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MA 6150: BASIC OPERATOR THEORY – ASSIGNMENT SHEET-2

1. Justify the following statements:

   (a) Every bounded operator of finite rank is a compact operator.

   (b) The identity operator on a normed linear space is a compact operator if and only if the space is of finite dimension.

   (c) If X_0 is a subspace of a normed linear space X, then the inclusion operator $I_0 : X_0 \to X$, i.e., $I_0 x = x$ for all $x \in X_0$, is a compact operator if and only if X_0 is finite dimensional.

   (d) If $P : X \to X$ is a bounded projection operator, then it is compact if and only if $rank P < \infty$.

2. Show that the right-shift and left-shift operators from ℓ^p to itself are not compact operators.

3. Let (λ_n) be a bounded sequence in $\mathbb{K}$ such that $\lambda_n \to \lambda \neq 0$, and let $A : \ell^p \to \ell^p$ be the *diagonal operator* associated with this sequence, i.e., A is defined by

$$(Ax)(i) = \lambda_i x(i), \quad i \in \mathbb{N};\ x \in \ell^p.$$

   Show that A is compact if and only if $\lambda = 0$.

4. For $k(\cdot,\cdot) \in C([a,b] \times [a,b])$, consider the integral operator K defined by

$$(Kx)(s) = \int_a^b k(s,t)x(t)\,d\mu(t), \quad x \in L^1[a,b].$$

   Show that, if X, Y any of the spaces $C[a,b]$, $L^p[a,b]$, $K : X \to Y$ is a compact operator.

5. Let X and Y be normed linear spaces $A \in \mathbb{K}(\mathbb{X}, \mathbb{Y})$ and X_0 be a subspace of X.

   (i) If A_0 is the restriction of A to X_0, i.e., $A_0 := A_{|X_0}$, then show that $A_0 : X_0 \to Y$ is a compact operator.

   (ii) Suppose $\|\cdot\|_0$ is a stronger norm on X_0, i.e., there exists $c_0 > 0$ such that $\|x\|_X \leq c\|x\|_0$ for all $x \in X_0$. If $\widetilde{X}_0 := X_0$ with $\|\cdot\|_0$ and $\widetilde{A}_0$ is the restriction of A to $\widetilde{X}_0$, then show that $\widetilde{A}_0 : \widetilde{X}_0 \to Y$ is a compact operator.

   (iii) Suppose $\widehat{Y}$ be a normed linear space such that Y is a subspace of $\widehat{Y}$ and with a norm weaker than that in Y, i.e., there exists $c > 0$ such that $\|y\|_{\widehat{Y}} \leq c\|y\|_Y$ for all $y \in Y$. If $\widehat{A} : X \to \widehat{Y}$ is the operator defined by $\widehat{A}x = Ax$ for all $x \in X$, then show that $\widehat{A} : X \to \widehat{Y}$ is a compact operator.

6. Let $A : X \to Y$ be an injective compact operator. Prove that $A^{-1} : R(A) \to X$ is continuous if and only if $rankA < \infty$.

7. Let $A : X \to Y$ be a compact operator of infinite rank. Then prove that A is not bounded below.

8. Let X and Y be Banach spaces, and $A : X \to Y$ be a compact operator. Then prove that $R(A)$ closed in Y if and only if A is of finite rank.

9. If X is an infinite dimensional normed linear space and $K \in \mathbb{K}(\mathbb{X})$, then show that, if λ is a nonzero scalar, then $\lambda I - K$ is not a compact operator . Deduce that the operator

$$A : (\alpha_1, \alpha_2, \ldots) \mapsto \Big(\alpha_1 + \alpha_2,\ \alpha_2 + \frac{\alpha_3}{2},\ \alpha_3 + \frac{\alpha_4}{3},\ \ldots\Big)$$

is not a compact operator on ℓ^p, $1 \le p \le \infty$.

10. For $1 \le p \le \infty$, let q be the conjugate exponent of p. Let (a_{ij}) be an infinite matrix with $a_{ij} \in \mathbb{K}$, $i, j \in \mathbb{N}$. Show, in each of the following cases, that $(Ax)(i) = \sum_{j=1}^{\infty} a_{ij}x(j)$, $x \in \ell^p$, $i \in \mathbb{N}$, is well defined and $A : \ell^p \to \ell^r$ is a compact operator.

    (a) $1 \le p \le \infty$, $1 \le r \le \infty$ and $\sum_{j=1}^{\infty} |a_{ij}| \to 0$ as $i \to \infty$.

