Assignment Sheet 3 on Legendre Polynimials
MA2020 Differential Equations (July - November 2012)

1. Using the governing differential equation of the Legendre polynomials show that
   (a) $P_n'(1) = \frac{n(n+1)}{2}$
   (b) $P_n'(-1) = (-1)^{n-1}\frac{n(n+1)}{2}$
2. Prove the Rodrigue's formula $P_n(x) \;=\; \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$
3. Prove that $P_n(x) \;=\; \sum_{r=0}^{N} \frac{(-1)^r\,(2n-2r)!}{2^n\,r!\,(n-r)!\,(n-2r)!}\,x^{n-2N}$ where N is $\frac{n}{2}$ or $\frac{n-1}{2}$ depending on n is even or odd
4. Prove that $(1-2xt+t^2)^{-\frac{1}{2}} \;=\; (1-t(2x-t))^{-\frac{1}{2}} \;=\; \sum_{n=0}^{\infty} t^n\, P_n(x)$
5. Using the generating function of $p_n(x)$, prove the following
   (a) $P_n(-1) = (-1)^n$
   (b) $P_n(-x) = (-1)^n P_n(x)$
   (c) $P_{2m}(0) = (-1)^m \frac{(2m)!}{2^{2m}(m!)^2}$
   (d) $P_{2m+1}(0) = 0$
6. Prove that
$$P_n(0) = \begin{cases} 0, & n \text{ is odd} \\ \frac{(-1)^{1/2}(n!)}{2^n((n/2)!)^2}, & n \text{ is even} \end{cases}$$
7. Prove the following recurrence formulae for $P_n(x)$
   (a) $(n+1)P_{n+1}(x) \;=\; (2n+1)xP_n(x) - nP_{n-1}(x)$
   (b) $nP_n(x) \;=\; xP_n'(x) - P_{n-1}'(x)$
   (c) $(2n+1)P_n(x) \;=\; P_{n+1}'(x) - P_{n-1}'(x)$
   (d) $P_n'(x) \;=\; xP_{n-1}'(x) + nP_{n-1}(x)$
   (e) $(1-x^2)P_n'(x) \;=\; n\,(P_{n-1}(x) - xP_n(x))$
8. If $P_n(\alpha) = 0$ then $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite sign
9. Prove the following orthogonal property of the Legendre polynomials $P_n(x)$
$$\int_{-1}^{1} P_m(x)\; P_n(x)\; dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$
10. Prove the following
    (a) $\int_{-1}^{1} x^2 P_{n-1}(x)\; P_{n+1}(x)\; dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$
    (b) $\int_{-1}^{1} x^2 P_n^2(x)\; dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$
    (c) $\int_{-1}^{1} P_3^2(x)\; dx = \frac{2}{7}$
    (d) $\int_{-1}^{1} xP_n^2(x)\; dx = 0$

11. Solve $(1-x^2)y''-2xy'=0$ (Legendre equation with $n=0$). Show that its general solution is $y(x)=c_1y_1(x)+c_2y_2(x)$ with $y_1(x)=P_0(x)=1$ and $y_2(x)=x+\frac{2}{3!}x^3+\frac{(-3)(-1)2.4}{5!}x^5+...=\frac{1}{2}\ln\frac{1+x}{1-x}$

12. Solve $(1-x^2)y''-2xy'+2y=0$. (Legendre equation with $n=1$)

13. Show that the coefficient of x^n in $P_n(x)$ is $\frac{(2n)!}{2^n(n!)^2}$.

14. Show that there are constants $\alpha_0,\alpha_1,...,\alpha_n$ such that

$$x^n=\alpha_0P_0(x)+\alpha_1P_1(x)+...+\alpha_nP_n(x)$$

15. Show that any polynomial of degree n is a linear combination of $P_0(x),P_1(x),...,P_n(x)$.

16. Find the first three terms of the Legendre series of

i) $f(x)=\begin{cases}0, & -1\le x<0\\ x, & 0\le x\le 1\end{cases}$ **ii)** $f(x)=e^x, -1\le x\le 1$

17. Let $P_n(x)$ be the Legendre polynomial. Show that

(i) $\int_{-1}^{1}x^mP_n(x)dx=\begin{cases}0, & m<n\\ \frac{2^{n+1}(n!)^2}{(2n+1)!}, & m=n\end{cases}$

(ii) $\int_{-1}^{1}(1-x^2)P_n'(x)P_n'(x)dx=\frac{2n(n+1)}{2n+1}$

18. The function on the left side of

$$\frac{1}{\sqrt{1-2xt+t^2}}=P_0(x)+P_1(x)t+P_2(x)t^2+...+P_n(x)t^n+...$$

is called the generating function of the Legendre polynomials. Assuming that this relation is true and use it

a) To verify that $P_n(1)=1$, $P_n(-1)=(-1)^n$.

b) $P_{2n+1}(0)=0$, $P_{2n}(0)=(-1)^n\frac{1.3...(2n-1)}{2^n(n!)}$

c) Differentiate (1) with respect to t and show that

$$(x-t)\sum_{n=0}^{\infty}P_n(x)t^n=(1-2xt+t^2)\sum_{n=1}^{\infty}nP_n(x)t^{n-1}$$

d) Equate the coefficients of t^n in (c) and obtain the recurrence relation

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$$

e) Use recurrence relation in (d) to find $P_2(x),P_3(x),P_4(x)$ and $P_5(x)$ assuming that $P_0(x)=1,P_1(x)=x$.

19. Find the first few terms of the Fourier-Legendre series.

(a) $f(x)=\cos\frac{\pi}{2}(x)$ **(b)** $f(x)=\begin{cases}0, & -1<x<0\\ x, & 0<x<1\end{cases}$