $[T_1]_{E_1,E_2}$, $[T_2]_{E_2,E_3}$, and $[T_2T_1]_{E_1,E_3}$, where $E_1 = E_2 = \{1, t, t^2, \ldots, t^n\}$ and $E_3 = \{1\}$.

51. Justify the statement: Let $T_1 : V_1 \to V_2$ be a linear transformation. Then T is bijective iff there exists a linear transformation $T_2 : V_2 \to V_1$ such that $T_1T_2 : V_2 \to V_2$ is the identity transformation on V_2 and $T_2T_1 : V_1 \to V_1$ is the identity transformation on V_1.

52. Let V_1 and V_2 be vector spaces with $dim V_1 = n < \infty$. Let $\{u_1, \ldots, u_n\}$ be a basis of V_1 and $\{v_1, \ldots, v_n\} \subset V_2$. Find a linear transformation $T : V_1 \to V_2$ such that $T(u_j) = v_j$ for $j = 1, \ldots, n$. Show that there is only one such linear transformation.

53. Let T be the linear transformation obtained as in the above problem. Show that

    (a) T is one-one if and only if $\{v_1, \ldots, v_n\}$ is linearly independent, and

    (b) T is onto if and only if span $(\{v_1, \ldots, v_n\}) = V_2$.

54. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which satisfies $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. Find the $T(2,3)$.

55. Does there exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,2) = ((1,1)$ and $T(1/2,0,1) = ((0,1)$?

56. Show that if V_1 and V_2 are finite dimensional vector spaces of the same dimension, then the there exists a bijective linear transformation from V_1 to V_2.

57. Find bases for $N(T)$ and $R(T)$ for the linear transformation T in each the following:

    (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1 - x_2, 2x_2)$,

    (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_1 + x_2, 0, 2x_3 - x_2)$,

    (c) $T : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $T(A) = trace(A)$. (Recall that trace of a square matrix is the sum of its diagonal elements.)

58. Let $T : V_1 \to V_2$ is a linear transformation. Given reasons for the following:

    (a) $\text{rank}(T) \leq \text{dim}V_1$.

    (b) T onto implies $\text{dim}V_2 \leq \text{dim}V_1$,

    (c) T one-one implies $\text{dim}V_1 \leq \text{dim}V_2$

    (d) Suppose $\text{dim}V_1 = \text{dim}V_2 < \infty$. Then T is one-one if and only T is onto.

59. Let V_1 and V_2 be finite dimensional vector spaces, and $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively. Show the following:

    (a) If $\{g_1, \ldots, g_m\}$ is the ordered dual basis of $\mathcal{L}(V_1, \mathbb{F})$ with respect to the basis E_2 of V_2, then $[T]_{E_1,E_2} = (g_i(Tu_j))$. for every $T \in \mathcal{L}(V_1, V_2)$.

    (b) If $A, B \in \mathcal{L}(V_1, V_2)$ and $\alpha \in \mathbb{F}$, then

    $$[A+B]_{E_1,E_2} = [A]_{E_1,E_2} + [B]_{E_1,E_2}, \qquad [\alpha A]_{E_1,E_2} = \alpha[A]_{E_1,E_2}.$$

    (c) Suppose $\{M_{ij} : i = 1 \ldots, m; j = 1, \ldots, n\}$ is a basis of $\mathbb{F}^{m \times n}$. If $T_{ij} \in \mathcal{L}(V_1, V_2)$ is the linear transformation such that $[T_{ij}]_{E_1,E_2} = M_{ij}$, then $\{T_{ij} : i = 1 \ldots, m; j = 1, \ldots, n\}$ is a basis of $\mathcal{L}(V_1, V_2)$.

60. Let V_1 and V_2 be finite dimensional vector spaces, and $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively. Let $F_1 = \{f_1, \ldots, f_n\}$ be the dual basis of $\mathcal{L}(V_1, \mathbb{F})$ with respect to E_1 and $F_2 = \{g_1, \ldots, g_n\}$ be the dual basis of $\mathcal{L}(V_2, \mathbb{F})$ with respect to E_2. For $i = 1, \ldots, n$; $j = 1, \ldots, m$, let $T_{ij} : V \to W$ defined by

    $$T_{ij}(x) = f_j(x)v_i, \qquad x \in V_1.$$

    Show that $\{T_{ij} : i = 1, \ldots, n; j = 1, \ldots, m\}$ is a basis of $\mathcal{L}(V_1, V_2)$.

61. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x_1, x_2, x_3) = (x_2 + x_3, x_3 + x_1, x_1 + x_2), \quad (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Find the matrix representation of T with respect to the basis given in each of the following.

(a) $E_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $E_2 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$

(b) $E_1 = \{(1, 0, 0), (1, 1, 0), (1, 1, 1)\}$, $E_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(c) $E_1 = \{(1, 1, -1), (-1, 1, 1), (1, -1, 1)\}$,
$E_2 = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)$

62. Let $T : \mathcal{P}^3 \to \mathcal{P}^2$ be defined by

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2.$$

Find the matrix representation of T with respect to the basis given in each of the following.

(a) $E_1 = \{1, t, t^2, t^3\}$, $E_2 = \{1 + t, 1 - t, t^2\}$

(b) $E_1 = \{1, 1 + t, 1 + t + t^2, t^3\}$, $E_2 = \{1, 1 + t, 1 + t + t^2\}$

(c) $E_1 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$, $\quad E_2 = \{t^2, t, 1\}$

63. Let $T : \mathcal{P}^2 \to \mathcal{P}^3$ be defined by

$$T(a_0 + a_1 t + a_2 t^2) = (a_0 t + \frac{a_1}{2} t^2 + \frac{a_2}{3} t^3).$$

Find the matrix representation of T with respect to the basis given in each of the following.

(a) $E_1 = \{1 + t, 1 - t, t^2\}$, $E_2 = \{1, t, t^2, t^3\}$,

(b) $E_1 = \{1, 1 + t, 1 + t + t^2\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, t^3\}$,

(c) $E_1 = \{t^2, t, 1\}$, $E_2 = \{1, 1 + t, 1 + t + t^2, 1 + t + t^2 + t^3\}$,

64. A linear transformation $T : V \to W$ is said to be of *finite rank* if $\operatorname{rank} T < \infty$.

Let $T : V_1 \to V_2$ be a linear transformation between vector spaces V_1 and V_2. Show that T is of finite rank if and only if there exists $\{v_1, \ldots, v_n\} \subset V_2$ and $\{f_1, \ldots, f_n\} \subset \mathcal{L}(V_1, \mathbb{F})$ such that $Ax = \sum_{j=1}^{n} f_j(x) v_j$ for all $x \in V_1$.

65. Check whether the following are inner product on the given vector spaces:

    (a) $\langle A, B\rangle := trace(A+B)$ on $\mathbb{R}^{2\times 2}$

    (b) $\langle A, B\rangle := trace(A^TB)$ on $\mathbb{R}^{3\times 3}$

    (c) $\langle x, y\rangle := \int_0^1 x'(t)y'(t)\,dt$ on $\mathcal{P}_n$ or on $C^1[0,1]$

    (d) $\langle x, y\rangle := \int_0^1 x(t)y(t)\,dt + \int_0^1 x'(t)y'(t)\,dt$ on $C^1[0,1]$

66. If $\{u_1, \ldots, u_n\}$ is an orthonormal basis of an inner product space V, then show that, for every $x, y \in V$,

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, u_i\rangle\langle u_i, y\rangle.$$

    Let $\mathbb{F}^n$ be endowed with the usual inner product. Then, deduce that there is a linear isometry from V onto $\mathbb{F}^n$, i.e., a linear operator $T : V \to \mathbb{F}^n$ such that $\|T(x)\| = \|x\|$ for all $x \in V$.

67. Let V_1 and V_2 be inner product spaces with inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ respectively. One $V = V_1 \times V_2$, define

$$\langle(x_1, x_2), (y_1, y_2)\rangle_V := \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2, \qquad \forall\,(x_1, x_2), (y_1, y_2) \in V.$$

    Show that $\langle\cdot,\cdot\rangle_V$ is an inner product on V.

68. Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are inner products on a vector space V. Show that

$$\langle x, y\rangle := \langle x, y\rangle_1 + \langle x, y\rangle_2, \qquad \forall\, x, y \in V$$

    defines another inner product on V.

