4

Improper Integrals

In Chapter 3, we defined definite integral of a function f for the case when f is a *bounded function* defined on a *closed interval* $[a, b]$. can we still have a notion of integral even when the above assumptions on f and the domain of integration are not satisfied? We consider a notion of integral, called *improper integral*, in a few cases.

4.1 Definitions

4.1.1 Integrals over infinite intervals

First we consider integrals of functions defined over infinite integrals of the form $[a, \infty)$, $(-\infty, b]$ and $(-\infty, \infty)$. Recall that Rieman integral was defined over intervals of the form $[a, b]$.

Definition 4.1 (i) Suppose f is defined on $[a, \infty)$ and integrable on $[a, t]$ for all $t > a$. If $\lim_{t\to\infty} \int_a^t f(x)\,dx$ exists, then we define the **improper integral of f over** $[a, \infty)$ as

$$\int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

(ii) Suppose f is defined on $(-\infty, b]$ and integrable on $[t, b]$ for all $t < b$. If $\lim_{t\to-\infty} \int_t^b f(x)\,dx$ exists, then we define the **improper integral of f over** $(-\infty, b]$ as

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

(iii) Suppose f is defined on $\mathbb{R} := (-\infty, \infty)$ and integrable on $[a, b]$ for every closed and bounded interval $[a, b] \subseteq \mathbb{R}$. If $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ exist for some $c \in \mathbb{R}$, then we define the **improper integral of f over** $(-\infty, \infty)$ as

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

□

In Definition 4.1, the following results is used without mentioning:

- Suppose f is defined on $(-\infty,\infty)$ and integrable on $[a,b]$ for every closed and bounded interval $[a,b]\subseteq\mathbb{R}$. If the integrals $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ exist for some $c\in\mathbb{R}$, then they exist for every $c\in\mathbb{R}$, and

$$\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx.$$

  for every $a\in\mathbb{R}$.

***Exercise* 4.1** Prove the above result. ◀

Remark 4.1 We may observe that the existence of $\lim_{t\to\infty}\int_{-t}^{t} f(x)\,dx$ does not, in general, imply the existence of $\int_{-\infty}^{\infty} f(x)\,dx$. To see this, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=x, \quad x\in\mathbb{R}.$$

Then we have $\int_{-t}^{t} f(x)\,dx = 0$ for every $t\in\mathbb{R}$, but the integrals $\int_{-\infty}^{c} f(x)\,dx$ and $\int_{c}^{\infty} f(x)\,dx$ do not exist for any $c\in\mathbb{R}$. ♦

Next we consider integrals of functions defined over infinite integrals of the form (a,∞) and $(-\infty,b)$.

Definition 4.2 (i) Suppose f is defined on (a,∞) and $\int_t^\infty f(x)dx$ exists for all $t>a$. If $\lim\limits_{t\to a}\displaystyle\int_t^\infty f(x)\,dx$ exists, then we define the **improper integral of f over** (a,∞) as

$$\int_a^\infty f(x)\,dx := \lim_{t\to a}\int_t^\infty f(x)\,dx.$$

(ii) Suppose f is defined on $(-\infty,b)$ and $\int_{-\infty}^{t} f(x)dx$ exists for all $t<b$. If $\lim\limits_{t\to b}\displaystyle\int_{-\infty}^{t} f(x)\,dx$ exists, then we define the **improper integral of f over** $(-\infty,b)$ as

$$\int_a^\infty f(x)\,dx := \lim_{t\to b}\int_{-\infty}^{t} f(x)\,dx.$$

□

Remark 4.2 In the case of (a,∞), the function may not be defined at the point a or may be unbounded on $(a,a+\delta)$ for some $\delta>0$ so that we cannot talk about the Riemann integral over $[a,a+\delta]$ for $\delta>0$. Analogous remark holds for functions defined on $(-\infty,b)$. ♦

4.1.2 Improper integrals over finite intervals

Now we consider the case when f is defined on a interval J of finite length, but either the function is not defined at any one of the end points or the function is not bounded on J.

