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MA 6150: BASIC OPERATOR THEORY
ASSIGNMENT SHEET-3

1. In the following, X is any of the sequence spaces c_{00}, c_0, c, ℓ^p.

    (a) Let (λ_n) be a bounded sequence of scalars. Let $A : X \to X$ be the *diagonal operator* defined by $(Ax)(j) = \lambda_j\, x(j)$, $x \in X$; $j \in \mathbb{N}$. Then show that $\sigma_{\rm eig}(A) = \{\lambda_1, \lambda_2, \ldots\}$.

    (b) Let $A : X \to X$ be the *right shift operator*. Show that $\sigma_{\rm eig}(A) = \varnothing$.

    (c) Let $A : X \to X$ be the *left shift operator*. Prove the following.

    i. If $X = c_{00}$, then $\sigma_{\rm eig}(A) = \{0\}$.

    ii. If $X = c_0$ or $X = \ell^p$ for $1 \le p < \infty$, then $\sigma_{\rm eig}(A) = \{\lambda : |\lambda| < 1\}$.

    iii. If $X = c$, then $\sigma_{\rm eig}(A) = \{\lambda : |\lambda| < 1\} \cup \{1\}$.

    iv. If $X = \ell^\infty$, then $\sigma_{\rm eig}(A) = \{\lambda : |\lambda| \le 1\}$.

2. If X is a normed linear space and $A \in \mathcal{B}(X)$, then show that $\sigma_{\rm eig}(A) \subseteq \{\lambda \in \mathbb{K} : |\lambda| \le \|A\|\}$.

3. Let X be a normed linear space and $A : X \to X$ be a linear operator. Suppose $\lambda \in \mathbb{K}$ is such that $A - \lambda I$ is injective. Show that $(A - \lambda I)^{-1} : R(A - \lambda I) \to X$ is continuous if and only if λ is not an approximate eigenvalue.

4. Give an example of an approximate eigenvalue which is neither an eigenvalue nor a limit of a sequence of eigenvalues.

5. Let $X = \ell^1$ and A be the *right shift operator* on ℓ^1. Find a sequence (x_n) in ℓ^1 such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Ax_n - x_n\| \to 0$.

6. Let X be a Banach space and $A \in \mathcal{B}(X)$. If $R(A)$ is not closed in X, then $0 \in \sigma_{\rm app}(A)$. Why?

7. Let $X = \ell^p$ for $1 \le p \le \infty$. Prove the following.

    (a) Let $A : X \to X$ be a diagonal operator, i.e., $(Ax)(i) = \lambda_i x(i)$ for all $x \in X$, $i \in \mathbb{N}$, where (λ_n) is a bounded sequence of scalars. Then $\sigma_{\rm app}(A) = \mathrm{cl}\{\lambda_{\rm n} : \mathrm{n} \in \mathbb{N}\}$.

    (b) Let A be the right shift operator on ℓ^p. Then $\sigma_{\rm app}(A) = \{\lambda \in \mathbb{K} : |\lambda| = 1\}$.

    (c) Let A be the *left shift operator* on ℓ^p. Then $\sigma_{\rm app}(A) = \{\lambda \in \mathbb{K} : |\lambda| \le 1\}$.

8. Let $X = C[a, b]$ with $\|\cdot\|_\infty$, and $A : X \to X$ be defined by $(Ax)(t) = tx(t)$, $\forall\, x \in X$; $t \in [a, b]$. Show that $\sigma_{\rm eig}(A) = \varnothing$ and $\sigma_{\rm app}(A) = [a, b]$.

9. If X is a finite dimensional space, then $\sigma_{\rm eig}(A) = \sigma_{\rm app}(A) = \sigma(A)$. Why?

10. If X is a Banach space and $A \in \mathcal{B}(X)$, then $\sigma(A) = \{\lambda \in \mathbb{K} : A - \lambda I \text{ is not bijective}\}$. Why?

11. Give an example of a Banach space X and $A \in \mathcal{B}(X)$ such that $\|A\| > 1$ and $I - A$ is bijective.

12. Let X be a Banach space and $A \in \mathcal{B}(X)$. Prove the following.

    (a) If $\|A\| < 1$, then the series $\sum_{n=0}^{\infty} A^n$ converges to $(I - A)^{-1}$.

