On Wronskian*

M.T.Nair

Department of Mathematics, IIT Madras

November 6, 2012

Abstract

We show that if φ and ψ are solution of a second order linear homogeneous ordinary differential equation with continuous coefficients, then its Wronskian $W(\varphi, \psi)$ is either identically zero or nonzero at every point.

Definition 1. Let φ and ψ be functions defined and differentiable in an interval I. Then **Wronskian** of φ and ψ is the function $W(\varphi, \psi) := \det \begin{bmatrix} \varphi & \psi \\ \varphi' & \psi \end{bmatrix}$ on I. $\diamondsuit$

Recall the following theorem from [1].

THEOREM 2. (Existence and uniqueness) *Consider the differential equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{$*$}$$

where p and q are continuous function on I. Then for every $(a, b) \in \mathbb{R}^2$ and $x_0 \in I$, there exists a unique function φ defined and differentiable on I such that

$$\varphi'' + p(x)\varphi' + q(x)\varphi = 0, \quad \varphi(x_0) = a, \quad \varphi'(x_0) = b.$$

THEOREM 3. (Wronskian property) *Let φ and ψ be any two solutions of the differential equation*

$$y'' + p(x)y' + q(x)y = 0,$$

where p and q are continuous function on I. Then one, and only one, of the following holds:

1. *For every $x \in I$, $W(\varphi, \psi)(x) = 0$.*

*November 6, 2012

2. For every $x \in I$, $W(\varphi, \psi)(x) \neq 0$.

Proof. Observe that one, and only one, of the following holds:

1. φ, ψ are linearly dependent.
2. φ, ψ are linearly independent.

Suppose case 1 occurs. Then there exists $\lambda \in \mathbb{R}$ such that $\psi = \lambda\varphi$. Clearly, for every $x \in I$, $W(\varphi, \psi)(x) = 0$.

Next, suppose case 2 occurs. Assume that there exists $x_0 \in I$ such that $W(\varphi, \psi)(x_0) = 0$. Then columns of the matrix $\begin{bmatrix} \varphi(x_0) & \psi(x_0) \\ \varphi'(x_0) & \psi(x_0) \end{bmatrix}$ are linearly dependent. Hence, there exists $\lambda \in \mathbb{R}$ such that

$$\psi(x_0) = \lambda\varphi(x_0), \quad \psi'(x_0) = \lambda\varphi'(x_0).$$

Let

$$a = \varphi(x_0), \quad b = \varphi'(x_0).$$

Now, by Theorem 2,

- φ is the unqiue solution of $(*)$ such that $\varphi(x_0) = a$, $\varphi'(x_0) = b$,
- ψ is the unqiue solution of $(*)$ such that $\psi(x_0) = \lambda a$, $\psi'(x_0) = \lambda b$,
- $\widetilde{\psi} := \psi/\lambda$ is the unqiue solution of $(*)$ such that $\widetilde{\psi}(x_0) = a$, $\widetilde{\psi}'(x_0) = b$,

Note that, in the above we use the fac that $\lambda \neq 0$. This is true, because, if $\lambda = 0$, then ψ is identically 0, which is not possible since φ and ψ are linearly independent functions.

Again by Theorem 2,

$$\widetilde{\psi}(x) = \varphi(x) \quad \forall\, x \in I,$$

i.e.,

$$\psi(x) = \lambda\varphi(x) \quad \forall\, x \in I.$$

This is a contradiction to the fact that φ and ψ are linearly independent functions. Thus, our assumption that that there exists $x_0 \in I$ such that $W(\varphi, \psi)(x_0) = 0$ is wrong. Hence, case 2 implies that for every $x \in I$, $W(\varphi, \psi)(x \neq 0$. □

Another proof for Theorem 3. Let $x_0 \in I$. By Theorem 2, there exists a unique solution φ for $(*)$ such that

$$\varphi(x_0) = 1, \quad \varphi'(x_0) = 0,$$

and there exists a unique solution ψ for $(*)$ such that

$$\psi(x_0) = 0, \quad \psi'(x_0) = 1.$$

Since columns of the matrix $\begin{bmatrix} \varphi(x_0) & \psi(x_0) \\ \varphi'(x_0) & \psi(x_0) \end{bmatrix}$ are linearly independent, the functions φ and ψ are linearly independent. Let

$$S := \{y \in C^1(I) : (*) \text{ holds } \}.$$

Note that S is a subspace of $C^1(I)$ and $\varphi, \psi \in S$. We claim that $\dim(S) = 2$. For this, let $f \in S$. Let

$$a = f(x_0), \quad b = f'(x_0).$$

Since

$$(a, b) = a(1, 0) + b(0, 1),$$

and since $g := a\varphi + \beta\psi$ is the unique solution of $(*)$ satisfying

$$g(x_0) = a, \quad g'(x_0) = b,$$

by Theorem 2, $g(x) = f(x)$ for all $x \in I$. Thus, we have shown that $\{\varphi, \psi\}$ is a basis of S.

Now, consider the function $T : S \to \mathbb{R}^2$ defined by

$$Tf = (f(x_0), f'(x_0), \quad f \in S.$$

Clearly, T is a linear transformation. Also, by Theorem 2, for every $(a, b) \in \mathbb{R}^2$, there exists a unique $f \in S$ such that

$$Tf = (a, b).$$

Hence, T is onto. Since $\dim(S) = \dim(\mathbb{R}^2) = 2$, T is one-one as well. Hence, for any two functions $f, g \in S$,

$$\{f, g\} \text{ linearly independent in } S \iff \{Tf, Tg\} \text{ linearly independent in } \mathbb{R}^2.$$

In other words

$$W(f, g)(x) = 0 \quad \forall x \in I \iff W(f, g)(x_0) = 0.$$

From this the proof follows. Indeed, for $\tau \in I$, one and only one of the following can hold:

- $W(f, g)(\tau) = 0$,
- $W(f, g)(\tau) \neq 0$.

Note that $W(f, g)(\tau) = 0$ implies, by taking $x_0 = \tau$, that $W(f, g)(x) = 0$ for all $x \in I$. Next suppose $W(f, g)(\tau) \neq 0$. If $W(f, g)(x_1) = 0$ for some x_1, then by taking $x_0 = x_1$ $W(f, g)(x) = 0$ for all $x \in I$; in particular, $W(f, g)(\tau) = 0$. Hence, we obtain $W(f, g)(\tau) \neq 0$ implies $W(f, g)(x) = 0$ for all $x \in I$. $\square$

References

[1] William E. Boycee and Richard C. DiPrima, (1986): *Elementary Differential Equations and Boundary Value Problems*, John Wiley and Sons.