Spectral Mapping Theorem for Self Adjoint Operators[1]

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THEOREM 1. *Let $A \in \mathcal{B}(X)$ be a self-adjoint operator on a Hilbert space X and $p(t)$ be a polynomial with coefficients in $\mathbb{F}$. Then,*

$$\sigma(p(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}.$$

Proof. By Theorem 10.14 in [1],

$$\{p(\lambda) : \lambda \in \sigma(A)\} \subseteq \sigma(p(A)),$$

and for $\mathbb{F} = \mathbb{C}$, we have the equality

$$\{p(\lambda) : \lambda \in \sigma(A)\} = \sigma(p(A)).$$

Hence, we need to prove

$$\sigma(p(A)) \subseteq \{p(\lambda) : \lambda \in \sigma(A)\}$$

for the case $\mathbb{F} = \mathbb{R}$. So, let $p(t)$ be a polynomial with real coefficients. The result is obvious if $p(t)$ is a constant polynomial. Hence, assume that $p(t)$ is not constant. Let $\mu \in \sigma(p(A))$. We observe that, since A is self adjoint and since coefficients of $p(t)$ are real numbers, $p(A)$ is also a self adjoint operator. Hence $\mu \in \mathbb{R}$. We consider two cases, namely,

Case(i): $p(t) - \mu$ does not have any non-real complex zeros, and

Case(i): $p(t) - \mu$ has at least one non-real complex zero.

Case(i): In this case, there are (not necessarily distinct) real numbers $c, t_1, \ldots, t_k$ such that

$$p(t) - \mu = c \prod_{j=1}^{k} (t - t_j).$$

Then

$$p(A) - \mu I = c \prod_{j=1}^{k} (A - t_j I).$$

Since $p(A) - \mu I$ is not invertible, there exists $\ell \in \{1, \ldots, k\}$ such that $A - t_\ell I$ is not invertible. Thus, $\lambda := t_\ell$ is a spectral value of A, and $p(\lambda) = \mu$.

Cae(ii): Suppose λ is a non-real complex zero of $p(t) - \mu$. Since coefficients of $p(t) - \mu$ are real numbers, $\bar{\lambda}$ is also a zero of $p(t) - \mu$. Hence, $(t - \lambda)(t - \bar{\lambda})$ is a factor of $p(t) - \mu$. Thus, $p(t) - \mu$ has

[1] A slight modified form of the proof given in [1], Theorem 12.12, Page 386.

the representation

$$p(t) - \mu = q(t) \prod_{j=1}^{m} (t - \lambda_j I)(t - \bar{\lambda}_j I),$$

where $\lambda_1, \ldots, \lambda_m$ are non-real (not necessarily distinct) complex numbers and $q(t)$ is a polynomial with real coefficients having no non-real zeros. Writing $\lambda_j = \alpha_j + i\beta_j$ with $\alpha_j, \beta_j \in \mathbb{R}$ and $\beta_j \neq 0$ and observing that

$$(t - \lambda_j)(t - \bar{\lambda}_j) = [(t - \alpha_j) - i\beta_j][(t - \alpha_j) + i\beta_j] = (t - \alpha_j)^2 + \beta_j^2,$$

we have

$$p(t) - \mu = q(t) \prod_{j=1}^{m} [(t - \alpha_j)^2 + \beta_j^2].$$

Hence,

$$p(A) - \mu I = q(A) \prod_{j=1}^{m} [(A - \alpha_j I)^2 + \beta_j^2 I].$$

Since each $(A - \alpha_j I)^2$ is a positive operator, $(A - \alpha_j I)^2 + \beta_j^2 I$ is invertible for every $j \in \{1, \ldots, m\}$. Hence, $q(A)$ is not invertible. Therefore, $q(t)$ is a non-constant polynomial and $q(t)$ is of the form

$$q(t) = c \prod_{j=1}^{k} (t - t_j),$$

for some real numbers $c, t_1, \ldots, t_k$. Consequently, we have

$$p(A) - \mu I = c \prod_{j=1}^{k} (A - t_j I) \prod_{j=1}^{m} [(A - \alpha_j I)^2 + \beta_j^2 I].$$

From this it follows, as in case (i), that there exists $\ell \in \{1, \ldots, k\}$ such that $A - t_\ell I$ is not invertible. Thus, $\lambda := t_\ell$ is a spectral value of A, and $p(\lambda) = \mu$. □

References

[1] M.T. Nair, *Functional Analysis: A First Course*, New Delhi: Printice-Hall of India, 2002 (Third Print, PHI Learning, 2010).