    (b) There exists $c > 0$ such that for every $\lambda \in \mathbb{K}$ with $|\lambda| > c$, $\lambda \in \rho(A)$ and $\|(A - \lambda I)^{-1}\| \to 0$ as $|\lambda| \to \infty$.

13. Let $X = \ell^p$ for $1 \leq p \leq \infty$. Show the following.

    (a) If A is the diagonal operator defined by $(Ax)(i) = \lambda_i x(i)$, $x \in X$, $i \in \mathbb{N}$, where (λ_n) is a bounded sequence of scalars, then $\sigma(A) = \mathrm{cl}\{\lambda_\mathrm{n} : \mathrm{n} \in \mathbb{N}\}$.

    (b) If A is the *right shift operator* on X, then $\sigma(A) = \{\lambda : |\lambda| \leq 1\}$.

    (c) If A is the *left shift operator* on X, then $\sigma(A) = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$.

14. Let $X = C[a, b]$ with $\|\cdot\|_\infty$, and let $A : X \to X$ be defined by $(Ax)(t) = tx(t)$, $x \in X$; $t \in [a, b]$. Show that $\sigma(A) = [a, b]$.

15. Give examples of Banach spaces and operators such that equality and strict inequality can hold in the relations $r_\sigma(A) \leq \|A\|$ and $r_\sigma(A) \leq \inf\limits_{k \in \mathbb{N}} \|A^k\|^{1/k}$.

16. Let X be a Banach space and $A \in \mathcal{B}(X)$. Then, prove that if $\mu \in \rho(A)$, then $\sigma\left((A - \mu I)^{-1}\right) = \{(\lambda - \mu)^{-1} : \lambda \in \sigma(A)\}$ and $r_\sigma\left((A - \mu I)^{-1}\right) = 1/\mathrm{dist}(\mu, \sigma(A))$.

17. Let X be a Banach space over $\mathbb{C}$, $A \in \mathcal{B}(X)$, and $\zeta \in \rho(A)$. Prove that $z \in \rho(A)$ and $R(z) = R(\zeta) \sum_{k=0}^{\infty} [R(\zeta)]^k (z - \zeta)^k$ if
    (i) $|z - \zeta| < 1/\|R(\zeta)\|$; (ii) $|z - \zeta| < \mathrm{dist}(\zeta, \sigma(A))$.

18. Let $X = C[0, 1]$ with $\|\cdot\|_\infty$ and $A : X \to X$ be the *Volterra operator* defined by $(Ax)(t) = \int_0^t x(s)ds$, $x \in C[0, 1]$. Prove that $\sigma_e(A) =$ and $\sigma(A) = \{0\} = \sigma_{app}(A)$.

19. Suppose A and B in $\mathcal{B}(X)$ such that AB is invertible in $\mathcal{B}(X)$ and $AB = BA$. Then show that both A and B are invertible in $\mathcal{B}(X)$.

20. Give an example of a normed linear space X and bijective operator A on a X such that $0 \in \sigma(A)$.

21. Let Ω be a compact subset of $\mathbb{K}$. Give an example of a Banach space X and an operator $A \in \mathcal{B}(X)$ such that $\sigma(A) = \Omega = \sigma_{\mathrm{app}}(A)$.

22. Let X be a Banach space. If $A \in \mathcal{B}(X)$ is invertible, and $B \in \mathcal{B}(X)$ is such that $\|A - B\| < 1/\|A^{-1}\|$, then show that B is invertible,

$$\|A^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A - B\| \, \|A^{-1}\|},$$

and

$$\|A^{-1} - B^{-1}\| \le \frac{\|A^{-1}\|^2 \|A - B\|}{1 - \|A - B\|\, \|A^{-1}\|}.$$

23. Let X be a Banach space and let $\mathcal{G}(X)$ be the set of all invertible operators in $\mathcal{B}(X)$. Show that $\mathcal{G}(X)$ is an open subset of $\mathcal{B}(X)$, and the map $A \mapsto A^{-1}$ is continuous on $\mathcal{G}(X)$.

