Department of Mathematics

Assignment 4, MA2020 Differential Equations

Solve the following problems near $x_0 = 0$, using Frobeneous method

(1) $9x(1-x)y'' - 12y' + 4y = 0$

(2) $2x^2y'' + xy' - (x^2+1)y = 0$

(3) $xy'' + y' - xy = 0$

(4) $x(x+1)y'' + 3xy' + y = 0$

(5) $x^2y'' + 6xy' + (6+x^2)y = 0$

Problems on Bessel's function & Sturm-Liouvellie Equations

(1) Show that, for integer values of n, $J_{-n}(x) = (-1)^n J_n(x)$ and $J_n(-x) = (-1)^n J_n(x)$.

(2) Show that

(a) $\sqrt{\frac{\pi x}{2}} J_{-1/2}(x) = \cos x$

(b) $J_{3/2}(x) = x^{-1}J_{1/2}(x) - J_{-1/2}(x)$.

(c) $\frac{d}{dx}[x^p J_p(x)] = x^p J_{p-1}(x)$

(d) $\frac{d}{dx}[x^{-p} J_p(x)] = -x^{-p} J_{p+1}(x)$

(e) $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$

(f) $J_p'(x) = \frac{-p}{x} J_p(x) + J_{p-1}(x)$

(g) $\int_0^\infty J_1(x)dx = 1$

(h) $\int_0^1 xJ_p(ax)J_p(bx)dx = \begin{cases} 0, & a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}(J_p'(a))^2, & a = b \end{cases}$

where a and b are zeros of $J_p(x)$.

(i) $J_0'(x) = -J_1(x)$

(j) $J_1'(x) = J_0(x) - \frac{1}{x}J_1(x)$

(k) $J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$

(l) $\int x^\nu J_{\nu-1}dx = x^\nu J_\nu(x) + c$

(m) $\int x^{-\nu} J_{\nu+1}dx = -x^{-\nu} J_\nu(x) + c$

(n) $\int J_{\nu+1}dx = \int J_{\nu-1}(x)dx - 2J_\nu(x)$

(3) Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ occur alternately. That is, between each pair of the consecutive zeros of either, there is exactly one zero of the other.

(4) Express $J_2(x)$, $J_3(x)$ and $J_4(x)$ in terms of $J_0(x)$, $J_1(x)$.

(5) Solve each of the following, using the indicated transformation, in terms of Bessel functions

(a) $y'' + 4xy = 0, \qquad (y = 4\sqrt{x},\ \frac{4}{3}x^{3/2} = z)$

(b) $x^2y'' + xy' + (4x^2 - \nu^2)y = 0, \qquad (2x = z)$

(c) $x^2y'' - 3xy' + 4(x^4 - 3)y = 0, \qquad (y = x^2u,\ x^2 = z)$

(d) $x^2y'' + xy' + (x^2 - \frac{1}{16})y = 0$

(6) Find the eigenvalues and eigenfunctions of the following Sturm-Liouville equations

(a) $y'' + \lambda y = 0; \qquad y(0) = 0, y'(1) = 0.$

(b) $y'' + \lambda y = 0; \qquad y(0) = y(2\pi), y'(0) = y'(2\pi).$

(c) $(xy')' + (\frac{\lambda}{x})y = 0; \qquad y(1) = 0, y(e) = 0.$

(d) $(e^{2x}y')' + e^{2x}(\lambda + 1)y = 0; \qquad y(0) = 0, y(\pi) = 0.$

Solve the following First Order Partial Differential Equations

(1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 2 - z, \quad z(x, 0) = \sin(x)$

(2) $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = y - z, \quad z(x, 0) = x^2$

(3) $x\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1, \quad z(1, y) = e^{-y}$

(4) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z, \quad z(1, y) = y^2$

(5) $x\frac{\partial z}{\partial x} + y^{-1}\frac{\partial z}{\partial y} = 1, \quad z(x, 0) = 5 - x$

(6) $\frac{\partial z}{\partial x} + \frac{1}{2y}\frac{\partial z}{\partial y} = 2, \quad z(x, 0) = \sin x - 2$