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OPERATOR THEORY – ASSIGNMENT SHEET-I

In the following X denotes a normed linear space over $\mathbb{K}$.

1. Show that $X' := \mathcal{B}(X, \mathbb{K})$ is a Banach space.

2. For each $x \in X$, let $\varphi_x(f) = f(x)$ for $f \in X'$. Show that $\varphi_x \in X''$, $\|f_x\| = \|x\|$ for every $x \in X$ and the map $x \mapsto \varphi_x$ is linear.

3. Let D be a dense subset of X' and (φ_n) be a sequence in X'' such that $(\|\varphi_n\|)$ is bounded and $(\varphi_n(f))$ converges for every $f \in D$. Show that $(\varphi_n(f))$ converges for every $f \in X'$, and φ defined by $\varphi(f) := \lim_{n\to\infty} \varphi_n(f)$, $f \in X'$, belongs to X''.

4. Let X be a finite dimensional normed linear space and $\{u_1, \ldots, u_k\}$ be a basis of X. For each $j \in \{1, \ldots, k\}$ and $(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, let $f_j\Big(\sum_{i=1}^{k} \alpha_i u_i\Big) := \alpha_j$. Show that $f_j \in X'$.

5. Let X and f_j be as in Problem 4. Let (x_n) be a sequence in X and $x \in X$. Show that $f_j(x_n) \to f_j(x)$ for each j if and only if $\|x_n - x\| \to 0$.

6. Let X be a finite dimensional normed linear space, (x_n) be a sequence in X and $x \in X$. Show that $f(x_n) \to f(x)$ for each $f \in X'$ if and only if $\|x_n - x\| \to 0$.

7. If X is an infinite dimensional Hilbert space, show that for a sequence (x_n) in X and $x \in X$,
$$f(x_n) \to f(x) \quad \forall f \in X' \quad \not\Rightarrow \quad \|x_n - x\| \not\to 0.$$

8. If $1 \le p < \infty$ and $x \in \ell^p$, then show that $\sum_{j=1}^{\infty} x(j) e_j$ converges. Is the conclusion true if $p = \infty$? - Justify your answer.

9. Let $1 < p < \infty$. For each $j \in \mathbb{N}$, let $f_j(x) = x(j)$ for $x \in \ell^p$ and $j \in \mathbb{N}$. Show that $f_j \in (\ell^p)'$ for every $j \in \mathbb{N}$. Show also that $\{f_j : j \in \mathbb{N}\}$ a Schauder basis of $(\ell^p)'$.

10. Let X be a Hilbert space. If (x_n) in X is such that (x_n) converges weakly to x and $\|x_n\| \to \|x\|$, then prove that $\|x_n - x\| \to 0$.

11. Show that

    (i) if X is a normed linear space having a dense separable subspace, then X is separable;

    (ii) if X is having a Schauder basis, then X is separable.

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12. Let X be an inner product space having a countable orthonormal basis. Is X separable?

13. Show that the following spaces are separable:

    (i) ℓ^p for $1 \leq p < \infty$, (ii) $C[a, b]$ with $1 \leq p \leq \infty$.

14. Show that the following spaces are not separable:

    (i) ℓ^∞ (ii) $L^\infty[a, b]$, (iii) $NBV[a, b]$.

15. Show that every normed liner space which is linearly isometric with a separable space is separable.

16. If X is reflexive, then X' is reflexive. Is the converse true? Justify your answer.

17. The following spaces are not reflexive. Why?

    (i) ℓ^1, (ii) ℓ^∞, (iii) $C[a, b]$, (iv) $L^1[a, b]$, (v) $L^\infty[a, b]$, (vi) $NBV[a, b]$.

18. Does every bounded sequence in ℓ^1 has a weakly convergent subsequence?

19. Let (x_n) is a bounded sequence in ℓ^p with $1 < p < \infty$, such that there exists $x \in \ell^p$ satisfying $x_n(j) \to x(j)$ for each $j \in \mathbb{N}$. Then (x_n) converges weakly to x - Why?

20. Let X and Y be normed linear spaces and $A : X \to Y$ be a linear operator. Prove that

    (i) If A is continuous then for every (x_n) which converges weakly to x, the sequence (Ax_n) converges weakly to Ax.

    (ii) A is continuous if and only if for (x_n) in X,

    $$\|x_n - x\| \to 0 \quad \text{implies} \quad (Ax_n) \quad \text{converges weakly to} \quad Ax.$$

21. Let X be Banach space, Y be a normed linear space and (A_n) be a sequence in $BL(X, Y)$. If for every $x \in X$, there exists a $y \in Y$ denoted by $y = A(x)$ such that $(A_n x)$ converges weakly to y, then prove that

    $$A \in BL(X, Y) \quad \text{and} \quad \|A\| \leq \sup_n \|A_n\| < \infty.$$

22. Let X be a normed linear space . For $S \subseteq X'$, let

    $$S^a = \{x \in X : f(x) = 0 \ \forall f \in S\}.$$

    Show, if X is reflexive and $S^a = \{0\}$, then $span(S)$ is dense in X'.

23. Suppose (f_n) is a sequence in X' which converges weakly to $f \in X'$. Show that $f_n(x) \to f(x)$ for every $x \in X$.

24. Is the converse of the assertion in Problem 23 true?

25. Is c_0 with $\| \cdot \|_\infty$ reflexive? Why?