24. Suppose X is a normed linear space (not necessarily a Banach space), and $A \in \mathcal{B}(X)$. If $\lambda \in \mathbb{K}$ is such that $|\lambda| > \|A\|$, then show that

    (a) $A - \lambda I$ is bounded below,

    (b) $R(A - \lambda I)$ dense in X, and

    (c) $(A - \lambda I)^{-1} : R(A - \lambda I) \to X$ is continuous.

25. Let X be a Banach space and $A \in \mathcal{B}(X)$. Then show that

    (a) $\lambda \in \sigma_{\mathrm{eig}}(A)$ if and only if there exists a nonzero $B \in \mathcal{B}(X)$ such that $(A - \lambda I)B = 0$,

    (b) $\lambda \in \sigma_{\mathrm{app}}(A)$ if and only if $A' - \lambda I$ is not surjective,

26. Let X be a Banach space and $A \in \mathcal{B}(X)$. Prove the following.

    (a) If $\lambda \in \rho(A)$, and if $B \in \mathcal{B}(X)$ is such that $\|A - B\| < 1/\|(A - \lambda I)^{-1}\|$, then $\lambda \in \rho(B)$.

    (b) If Ω is a compact subset of $\mathbb{K}$ contained in $\rho(A)$, then there exists $\varepsilon > 0$ such that $\Omega \subseteq \rho(B)$ for every $B \in \mathcal{B}(X)$ with $\|A - B\| < \varepsilon$.

27. Let X be a normed linear space and $A : X \to X$ be a compact operator. For a nonzero scalar λ and $k \in \mathbb{N}$, let $N_k := N\left((A - \lambda I)^k\right)$ and $R_k := R\left((A - \lambda I)^k\right)$. Prove the following.

    (a) There exist non-negative integers m and n such that

    $$N_m = N_{m+j}, \quad R_n = R_{n+j} \quad \forall j \in \mathbb{N}.$$

    (b) If $\ell := \min\{m : N_m = N_{m+j}\ \forall j \in \mathbb{N}\}$, $\kappa := \min\{n : R_n = R_{n+j}\ \forall j \in \mathbb{N}\}$, then $\ell = \kappa$, and $X = R((A - \lambda I)^r) \oplus N((A - \lambda I)^r)$, where $r = \kappa = \ell$.

28. Let X and Y be an inner product spacesand $A : X \to X$ be a linear operator. Show that there exists atmost one linear operator $B : Y \to X$ such that $\langle Ax, y\rangle = \langle x, By\rangle$ for all $x \in X$, $y \in Y$.

29. Suppose $\dim X < \infty$ and $A : X \to Y$ is a linear operator. If $\{u_1, \ldots, u_n\}$ is an orthonormal basis of X, then show that

    $$\|A\| \le \Big(\sum_{j=1}^{n} \|Au_j\|^2\Big)^{1/2}.$$

30. Let X and Y be $C[a,b]$ or $L^2[a,b]$ with $\|\cdot\|_2$. Find A^* in the following cases.

 (a) For $u \in C[a,b]$, $A : X \to X$ is defined by

$$(Ax)(t) = u(t)x(t), \quad x \in X,\ t \in [a,b].$$

 (b) For $k(\cdot,\cdot) \in C([a,b]\times[a,b])$, $A : X \to X$ is defined by

$$(Ax)(s) = \int_a^b k(s,t)x(t)d\mu(t), \quad x \in L^2[a,b];\ s \in [a,b].$$

31. Let $X = L^2[a,b]$. Find A^* if $A : X \to X$ is defined as in the following.

 (a) $Ax := \phi x,\ x \in X$ for some $\phi \in L^\infty[a,b]$.
 (b) $(Ax)(s) = \int_0^s x(t)dt$, $s \in [0,1]$, $x \in L^2[0,1]$.

32. Let X, Y be inner product spacesand $A : X \to Y$ be a linear operator for which A^* exists. Then prove that $A \in \mathcal{B}(X,Y)$ if and only if $A^* \in \mathcal{B}(Y,X)$, and in that case $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2$.

