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BASIC OPERATOR THEORY - ASSIGNMENT PROBLEMS

In the following X denotes a normed linear space over $\mathbb{K}$.

1. Show that $X' := \mathcal{B}(\mathcal{X}, \mathbb{K})$ is a Banach space.

2. For each $x \in X$, let $\varphi_x(f) = f(x)$ for $f \in X'$. Show that $\varphi_x \in X''$, $\|f_x\| = \|x\|$ for every $x \in X$ and the map $x \mapsto \varphi_x$ is linear.

3. Let D be a dense subset of X' and (φ_n) be a sequence in X'' such that $(\|\varphi_n\|)$ is bounded and $(\varphi_n(f))$ converges for every $f \in D$. Show that $(\varphi_n(f))$ converges for every $f \in X'$, and φ defined by $\varphi(f) := \lim_{n\to\infty} \varphi_n(f)$, $f \in X'$, belongs to X''.

4. Let X be a finite dimensional normed linear space and $\{u_1, \ldots, u_k\}$ be a basis of X. For each $j \in \{1, \ldots, k\}$ and $(\alpha_, \ldots, \alpha_k) \in \mathbb{R}^k$, let $f_j\Big(\sum_{i=1}^k \alpha_i u_i\Big) := \alpha_j$. Show that $f_j \in X'$.

5. Let X and f_j be as in Problem 4. Let (x_n) be a sequence in X and $x \in X$. Show that $f_j(x_n) \to f_j(x)$ for each j if and only if $\|x_n - x\| \to 0$.

6. Let X be a finite dimensional normed linear space, (x_n) be a sequence in X and $x \in X$. Show that $f(x_n) \to f(x)$ for each $f \in X'$ if and only if $\|x_n - x\| \to 0$.

7. If X is an infinite dimensional Hilbert space, show that there exist a sequence (x_n) in X and $x \in X$ such that $f(x_n) \to f(x)$ for every $f \in X$, but $\|x_n - x\| \not\to 0$.

8. If $1 \le p < \infty$ and $x \in \ell^p$, then show that $\sum_{j=1}^\infty x(j)e_j$ converges. Is the conclusion true if $p = \infty$? - Justify your answer.

9. Let $1 < p < \infty$. For each $j \in \mathbb{N}$, let $f_j(x) = x(j)$ for $x \in \ell^p$ and $j \in \mathbb{N}$. Show that $f_j \in (\ell^p)'$ for every $j \in \mathbb{N}$. Show also that $\{f_j : j \in \mathbb{N}\}$ a Schauder basis of $(\ell^p)'$.

10. Let X be a Hilbert space. If (x_n) in X is such that (x_n) converges weakly to x and $\|x_n\| \to \|x\|$, then prove that $\|x_n - x\| \to 0$.

11. Show that

    (i) if X is a normed linear space having a dense separable subspace, then X is separable;

    (ii) if X is having a Schauder basis, then X is separable.

12. Let X be an inner product space having a countable orthonormal basis. Is X separable? (see[1])

[1]Example of a nonseparable inner product space with all its orthonormal sets countable, http://mat.iitm.ac.in/home/mtnair/public$_\text{h}$tml/

13. Show that the following spaces are separable:

    (i) ℓ^p for $1 \leq p < \infty$, (ii) $C[a,b]$ with $1 \leq p \leq \infty$.

14. Show that the following spaces are not separable:

    (i) ℓ^∞ (ii) $L^\infty[a,b]$, (iii) $NBV[a,b]$.

15. Show that every normed liner space which is linearly isometric with a separable space is separable.

16. If X is reflexive, then X' is reflexive. Is the converse true? Justify your answer.

17. Show that the following spaces are not reflexive:

    (i) ℓ^1, (ii) ℓ^∞, (iii) $C[a,b]$, (iv) $L^[a,b]$, (v) $L^\infty[a,b]$, (vi) $NBV[a,b]$.

