Notes on Analysis

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Assignment Problems-I

In the following, unless otherwise mentioned explicitly, E denotes a subset of a metric space with a metric d.

(1) Give an example of E and a sequence (f_n) of continuous functions defined on E such that $\{f_n(x)\}$ converges for every $x \in E$, but the limit function f defined by $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in E$, is not continuous.

(2) Give an example of E and a sequence (f_n) of continuous functions defined on E such that $\{f_n(x)\}$ converges for every $x \in E$ to a continuous function, but the convergence is not uniform.

(3) In Problem 2, is the sequence (f_n) uniformly bounded? If not, give an example of E and a uniformly bounded sequence (f_n) of continuous functions defined on E such that $\{f_n(x)\}$ converges for every $x \in E$ to a continuous function, but the convergence is not uniform.

(4) Give an example of an interval $[a, b]$ and a sequence (f_n) of differentiable functions defined on E which converges uniformly to a function f, but f is not differentiable on $[a, b]$.

[Hint: Try with $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ for $n \in \mathbb{N}$ and $x \in [-1, 1]$.]

(5) Suppose (f_n) is a sequence of functions defined on E.

(a)When do you say that (f_n) is pointwise bounded?

(b) When do you say that (f_n) is uniformly bounded?

(c) Suppose (f_n) converges uniformly to f on E and each f_n is bounded on E. Show that (f_n) is uniformly bounded on E.

(6) Suppose that (f_n) is a sequence functions defined on E which converges pointwise to f on E. Suppose there exists a sequence (x_n) in E and $c \neq 0$ such that

$f_n(x_n) - f(x_n) \to c$ as $n \to \infty$. Show that the convergence of (f_n) to f is not uniform.

(7) Let (f_n) be a sequence of continuous functions which converges uniformly to a function f. If (x_n) is a sequence in E such which converges to a point $x \in E$, then show that $\lim n \to \infty f_n(x_n) = f(x)$.

(8) Suppose that f_n is a sequence of continuous functions defined on E which converges pointwise to f on E. Show that, if either (a) E is compact or (b) (f_n) is equicontinuous, then the convergence of (f_n) to f is uniform. Given an examples to show that unform convergence is not guaranteed if either (a) or (b) is dropped.

(9) Let $f_n(x) = n^2 x(1-x^2)^n$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Show that (f_n) converges pointwise to a continuous function f, but $\int_0^1 f_n(x)dx \not\to \int_0^1 f(x)dx$.

(10) Let $f_n(x) = \frac{1}{n}\sin(n^3 \cos n^2 x)$ for $n \in \mathbb{N}$ and $x \in [0,1]$. Then (f_n) converges pointwise to the zero function and $\int_0^1 f_n(x)dx \to 0$ as $n \to \infty$ - Why?

(11) If (f_n) and (g_n) are sequence of bounded functions which converge uniformly to f and g respectively on E, then show that $(f_n g_n)$ converge uniformly to fg on E.

(12) Let $f_n(x) = \frac{x}{1+nx^2}$ for $n \in \mathbb{N}$ and $x \in (-1,1)$. Show that (f_n) converges uniformly to a function f. Does $\{f_n'(x)\}$ converge to $f'(x)$ for all $x \in (-1.1)$?

(13) If f is continuous on $[0,1]$ and if $\int_0^1 f(x)x^n dx = 0$ for all $n \in \mathbb{N} \cup \{0\}$, then prove that $f(x) = 0$ fir all $x \in [0,1]$.

(14) Prove that for every interval $[-a,a]$, there exists a sequence (P_n) of polynomials which converges uniformly to the function f defined by $f(x) = |x|$, $x \in [-a,a]$, and satisfying $P_n(0) = 0$ for all $n \in \mathbb{N}$.

(15) Suppose a power series $\sum_{n=0}^{\infty} c_n x^n$ converges at a point $x_0 \in \mathbb{R}$ and diverges at a point $y_0 \in \mathbb{R}$. Show that $\sum_{n=0}^{\infty} c_n x^n$ converges for all $x \in (-|x_0|, |x_0|)$ and $\sum_{n=0}^{\infty} c_n x^n$ diverges for all $x \notin [-|y_0|, |y_0|]$.

(16) Prove that the set $\mathcal{R}(\alpha)$ of all Riemann-Stieltjes integrable functions on $[a,b]$ is a vector space over $\mathbb{R}$ and the map $\int_a^b f d\alpha$ is a linear functional on $\mathcal{R}(\alpha)$.