33. Let X and Y be Hilbert spaces with orthonormal bases $\mathcal{U} = \{u_1, u_2, \ldots\}$ and $\mathcal{V} = \{v_1, v_2, \ldots\}$ respectively. Let (a_{ij}) be an infinite matrix of scalars such that

$$Ax := \sum_i \Big(\sum_j a_{ij}\langle x, u_j\rangle\Big)v_i, \quad x \in X,$$

defines a bounded linear operator from X to Y. Find A^*.

34. Suppose X, Y are Hilbert spaces and $A : X \to Y$ is a linear operator. Then show the following:

 (a) If the adjoint A^* exists, then both A and A^* are continuous.
 (b) If $X = Y$ and A is symmetric, i.e., $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in X$, then $A^* = A \in \mathcal{B}(X)$. (*Hint:* Closed graph theorem.)

35. Give an example of a Hilbert space X and $A \in \mathcal{B}(X)$ such that $A^*A = I$, but $AA^* \neq I$. Can this happen if $\dim X < \infty$?

36. Find adjoints of the right shift operator and left operator on ℓ^2.

37. Suppose X, Y are Hilbert spaces and $A : X \to Y$ is a linear operator such that $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x \in X$, $y \in Y$. Show that if A is injective and $R(A)$ is closed, then A is bijective and both A and A^{-1} are continuous.

38. Let (P_n) be a sequence of orthogonal projections on a Hilbert space X and K be a compact operator on X. If $P_n x \to x$ as $n \to \infty$ for every $x \in X$, then show that $\|K(I - P_n)\| \to 0$ as $n \to \infty$.

39. Let X, Y be Hilbert spaces and $\psi : X \times Y \to \mathbb{K}$ be a bounded sesquilinear functional, i.e., $\psi : X \times Y \to \mathbb{K}$ satisfies the following conditions:

$$\begin{aligned}\psi(\alpha x + \beta y, u) &= \alpha\psi(x, u) + \beta\psi(y, u),\\ \psi(x, \alpha u + \beta v) &= \bar{\alpha}\psi(x, u) + \bar{\beta}\psi(x, v),\end{aligned}$$

for every $x, y \in X$ and $u, v \in Y$, and $\alpha, \beta \in \mathbb{K}$, and there exists $c > 0$ such that

$$|\psi(x, y)| \leq c\|x\|\,\|y\| \quad \forall\,(x, y) \in X \times Y.$$

Prove that there exists a unique bounded linear operator $A : X \to Y$ such that

$$\psi(x, y) = \langle Ax, y\rangle$$

for all $(x, y) \in X \times Y$.

40. Show that a sesquilinear functional $\psi : X \times X \to \mathbb{K}$ is bounded if and only if it is continuous.

41. Let X be a complex Hilbert space and $\psi : X \times X \to \mathbb{K}$ be a sesquilinear functional. Let $q : X \to \mathbb{C}$ be the associated *quadratic form*, i.e., $q(x) = \psi(x, x)$ for all $x \in X$. Verify the following:

(a) $4\,\psi(x, y) = q(x + y) - q(x - y) + iq(x + iy) - iq(x - iy)$.

(b) ψ is *symmetric*, i.e., $\psi(x, y) = \overline{\psi(y, x)}$, if and only if q is real-valued.

42. Find the adjoint of the unbounded operators in the following cases:

(a) Let $X = \ell^2$ and (λ_n) be a sequence of positive scalars such that $\lambda_n \to 0$ as $n \to \infty$. Let $X_0 = \left\{x \in \ell^2 : \sum_{j=1}^{\infty} \frac{|x(j)|^2}{\lambda_j^2} < \infty\right\}$ and $A : X_0 \to X$ be defined by

$$(Ax)(j) = \frac{x(j)}{\lambda_j}, \quad x \in X;\ j \in \mathbb{N}.$$

(b) Let $X = L^2[0, 1]$ and $X_0 = \{x \in C^1[0, 1] : x(0) = x(1)\}$. Let $A : X_0 \to X$ be defined by

$$Ax = x', \quad x \in X_0.$$

(c) Let $X = L^2[0, 1]$ be over $\mathbb{C}$, and let X_0 be as above. Let $A : X_0 \to X$ be defined by

$$Ax = i\,x', \quad x \in X_0.$$

43. Let X and Y be Hilbert spaces. Prove the following:

(a) If $A : X_0 \subseteq X \to Y$ is a densely defined operator, then the adjoint operator $A^* : Y_0 \subseteq Y \to X$ is a closed operator.