69. Let V be an n-dimensional inner product space and $\{u_1, \ldots, u_n\}$ be an orthonormal basis of V. Show that every linear functional $f : V \to \mathbb{F}$ can be written as $f = \sum_{j=1}^{n} f(u_j)f_j$, where, for each $j \in \{1, \ldots, n\}$, $f_j : V \to \mathbb{F}$ is the linear functional defined by $f_j(x) = \langle x, u_j\rangle$, $x \in V$.

70. For x, y in an inner product space V, show that $(x+y) \perp (x-y)$ if and only if $\|x\| = \|y\|$.

71. Let V be an inner product space. For $S \subset V$, let

$$S^\perp := \{x \in V : \langle x, u\rangle = 0 \quad \forall u \in S\}.$$

Show that

(a) $S^\perp$ is a subspace of V.

(b) $V^\perp = \{0\}, \quad \{0\}^\perp = V$.

(c) $S \subset S^{\perp\perp}$.

(d) If V is finite dimensional and V_0 is a subspace of V, then $V_0^{\perp\perp} = V_0$.

72. Find the best approximation of $x \in V$ from V_0 where

(a) $V = \mathbb{R}^3,\ x := (1, 2, 1),\ V_0 := \operatorname{span}\{(3, 1, 2), 1, 0, 1)\}$.

(b) $V = \mathbb{R}^3,\ x := (1, 2, 1),$ and V_0 is the set of all $(\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{R}^4$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 0\}$.

(c) $V = \mathbb{R}^4,\ x := (1, 0, -1, 1)\ V_0 := \operatorname{span}\{(1, 0, -1, 1),\ (0, 0, 1, 1)\}$.

(d) $V = C[-1, 1],\ x(t) = e^t,\ V_0 = \mathcal{P}_3$.

73. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$. Show that, there exists $x \in \mathbb{R}^n$ such that $\|Ax - y\| \le \|Au = y\|$ for all $u \in \mathbb{R}^n$, if and only if $A^T Ax = A^T y$.

74. Let $A \in \mathbb{R}^{m\times n}$ and $y \in \mathbb{R}^m$. If columns of A are linearly independent, then show that there exists a unique $x \in \mathbb{R}^n$ such that $A^T Ax = A^T y$.

75. Find the best approximate solution (least square solution) for the system $Ax = y$ in each of the following:

(a) $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}; \qquad y = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}; \qquad y = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$

(c) $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & 5 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}; \qquad y = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}.$

76. (a) Show that $\|x\|_\infty \le \|x\|_2 \le \|x\|_1$ for every $x \in \mathbb{R}^k$.

    (b) Find $c_1, c_2, c_3, c_4 > 0$ such that

    $$c_1\|x\|_2 \le \|x\|_\infty \le c_2\|x\|_2, \quad c_3\|x\|_1 \le \|x\|_1 \le c_4\|x\|_\infty$$

    for all $x \in \mathbb{R}^k$.

    (c) Compute $\|x\|_\infty$, $\|x\|_2$, $\|x\|_1$ for $x = (1,1,1) \in \mathbb{R}^3$.

77. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and and $A \in \mathbb{R}^{n\times n}$. Suppose $c > 0$ is such that $\|Ax\| \le c\,\|x\|$ for all $x \in \mathbb{R}^n$, and there exists $x_0 \ne 0$ in $\mathbb{R}^n$ such that $\|Ax_0\| = c\,\|x_0\|$. Then show that $\|A\| = c$.

78. Find $\|A\|_1$, $\|A\|_\infty$, for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

79. Suppose A, B in $\mathbb{R}^{n\times n}$ are invertible matrices, and $b, \tilde{b}$ are in $\mathbb{R}^n$. Let $x, \tilde{x}$ are in $\mathbb{R}^n$ be such that $Ax = b$ and $B\tilde{x} = \tilde{b}$. Show that

    $$\frac{\|x-\tilde{x}\|}{\|x\|} \le \|A\|\,\|B^{-1}\|\Big(\frac{\|A-B\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\Big).$$

    [Hint: Use the fact that $B(x-\tilde{x}) = (B-A)x + (b-\tilde{b})$, and use the fact that $\|(B-A)x\| \le \|B-A\|\,\|x\|$, and $\|b-\tilde{b}\| = \|b-\tilde{b}\|\|Ax\|/\|b\| \le \|b-\tilde{b}\|\|A\|\,\|x\|/\|b\|$.]

80. Let $B \in \mathbb{R}^{n\times n}$. If $\|B\| < 1$, then show that $I - B$ is invertible, and $\|(I-B)^{-1}\| \le 1/(1-\|B\|)$.