18. Does every bounded sequence in ℓ^1 has a weakly convergent subsequence?

19. Let (x_n) is a bounded sequence in ℓ^p with $1 < p < \infty$, such that there exists $x \in \ell^p$ satisfying $x_n(j) \to x(j)$ for each $j \in \mathbb{N}$. Then (x_n) converges weakly to x - Why?

20. Let X and Y be normed linear spaces and $A : X \to Y$ be a linear operator. Prove:

    (i) If A is continuous then for every (x_n) which converges weakly to x, the sequence (Ax_n) converges weakly to Ax.

    (ii) A is continuous if and only if for (x_n) in X, $\|x_n - x\| \to 0$ implies (Ax_n) converges weakly to Ax.

21. Let X be Banach space, Y be a normed linear space and (A_n) be a sequence in $BL(X,Y)$. If for every $x \in X$, there exists a $y \in Y$ denoted by $y = A(x)$ such that $(A_n x)$ converges weakly to y, then prove that

    $$A \in BL(X,Y) \quad \text{and} \quad \|A\| \leq \sup_n \|A_n\| < \infty.$$

22. Let X be a normed linear space . For $S \subseteq X'$, let

    $$S^a = \{x \in X : f(x) = 0 \ \forall f \in S\}.$$

    Show, if X is reflexive and $S^a = \{0\}$, then $span(S)$ is dense in X'.

23. Suppose (f_n) is a sequence in x' which converges weakly to $f \in X'$. Show that $f_n(x) \to f(x)$ for every $x \in X$.

24. Is the converse of the assertion in Problem 23 true?

25. Is c_0 with $\|\cdot\|_\infty$ reflexive? Why?

26. State whether the operator $A : \ell^p \to \ell^p$, $1 \le p \le \infty$, in each of the following is compact or not. Justify the answer.

    (i) $A : (a_1, a_2, a_3, \ldots) \mapsto \big(\alpha_1 + \alpha_2, \alpha_2 + \frac{1}{2}\alpha_3, \alpha_3 + \frac{1}{3}\alpha_4, \ldots\big)$.

    (ii) $A : (\alpha_1, \alpha_2, \alpha_3, \ldots) \mapsto \big(\frac{1}{2}\alpha_1, \frac{2}{3}\alpha_2, \frac{3}{4}\alpha_3, \ldots\big)$.

27. Suppose $P \in BL(X)$ be a projection operator, i.e., $P^2 = P$. Then prove that P is compact if and only if *rank* $P < \infty$.

28. Let (a_{ij}) be an infinite matrix of scalars such that $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$. Show that the map $(\alpha_i) \mapsto (\beta_i)$ with $\beta_i = \sum_j k_{ij}\alpha_j, \quad i = 1, 2, \ldots,$ defines a compact operator on ℓ^2.

29. Let X and Y be normed linear spaces and $A : X \to Y$ be an injective compact operator. Show that $A^{-1} : R(A) \to X$ is continuous iff A is of finite rank.

30. Is the set $\{(\alpha_1\alpha_2, \ldots) \in \ell^2 : \sum_{n=1}^{\infty} n^2|\alpha_n|^2 < \infty\}$ a closed subspace of ℓ^2 ? – Justify the answer.

31. State whether the operator in each of the following is compact or not. Justify your answer:

    (i) The diagonal operator on ℓ^p associated with the sequence $(\frac{1}{n})$.

    (ii) The operator $I + K$, where K is a compact operator.

32. Suppose $A : X \to Y$ is a compact operator of infinite rank. Then show that there exists a sequence (x_n) in X such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$, and $\|Ax_n\| \to 0$ as $n \to \infty$.

33. Give an example of a bounded operator on a complex Banach space having no eigenvalues. Justify your claim.

34. Give an example of a normal operator A on ℓ^2 such that $\sigma_{\text{app}}(A) = [0, 1]$. Justify your claim.

35. If $P : X \to X$ is a projection operator on a Banach space X, and if $0 \neq P \neq I$, then show that $\sigma(P) = \{0, 1\}$.