(b) If $A : X_0 \subseteq X \to Y$ is a closed densely defined operator, then its adjoint $A^* : Y_0 \subseteq Y \to X$ is a closed densely defined operator, and in that case $D(A^{**}) = D(A)$ and $A = A^{**}$. Here, A^{**} is the adjoint of A^*.

44. Let X be a separable Hilbert space and $\{u_j : j \in \mathbb{N}\}$ be an orthonormal basis of X. Let $\{\lambda_j : j \in \mathbb{N}\}$ be a bounded set of scalars, and let

$$Ax = \sum_j \lambda_j \langle x, u_j \rangle u_j, \quad x \in X.$$

Find conditions on (λ_n) such that A is a (i) normal operator, (ii) self-adjoint operator, (iii) unitary operator.

45. Let $X = L^2[a,b]$, $\phi \in L^\infty[a,b]$ and A be defined by

$$Ax = \phi\, x, \quad x \in X.$$

Find conditions on (λ_n) such that A is a (i) normal operator, (ii) self-adjoint operator, (iii) unitary operator.

46. Let X be a Hilbert space over $\mathbb{C}$ and $A \in \mathcal{B}(X)$. Then $A = 0$ if and only if $\langle Ax, x \rangle = 0$ for all $x \in X$. Justify.

47. A linear operator $A : X \to X$ is said to be a **positive operator**if $\langle Ax, x \rangle \geq 0$ for all $x \in X$.

    Suppose $A \in \mathcal{B}(X)$ is a positive operator. Then prove that,

    (a) if $\mathbb{K} = \mathbb{C}$, then A is a self-adjoint operator, and

    (b) if $\mathbb{K} = \mathbb{R}$, then A need not be self-adjoint,

48. Let $A \in \mathcal{B}(X)$ and M be a subspace of X. Show that M is invariant under A if and only if $M^\perp$ is invariant under A^*.

    [A subspace X_0 is said to be *invariant* under an operator A if $A(X_0) \subseteq X_0$.]

49. Let $A \in \mathcal{B}(X,Y)$. Show that $R(A)$ is dense in Y if and only if A^* is injective.

50. Show that for $A \in \mathcal{B}(X,Y)$, if $R(A) = Y$, then A^* is bounded below. [*Hint*: One may use the closed range theorem and the open mapping theorem.]

51. Let $P \in \mathcal{B}(X)$ be a projection operator. Show that the following are equivalent.

    (i) P is orthogonal, (ii) $P^* = P$, (iii) $\|P\| = 1$,

    and in that case, P is a positive operator.

52. Let $A \in \mathcal{B}(X)$ be a self-adjoint operator and $P \in \mathcal{B}(X)$ be an orthogonal projection. Show that $PA|_{R(P)} : R(P) \to R(P)$ is a self-adjoint operator on $R(P)$.

53. Let $A \in \mathcal{B}(X)$ be a self-adjoint operator. Show that $\|A^n\| = \|A\|^n$ for all $n \in \mathbb{N}$.

54. Let $A \in \mathcal{B}(X)$ be a normal operator. Show that $(A^*A)^n = (A^*)^n A^n$ for all $n \in \mathbb{N}$. Deduce that $\|A^n\| = \|A\|^n$ for all $n \in \mathbb{N}$.

    [*Hint*: Use the self-adjointness of A^*A and the previous problem.]

55. For $A \in \mathcal{B}(X)$, let $\exp(A) := \sum_{n=1}^\infty A^n/n!$. Show that, if A is a self-adjoint operator, then $\exp(iA)$ is a unitary operator.