Definition 4.3 (i) Suppose f is defined on $(a, b]$. If $\int_t^b f(x)\, dx$ exists for every $t \in (a, b)$, and if $\lim_{t \to a} \int_t^b f(x)\, dx$ exists, then we define the **improper integral of** f **over** $(a, b]$ as

$$\int_a^b f(x)\, dx = \lim_{t \to a} \int_t^b f(x)\, dx.$$

(ii) Suppose f is defined on $[a, b)$. If $\int_a^t f(x)\, dx$ exists for every $t \in (a, b)$, and if $\lim_{t \to b} \int_a^t f(x)\, dx$ exists, then we define the **improper integral of** f **over** $[a, b)$ as

$$\int_a^b f(x)\, dx = \lim_{t \to b} \int_a^t f(x)\, dx.$$

(iii) Suppose f is defined on $[a, c)$ and $(c, b]$. If $\int_a^c f(x)\, dx$ and $\int_c^b f(x)\, dx$ exist, then we define the **improper integral of** f **over** $[a, b]$ as

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

□

Remark 4.3 In the case of improper integrals over $(a, b]$, the function may not be defined at the point a or may be unbounded on $(a, a + \delta)$ for some $\delta > 0$ so that we cannot talk about the Riemann integral over $[a, a+\delta]$ for $\delta > 0$. Analogous statement holds for case of improper integrals over $[a, b)$. In the case of improper integrals over $[a, b]$, the function may not be defined at the point c or may be unbounded on $[a, c)$ and $(c, b]$ for some $\delta > 0$ so that we cannot have the Riemann integral over $[a, b]$. ♦

Definition 4.4 If improper integral of a function f over an interval J (of finite or infinite length) exists, then we say that the the **improper integral exists** or **improper integral converges**; otherwise we say that the **improper integral does not exist** or **improper integral diverges**. □

4.1.3 Typical examples

Example 4.1 Consider the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$. Note that

$$\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t \to \infty \quad \text{as} \quad t \to \infty.$$

Hence, $\int_1^\infty \frac{1}{x}\,dx$ diverges. □

Example 4.2 Consider the improper integral $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ Note that

$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t} \to 1 \quad \text{as} \quad t \to \infty.$$

Hence, $\int_1^\infty \frac{1}{x^2}\,dx$ converges. □

Example 4.3 For $p \neq 1$, consider the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$. In this case, we have

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{t^{-p+1}-1}{-p+1}.$$

Note that,

$$p > 1 \quad \Longrightarrow \quad \frac{t^{-p+1}-1}{-p+1} \to \frac{1}{p-1} \quad \text{as} \quad t \to \infty,$$

and

$$p < 1 \quad \Longrightarrow \quad \frac{t^{-p+1}-1}{-p+1} \to \infty \quad \text{as} \quad t \to \infty,$$

The above observations combined with Example 4.1 show that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges for}\, p > 1, \\ \text{diverges for}\, p \leq 1. \end{cases}$$

□

Example 4.4 (i) We consider the improper integral $\displaystyle\int_0^1 \frac{1}{x}\,dx$: Note that for $0 < \delta < 1$,

$$\int_\delta^1 \frac{1}{x}\,dx = [\log x]_\delta^1 = \log 1 - \log\delta = -\log\delta = \log\left(\frac{1}{\delta}\right) \to \infty \quad \text{as} \quad \delta \to 0.$$

Thus, the integral diverges.