36. Let X be a complex Banach space, $A \in BL(X)$. Prove the following.

    (i) For every $\mu \in \rho(A)$, $\sigma(A - \mu I)^{-1} = \{\frac{1}{\lambda - \mu} : \lambda \in \sigma(A)\}$.

    (ii) If X is over $\mathbb{C}$ and $p(t)$ is a polynomial, then $\{p(\lambda) : \lambda \in \sigma(A)\} = \sigma(p(A))$.

37. Let A be a normal operator on a Hilbert space X. Show that if $\mu \in \rho(A)$, then $\|(A - \mu I)^{-1}\| = 1/\text{dist}(\mu, \sigma(A))$.

38. Let H be a Hilbert space, $\mathcal{S}$ be the set of all self adjoint operators on H, $\mathcal{N}$ be the set of all normal operators on H, and $\mathcal{U}$ be the set of all unitary operators on H. Show that

    (i) $\mathcal{S} \subseteq \mathcal{N}$ and (ii) $\mathcal{U} \subseteq \mathcal{N}$.
    Give examples to show that the inclusions are proper if $\dim(H) \ge 2$.

39. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 0 & 2 \\ 3 & 0 & 3 & 1 \\ 4 & 2 & 1 & 4 \end{bmatrix}$. State whether the following statements are true or false, and justify the answers :

(i) There exists an eigenvalue λ of A such that $\lambda \leq 1$

(ii) If λ is an eigenvalue of A, then $\lambda \leq 1$.

(iii) There exists an eigenvalue λ of A such that $\lambda \geq 4$

(iv) If λ is an eigenvalue of A, then $\lambda \geq 4$.

40. Show that the approximate eigen spectrum of a bounded operator is a closed set.

41. Find the eigen-spectrum, approximate eigen-spectrum, and spectrum of the operator A defined below: $X = C[a, b]$ with $\|\cdot\|_\infty$, and $(Ax)(t) = tx(t)$ for $t \in [a, b]$, $x \in X$.

42. In each f the following, give an example and justify your claim:

(i) An approximate eigenvalue (of an operator) which is neither an eigenvalue nor the limit of a sequence of eigenvalues.

(ii) A self adjoint operator having no eigenvalues.

(iii) A compact operator on $L^2[0, 1]$ having no eigenvalues.

(iv) A bounded operator on an inner product space having no adjoint.

43. If A is a compact operator on a Hilbert space and $\lambda \neq 0$, then show that there exists $c > 0$ such that $\|Ax - \lambda x\| \geq c\|x\|$ for all $x \in N(A - \lambda I)^\perp$.

44. Let X and Y be Hilbert spaces and $T \in \mathcal{K}(X, Y)$. Show that there exists a sequence (T_n) in $\mathcal{B}(X, Y)$ of finite rank operators such that $\|T - T_n\| \to 0$ as $n \to \infty$.

45. Let X and Y be Hilbert spaces, $T \in BL(X, Y)$ and $y \in Y$. Show that there exists $x \in X$ such that $\|Tx - y\| = \inf\{\|Tu - y\| : u \in X\}$ if and only if there exists $x_0 \in X$ such that $T^*Tx_0 = T^*y$.

46. Let X and Y be Hilbert spaces and $T \in BL(X, Y)$. Let $P : Y \to Y$ be the orthogonal projection onto the closure of $R(A)$. Prove the following:

(i) $D := \{y \in Y : Py \in R(A)\}$ is dense in Y.

(ii) For each $y \in D$, there exists a unique $x^\dagger \in N(A)^\perp$ such that $Tx^\dagger = Py$.

(iii) The map $T^\dagger : D \to X$ defined by $T^\dagger y = x^\dagger(y)$, $y \in D$ is a closed operator.

(iv) $T^\dagger$ is continuous iff and only if $R(A)$ is closed in Y.