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OPERATOR THEORY – ASSIGNMENT SHEET-I

In the following X denotes a normed linear space over $\mathbb{K}$ which is either $\mathbb{R}$ or $\mathbb{C}$.

1. For each $x \in X$, let $\varphi_x(f) = f(x)$ for $f \in X'$. Show that $\varphi_x \in X''$, $\|f_x\| = \|x\|$ for every $x \in X$ and the map $x \mapsto \varphi_x$ is linear.

2. Let D be a dense subset of X' and (φ_n) be a sequence in X'' such that $(\|\varphi_n\|)$ is bounded and $(\varphi_n(f))$ converges for every $f \in D$. Show that $(\varphi_n(f))$ converges for every $f \in X'$, and φ defined by $\varphi(f) := \lim\limits_{n\to\infty} \varphi_n(f)$, $f \in X'$, belongs to X''.

3. Let X be a finite dimensional normed linear space and $\{u_1, \ldots, u_k\}$ be a basis of X. For each $j \in \{1, \ldots, k\}$ and $(\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, let $f_j\Big(\sum\limits_{i=1}^{k} \alpha_i u_i\Big) := \alpha_j$. Show that $f_j \in X'$.

4. Let X and f_j be as in Problem 3. Let (x_n) be a sequence in X and $x \in X$. Show that the following are equivalent:
   (a) $f_j(x_n) \to f_j(x)$ for each j.
   (b) $f(x_n) \to f(x)$ for every $f \in X'$.
   (c) $\|x_n - x\| \to 0$.

5. If X is an infinite dimensional Hilbert space, show that there exists a sequence (x_n) in X and $x \in X$ such that $f(x_n) \to f(x) \,\forall\, f \in X'$ but $x_n \not\to x$.

6. If $1 \le p < \infty$ and $x \in \ell^p$, then show that $\sum\limits_{j=1}^{\infty} x(j) e_j$ converges. Is the conclusion true if $p = \infty$? - Justify your answer.

7. Let $1 < p < \infty$. For each $j \in \mathbb{N}$, let $f_j(x) = x(j)$ for $x \in \ell^p$ and $j \in \mathbb{N}$. Show that $f_j \in (\ell^p)'$ for every $j \in \mathbb{N}$, and $\{f_j : j \in \mathbb{N}\}$ is a Schauder basis of $(\ell^p)'$.

8. Let $X = c_0$ with $\|\cdot\|_\infty$ and $x \in X$. For each $j \in \mathbb{N}$, let $f_j(x) = x(j)$ for $x \in X$ and $j \in \mathbb{N}$. Show that $f_j \in X'$ for every $j \in \mathbb{N}$, and $\{f_j : j \in \mathbb{N}\}$ is a Schauder basis of X'.

9. Let X be a Hilbert space. If (x_n) in X is such that (x_n) converges weakly to x and $\|x_n\| \to \|x\|$, then prove that $\|x_n - x\| \to 0$.

10. Let X_0 be a subspace of X and (x_n) be a sequence in X which converges weakly to x in X_0. Show that (x_n) converges weakly to x in X.

11. If (x_n) in X is such that $(f(x_n))$ converges for every $f \in X'$, then (x_n) need not converge weakly - Justify.

12. Prove that, if X is reflexive and (x_n) in X is such that $(f(x_n))$ converges for every $f \in X'$, then (x_n) converges weakly.

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13. Prove:

    (a) If X is a normed linear space having a dense separable subspace, then X is separable;

    (b) If X is having a Schauder basis, then X is separable.

14. Let X be an inner product space having a countable orthonormal basis. Is X separable?

15. Show that the following spaces are separable:

    (i) ℓ^p for $1 \leq p < \infty$, (ii) $C[a,b]$ with $1 \leq p \leq \infty$.

16. Show that the following spaces are not separable:

    (i) ℓ^∞ (ii) $L^\infty[a,b]$, (iii) $NBV[a,b]$.

17. Show that every normed liner space which is linearly isometric with a separable space is separable.

18. If X is reflexive, then X' is reflexive. Is the converse true? Justify your answer.

19. The following spaces are not reflexive. Why?

    (i) ℓ^1, (ii) ℓ^∞, (iii) $C[a,b]$, (iv) $L^1[a,b]$, (v) $L^\infty[a,b]$, (vi) $NBV[a,b]$.

20. Does every bounded sequence in ℓ^1 has a weakly convergent subsequence?

21. Let X be ℓ^p with $1 < p < \infty$ or c_0 with $\cdot\|_\infty$. Let (x_n) be a bounded sequence in X such that there exists $x \in X$ satisfying $x_n(j) \to x(j)$ for each $j \in \mathbb{N}$. Prove that (x_n) converges weakly to x.

    Show that the conclusion need not hold if X is either ℓ^1 or ℓ^∞.

22. Let X and Y be normed linear spaces and $A : X \to Y$ be a linear operator. Prove that

    (i) If A is continuous then for every (x_n) which converges weakly to x, the sequence (Ax_n) converges weakly to Ax.

    (ii) A is continuous if and only if for (x_n) in X,

    $$\|x_n - x\| \to 0 \quad \text{implies} \quad (Ax_n) \quad \text{converges weakly to} \quad Ax.$$

23. Let X be Banach space, Y be a normed linear space and (A_n) be a sequence in $BL(X,Y)$ such that for every $x \in X$, $(A_n x)$ converges weakly, to say y_x. Let $Ax = y_x$, $x \in X$. Prove that

    $$A \in BL(X,Y) \quad \text{and} \quad \|A\| \leq \sup_n \|A_n\| < \infty.$$

24. Let X be a normed linear space . For $S \subseteq X'$, let $S^a = \{x \in X : f(x) = 0 \; \forall f \in S\}$. Show, if X is reflexive and $S^a = \{0\}$, then $span(S)$ is dense in X'.

25. Suppose (f_n) is a sequence in X' which converges weakly to $f \in X'$. Show that $f_n(x) \to f(x)$ for every $x \in X$.

    Is the converse of the assertion in Problem 25 true?