(ii) For $p \neq 1$, consider the improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. In this case, we have

$$\int_\delta^1 \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_\delta^1 = \frac{1-\delta^{-p+1}}{-p+1}.$$

Note that,

$$p > 1 \quad \Longrightarrow \quad \frac{\delta^{-p+1} - 1}{-p+1} \to \infty \quad \text{as} \quad \delta \to 0,$$

and

$$p < 1 \quad \Longrightarrow \quad \frac{\delta^{-p+1} - 1}{-p+1} \to \frac{1}{1-p} \quad \text{as} \quad \delta \to 0.$$

The above observations combined with (i) above give

$$\int_0^1 \frac{1}{x^p}\, dx \quad \begin{cases} \text{converges for}\, p < 1, \\ \text{diverges for}\, p \geq 1. \end{cases}$$

□

Example 4.5 Let $a < b$ and $\alpha < 1$. Then $\int_a^b \frac{dx}{(b-x)^\alpha}$ converges:

We observe that for $a < t < b$,

$$\int_a^t \frac{dx}{(b-x)^\alpha} = \int_{b-t}^{b-a} \frac{du}{u^\alpha}.$$

Now,

$$\begin{aligned} \lim_{t\to b} \int_a^t \frac{dx}{(b-x)^\alpha} \text{ exists} \quad & \Longleftrightarrow \quad \lim_{t\to b} \int_{b-t}^{b-a} \frac{du}{u^\alpha} \text{ exists} \\ & \Longleftrightarrow \quad \lim_{\varepsilon\to 0} \int_{\varepsilon}^{b-a} \frac{du}{u^\alpha} \text{ exists} \\ & \Longleftrightarrow \quad \alpha < 1. \end{aligned}$$

□

***Exercise* 4.2** Suppose $f \geq 0$ on $[a, b)$ and the integral $\int_a^t f(x)dx$ exists for every $t \in [a, b)$. If $\lim_{x\to b}(b-x)^\alpha f(x)$ converges for some $\alpha < 1$, then show that $\int_a^b f(x)dx$ also converges.

[*Hint:* Observe that for any $\varepsilon > 0$, there exists $x_0 \in [a, b)$ such that the number $\beta := \lim_{x\to b}(b-x)^\alpha f(x)$ satisfies $0 \leq f(x) \leq \frac{\beta+\varepsilon}{(b-x)^\alpha}$ for all $x \in [x_0, b)$.]

4.2 Integrability by Comparison

We state a result which will be useful in asserting the existence of certain improper integral by comparing it with certain other improper integral.

Suppose J is either an interval of finite or infinite length. Suppose f is defined on J, except possibly at a finite number of point in J. We denote the improper integral of f over J by

$$\int_J f(x)\, dx.$$

We say that the improper integral $\int_J f(x)\ dx$ converges whenever it exists, and diverges if it does not exist.

For example, if $J = [a, b]$, then f may not be defined at a or at b or at some point $c \in (a, b)$, and the corresponding improper integrals, by definition, are

$$\lim_{t\to a}\int_t^b f(x)\,dx, \quad \lim_{t\to b}\int_a^t f(x)\,dx, \quad \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx$$

respectively.

Theorem 4.1 *Suppose f and g are defined on J.*

(i) If $0 \le f(x) \le g(x)$ for all $x \in J$, and $\int_J g(x)\ dx$ exists, then $\int_J f(x)\ dx$ exists.

(ii) If $\int_J |f(x)|\ dx$ exists, then $\int_J f(x)\ dx$ exists.

Example 4.6 Since

$$\left|\frac{\sin x}{x^p}\right| \le \frac{1}{x^p}, \qquad \left|\frac{\cos x}{x^p}\right| \le \frac{1}{x^p}$$

it follows from Example 4.3 and Theorem 4.1(ii) that the improper integrals

$$\int_1^\infty \frac{\sin x}{x^p}\ dx \quad \text{and} \quad \int_1^\infty \frac{\cos x}{x^p}\ dx$$

converge for all $p > 1$. □

In fact $\int_1^\infty \frac{\sin x}{x^p}\ dx$ and $\int_1^\infty \frac{\cos x}{x^p}\ dx$ converge for all $p > 0$ as we see in the next example.