    [Hint: Show that $I - B$ is injective, by showing that for every x, $\|(I-B)x\| \ge (1-\|B\|)\|x\|$, and then deduce the result.]

81. Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A-B\| < 1/\|A^{-1}\|$. Then, show that, B is invertible, and

    $$\|B^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A-B\|\,\|A^{-1}\|}.$$

[Hint: Observe that $B = A - (A - B) = [I - (A - B)A^{-1}]A$, and use the previous problem.]

82. Let $A, B \in \mathbb{R}^{n\times n}$ be such that A is invertible, and $\|A - B\| < 1/2\|A^{-1}\|$. Let $b, \tilde{b}, x, \tilde{x}$ be as in Problem 79. Then, show that, B is invertible, and

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le 2\kappa(A)\Big(\frac{\|A - B\|}{\|A\|} + \frac{\|b - \tilde{b}\|}{\|b\|}\Big).$$

[Hint: Apply conclusion in Problem 81 to that in Problem 79]

83. Consider the system $Ax = b$ with

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix}.$$

(a) Show that the Jacobi method and Gauss-Siedel method for the above system converges.

(b) Obtain an error estimate for the k the iterate (for both the methods) w.r.t. the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ with initial approximation as $x^{(0)}$ as the zero vector.

84. Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$. Let $\beta_1, \ldots, \beta_n$ are real numbers. Then show that there exists a unique $\varphi \in \operatorname{span}\{u_1, \ldots, u_n\}$ satisfying $\varphi(t_i) = \beta_i$ for $i = 1, \ldots, n$ if and only if the matrix $[u_j(t_i)]$ is invertible.

85. Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$. Show that, if the matrix $[u_j(t_i)]$ is invertible, then $u_1, \ldots, u_n$ are linearly independent.

[*Hint*: A square matrix is invertible if and only if its columns are linearly independent.]

86. Suppose $u_1, \ldots, u_n$ are functions defined on $[a, b]$, and $t_1, \ldots, t_n$ are points in $[a, b]$ such that the matrix $[u_j(t_i)]$ is invertible. If $v_1, \ldots, v_n$ are linearly independent functions in $\operatorname{span}\{u_1, \ldots, u_n\}$, then show that the matrix $[v_j(t_i)]$ is also invertible.

[*Hint*: Let $X_0 := \operatorname{span}\{u_1, \ldots, u_n\}$ and $[u_j(t_i)]$ is invertible. Then observe that, the function $J : X_0 \to \mathbb{R}^n$ defined by $J(x) = [x(t_1), \ldots, x(t_n)]^T$ is bijective.]

87. Let $t_1, \ldots, t_n$ be distinct points in $\mathbb{R}$, and let

$$\ell_j(t) = \prod_{i \neq j} \frac{t - t_i}{t_j - t_i}, \quad j = 1, 2, \ldots, n.$$

Then show that $\{\ell_1, \ldots, \ell_n\}$ is a basis of $\mathcal{P}_{n-1}$, and it satisfies $\ell_j(t_i) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Deduce from the previous exercise that the matrix $[t_j^{i-1}]$ is invertible.

88. Let $t_1, \ldots, t_n$ be distinct points in $[a, b]$ and $u_1, \ldots, u_n$ are in $C[a, b]$ such that $u_i(t_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Show that

$$Px = \sum_{j=1}^{n} x(t_j) u_j, \quad x \in C[a, b],$$

defines a linear transformation $C[a, b]$ into itself, and it satisfies (a) $(Px)(t_i) = x(t_i)$, (b) $Px = x$ for all $x \in R(P)$ amd (c) $P^2 = P$.

89. Let $t_1, \ldots, t_n$ be distinct points in $[a, b]$. Show that for every $x \in C[a, b]$, there exists a unique polynomial $p(t)$ of degree atmost $n - 1$ such that $p(t_j) = x(t_j)$ for $j = 1, \ldots, n$.

90. Apply trapezoidal rule, composite trapezoidal rule, Simpson's rule, and composite Simpson's rule for approximating the the following integrals:

$$\int_0^1 e^{-t^2} dt, \qquad \int_0^4 \frac{dt}{1 + t^2} dt, \qquad \int_0^{2\pi} \frac{dt}{2 + \cos(t)} dt.$$