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MA 6150: OPERATOR THEORY – ASSIGNMENTS II

1. Let $P \in \mathfrak{B}(X)$ be a projection operator, i.e., $P^2 = P$. Then prove that P is compact if and only if $\text{rank}(P) < \infty$.

2. Let (a_{ij}) be an infinite matrix of scalars. In each of the following, show that the operator $A : X \to Y$ is compact, where X and Y are sequence spaces, and $A(\alpha_i) = (\beta_i)$ with $\beta_i = \sum_j a_{ij}\alpha_j$:

   (a) $\alpha_j := \sum_{i=1}^{\infty} |a_{ij}| \to 0$ as $j \to \infty$, $\quad X = Y = \ell^1$.

   (b) $\beta_i := \sum_{j=1}^{\infty} |a_{ij}| \to 0$ as $i \to \infty$, $\quad X = Y = \ell^\infty$.

   (c) $\alpha_j \to 0$, $\beta_i \to 0$ as $i, j \to \infty$, $1 < p < \infty$, $\quad X = Y = \ell^p$.

   (d) $1 < p < \infty$, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |a_{ij}|^q < \infty$, $\quad X = Y = \ell^p$.

3. Let $C[a,b]$ be with $\|\cdot\|_\infty$, $k(\cdot,\cdot) \in C([a,b]\times[a,b])$ and $1 \le p, r \le \infty$. For $x \in L^1[a,b]$, let $(Ax)(s) = \int_a^b k(s,t)x(t)dt$. Prove that if X and Y are any of the spaces $C[a,b]$, $L^p[a,b]$, $L^r[a,b]$, then $A : X \to Y$ is a compact operator.

4. Let $1 < p < \infty$, $k(\cdot,\cdot) \in L^q([a,b]\times[a,b])$, i.e., $\int_a^b\int_a^b |k(s,t)|^q dm(s)dm(t) < \infty$. For $x \in L^p[a,b]$, let $(Ax)(s) = \int_a^b k(s,t)x(t)dt$. Prove that $L^p[a,b] \to L^p[a,b]$ is a compact operator.

5. Let X and Y be normed linear spaces and $A : X \to Y$ be an injective compact operator. Show that $A^{-1} : R(A) \to X$ is continuous iff A is of finite rank.

6. Is the set $\{(\alpha_1\alpha_2,\ldots) \in \ell^2 : \sum_{n=1}^{\infty} n^2|\alpha_n|^2 < \infty\}$ a closed subspace of ℓ^2 ? – Justify the answer.

7. State whether the operator in each of the following is compact or not. Justify your answer:

   (i) The diagonal operator on ℓ^p associated with the sequence $(\frac{n}{n+1})$.

   (ii) The operator $I + K$, where K is a compact operator.

8. Suppose $A : X \to Y$ is a compact operator of infinite rank.

   Show that, given any sequence (λ_n) of positive real numbers with $\lambda_n \to 0$, thee exists a sequence (x_n) in X such that $\|Ax_n\| \le \lambda_n$ and $\|x_n\| \ge 1/\lambda_n$ for all $n \in \mathbb{N}$.

9. Let X and Y Banach spaces and (A_n) be a sequence in $\mathfrak{B}(X,Y)$ such that there exists $A \in \mathfrak{B}(X,Y)$ satisfying $A_n x \to Ax$ as $n \to \infty$ for every $x \in X$. If $K \in \mathcal{K}(Y,X)$, then show that $\|(A - A_n)K\| \to 0$.

10. Let X be a normed linear space and $A : X \to X$ be a compact operator. For a nonzero scalar λ and $k \in \mathbb{N}$, let $N_k := N\left((A - \lambda I)^k\right)$ and $R_k := R\left((A - \lambda I)^k\right)$. Prove that there exist non-negative integers m and n such that

$$N_m = N_{m+j}, \quad R_n = R_{n+j} \quad \forall j \in \mathbb{N}.$$

Let $\ell := \min\{m : N_m = N_{m+j}\ \forall\, j \in \mathbb{N}\}$ and $\kappa := \min\{n : R_n = R_{n+j}\ \forall\, j \in \mathbb{N}\}$. Prove that $\ell = \kappa$ and

$$X = R\left((A - \lambda I)^r\right) \oplus N\left((A - \lambda I)^r\right), \quad r = \kappa = \ell.$$

11. Let X be any of the spaces $(c_{00}, \|\cdot\|_p)$, c_0, c, ℓ^p. Find $\sigma_{\rm eig}(A)$, $\sigma_{\rm app}(A)$, $\sigma(A)$, in the following cases, :

    (a) A is a diagonal operator, with diagonal entries $\lambda_1, \lambda_2, \ldots$, where (λ_n) is a bounded sequence of scalars, i.e., $(Ax)(j) = \lambda_j x(i)$, $j \in \mathbb{N}$, $x \in X$.

    (b) A is the right shift operator.

    (c) A is the left shift operator.

12. Find $\sigma_{\rm eig}(A)$, $\sigma_{\rm app}(A)$, $\sigma(A)$, where $A : C[a, b] \to C[a, b]$ is the multiplication operator, $(Ax)(t) := tx(t)$. Is it a compact operator?

13. Describe $\sigma_{\rm eig}(A)$, $\sigma_{\rm app}(A)$, $\sigma(A)$, if $A : C[a, b] \to C[a, b]$ is the multiplication operator, $(Ax)(t) := u(t)x(t)$ for some $u \in C[a, b]$.

14. Give an example of a normal operator A on ℓ^2 such that $\sigma_{\rm app}(A) = [0, 1]$. Justify your claim.

15. If $P : X \to X$ is a projection operator on a Banach space X, and if $0 \neq P \neq I$, then show that $\sigma(P) = \{0, 1\}$.

16. Let A be a bounded operator on a Banach space X. Show that if $\mu \in \rho(A)$, then $r_\sigma((A - \mu I)^{-1}) = 1/dist(\mu, \sigma(A))$.

17. If X is a Banach space and $A \in \mathfrak{B}(X)$ with $\sigma(A)$ is nonempty, then $\sigma_{\rm app}(A)$ is also non-empty – Why?

18. Give examples of X, $A \in \mathfrak{B}(X)$ such that (i) $\{\lambda^n : \lambda \in \sigma(A)\}$ is a proper subset of $\sigma(A^n)$ (ii) $r_\sigma(A) < \inf_n \|A^n\|^{1/n}$.

19. Let X be a Banach space over $\mathbb{C}$, $A \in \mathfrak{B}(X)$. For $z \in \rho(A)$, let $R(z) := (A - zI)^{-1}$. Prove the following:

    (a) $z \mapsto R(z)$ is continuous on $\rho(A)$.

    (b) For every $z_0 \in \rho(A)$, $\displaystyle\lim_{z \to z_0} \frac{R(z) - R(z_0)}{z - z_0}$ exists.

    (c) $z_0 \in \rho(A)$ and $z \in \mathbb{C}$ is such that $|z - z_0| < 1/\|R(z_0)\|$, then $z \in \rho(A)$ and

$$R(z) = R(z_0) \sum_{k=0}^{\infty} [R(z_0)]^k (z - z_0)^k.$$