Example 4.7 Let $p > 0$. Then for $t > 0$,

$$\begin{aligned}\int_1^t \frac{\sin x}{x^p}\ dx &= \left[\frac{1}{x^p}(-\cos x)\right]_1^t - p\int_1^t \frac{1}{x^{p+1}}\cos x\ dx\\ &= \left[\cos 1 - \frac{\cos t}{t^p}\right] - p\int_1^t \frac{\cos x}{x^{p+1}}\ dx.\end{aligned}$$

By the result in Example 4.6, $\int_1^\infty \frac{\cos x}{x^{p+1}}\ dx$ converges for all $p > 0$. Also, $\frac{\cos t}{t^p} \to 0$ as $t \to \infty$. Hence,

$$\int_1^\infty \frac{\sin x}{x^p}\ dx \quad \text{converges for all} \quad p > 0.$$

Similarly, we see that

$$\int_1^\infty \frac{\cos x}{x^p}\ dx \quad \text{converges for all} \quad p > 0.$$

□

Example 4.8 Since

$$\left|\frac{\sin x}{x^p}\right| = \left|\frac{\sin x}{x}\right| \frac{1}{x^{p-1}} \le \frac{1}{x^{p-1}}, \qquad \left|\frac{\cos x}{x^p}\right| \le \frac{1}{x^p}$$

it follows from Example 4.4 above and Theorem 4.1(ii)that

$$\int_0^1 \frac{\sin x}{x^p}\,dx \quad \text{converges for all} \quad p < 2,$$

$$\int_0^1 \frac{\cos x}{x^p}\,dx \quad \text{converges for all} \quad p < 1.$$

□

Example 4.9 Observe that

$$\frac{\sin x}{x^p} = \frac{\sin x}{x}\frac{1}{x^{p-1}} \ge \frac{\sin 1}{x^{p-1}} \quad \forall x \in (0,1].$$

Since $\int_0^1 \frac{1}{x^{p-1}}\,dx$ diverges for $p - 1 \ge 1$, i.e., for $p \ge 2$, it follows that

$$\int_0^1 \frac{\sin x}{x^p}\,dx \quad \text{diverges for all} \quad p \ge 2,$$

□

Example 4.10 From Examples 4.8, 4.9, 4.7,

$$\int_0^\infty \frac{\sin x}{x^p}\,dx \quad \text{converges for} \quad 0 < p < 2.$$

□

4.3 Integrability Using Limits

Now some more results which facilitate the assertion of convergence/divergence of improper integrals, whose proofs follow from the definition of limits.

Theorem 4.2 *Suppose $f(x) \ge 0$, $g(x) \ge 0$ for all $x \in [a,\infty)$, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exists for every $b > a$. Suppose further that $\frac{f(x)}{g(x)} \to \ell$ as $x \to \infty$ for some $\ell \ge 0$.*

(i) *If $\ell \ne 0$, then $\int_a^\infty f(x)dx$ converges $\iff$ $\int_a^\infty g(x)dx$ converges.*

(ii) *If $\ell = 0$, then $\int_a^\infty g(x)dx$ converges $\Longrightarrow$ $\int_a^\infty f(x)dx$ converges.*

Further, if $\frac{f(x)}{g(x)} \to \infty$ as $x \to \infty$, then

$$\int_a^\infty g(x)dx \;\; \textit{converges} \;\; \Longrightarrow \int_a^\infty f(x)dx \;\; \textit{converges.}$$

Proof. Suppose further that $\dfrac{f(x)}{g(x)} \to \ell$ as $x \to \infty$ for some $\ell \geq 0$.