    (b) $1 \le p \le \infty$, $1 \le r < \infty$ and $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} |a_{ij}|)^r < \infty$.

    (c) $1 < p \le \infty$, $1 \le r \le \infty$ and $\sum_{j=1}^{\infty} |a_{ij}|^q \to 0$ as $i \to \infty$.

    (d) $1 < p \le \infty$, $1 \le r < \infty$ and $\sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} |a_{ij}|^q)^{r/q} < \infty$.

11. Let (λ_n) be a bounded sequence in $\mathbb{K}$ such that $\{\lambda_n : n \in \mathbb{N}\}$ has a nonzero accumulation point and $A : \ell^p \to \ell^p$ be the diagonal operator associated (λ_n), i.e., $(Ax)(i) = \lambda_i x(i)$ for every $i \in \mathbb{N}$ and $x \in \ell^p$. Show that A is not a compact operator.

12. Show that the range of a compact operator is separable.

    [*Hint*: Use the facts (to be proved) that totally bounded sets are separable, and the range of a compact operator is a countable union of totally bounded sets.]

13. Let X_0, X_1, X_2, X_3 be normed linear spaces such that $X_0 \subseteq X_1$ and $X_2 \subseteq X_3$. Suppose further that the above inclusions are imbedings, i.e., there exists $c > 0$ and c' such that

$$\|x\|_1 \le c\|x\|_0, \quad \|y\|_3 \le c'\|y\|_2 \quad \forall\, x \in X_0,\ y \in X_2.$$

    If $A : X_1 \to X_2$ is a compact operator, then prove that A as an operator from X_0 to , X_2, from X_1 to X_3 and from X_0 to X_3 are compact.

14. Show that the *Volterra integral operator* V defined by

$$(Vx)(s) = \int_a^s x(t)\,dt, \quad x \in C[a,b],\ s \in [a,b]$$

is a compact operator from $C[a,b]$ into itself with respect to the norm $\|\cdot\|_\infty$.

15. Let $X_0 = C^1[a,b]$ with $\|x\|_0 := \|x\|_\infty + \|x'\|_\infty$ for $x \in X_0$ and $X = C[a,b]$ with $\|\cdot\|_\infty$. Let $A_0 : X_0 \to X$ be defined by

$$A_0 x = x(a)u_0, \quad x \in X_0,$$

where $u_0(t) = 1$ for all $t \in [a,b]$, and $B : X_0 \to X$ be the differential operator, i.e., $Bx = x'$ for every $x \in X_0$.

(a) Show that A_0 is a compact operator and B is a bounded operator.

(b) If $I_0 : X_0 \to X$ is the inclusion operator, i.e., $I_0 x = x$ for all $x \in X_0$, and V is the Volterra operator on $C[a,b]$ defined as in Problem 14, then show that $I_0 = A_0 + VB$.

16. Does compactness of I_0 follow from the above representation?

17. Let X be a separable (infinite dimensional) Hilbert space and $\{u_n : n \in \mathbb{N}\}$ be an orthonomal basis for X. Let (λ_n) be bounded sequence in $\mathbb{K}$. Let $A : X \to X$ be defined by

$$Ax = \sum_{j=1}^{\infty} \lambda_n \langle x, u_j \rangle u_j, \quad x \in H.$$

Show that A is a compact operator if and only if $\lim_{n\to\infty} \lambda_n = 0$.

18. Let $X = C[a,b]$ with $\|\cdot\|_\infty$ and $u \in X$. Show that the multiplication operator $A : X \to X$ defined by $(Ax)(t) = u(t)x(t)$ for $x \in X$ and $t \in [a,b]$ is not a compact operator, unless u is identically zero.

[*Hint:* Construct an appropriate subspace X_0 of X such that the restriction of A to X_0 is bounded below.]

19. Suppose X is a Hilbert space and $A \in \mathcal{B}(X)$. Show that $A : X \to X$ is compact if and only if for every sequence (x_n) in X,

$$\langle x_n, u \rangle \to \langle x, u \rangle \quad \forall\, u \in X \implies Ax_n \to Ax.$$

20. Let $1 \le p < \infty$. Is the set

$$\left\{ x = (\alpha_1 \alpha_2, \ldots) \in \ell^p : \sum_{j=1}^{\infty} j^p |\alpha_j|^p < \infty \right\}$$

a closed subspace of ℓ^p?