(i) Suppose $\ell \neq 0$. Then $\ell > 0$, and for $\varepsilon > 0$ with $\ell - \varepsilon > 0$, there exists $x_0 \geq a$ such that

$$\ell - \varepsilon < \frac{f(x)}{g(x)} < \ell + \varepsilon \quad \forall x \geq x_0.$$

Hence

$$(\ell - \varepsilon)g(x) < f(x) < (\ell + \varepsilon)g(x) \quad \forall x \geq x_0.$$

Consequently, $\int_{x_0}^{\infty} f(x)dx$ converges iff $\int_{x_0}^{\infty} g(x)dx$ converges. As $\int_a^{x_0} f(x)dx$ and $\int_a^{x_0} g(x)dx$ exist, the result in (i) follows.

(ii) Suppose $\ell = 0$. Then for $\varepsilon > 0$, there exists $x_0 \geq a$ such that

$$\frac{f(x)}{g(x)} < \varepsilon \quad \forall x \geq x_0.$$

Thus, $f(x) < \varepsilon g(x)$ for all $x \geq x_0$. Hence, convergence of $\int_{x_0}^{\infty} g(x)dx$ implies the convergence of $\int_{x_0}^{\infty} f(x)dx$. From this the result in (ii) follows.

Next, suppose further that $\dfrac{f(x)}{g(x)} \to \infty$ as $x \to \infty$. Then for $M > 0$, there exists $x_0 \geq a$ such that

$$0 \leq \frac{f(x)}{g(x)} \leq M \quad \forall x \geq x_0.$$

Hence

$$0 \leq f(x) \leq Mg(x) \quad \forall x \geq x_0.$$

Consequently, $\int_{x_0}^{\infty} g(x)dx$ converges implies $\int_{x_0}^{\infty} f(x)dx$ converges. As $\int_a^{x_0} f(x)dx$ and $\int_a^{x_0} g(x)dx$ exist, the proof is over. ▮

***Exercise* 4.3** Suppose f and g are non-negative continuous functions on J. Then

$$\int_a^b f(x)dx \text{ exists} \iff \int_a^b g(x)dx \text{ exists}$$

in the following cases:

1. $J = (a, b]$ and $\displaystyle\lim_{x \to a} \frac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
2. $J = [a, b)$ and $\displaystyle\lim_{x \to b} \frac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
3. $J = [a, \infty)$ and $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
4. $J = (-\infty, b]$ and $\displaystyle\lim_{x \to -\infty} \frac{f(x)}{g(x)} = \ell$ and $\ell > 0$.

In 1-4 above, if $\ell = 0$, then $\displaystyle\int_a^b g(x)dx$ exists $\Longrightarrow \displaystyle\int_a^b g(x)dx$ exists . ◀

4.4 Gamma and Beta Functions

Gamma and Beta Functions are certain improper integrals which appear in many applications.

Gamma function

We show that for $x > 0$, the improper integral

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t}\, dt$$

converges. The function $\Gamma(x)$, $x > 0$, is called the **gamma function**.

Note that for $t^{x-1}e^{-t} \leq t^{x-1}$ for all $t > 0$, and $\int_0^1 t^{x-1}\, dt$ converges for $x > 0$. Hence, by Theorem 4.1,

$$\int_0^1 t^{x-1} e^{-t}\, dt \quad \text{converges for} \quad x > 0.$$

Also, we observe that $\dfrac{t^{x-1}e^{-t}}{t^{-2}} \to 0$ as $t \to \infty$, and $\int_1^\infty t^{-2} dt$ converges. Hence, by Theorem 4.2, $\int_1^\infty t^{x-1}e^{-t}\, dt$ converges. Thus,

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt = \int_0^1 t^{x-1}e^{-t}\, dt + \int_1^\infty t^{x-1}e^{-t}\, dt$$

converges for every $x > 0$.

Beta function

We show that for $x > 0$, $y > 0$, the improper integral

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

converges. The function $\beta(x, y)$ for $x > 0, y > 0$ is called the **beta function**.

Clearly, the above integral is *proper* for $x \geq 1$, $y \geq 1$. Hence it is enough to consider the case of $0 < x < 1$, $0 < y < 1$. In this case both the points $t = 0$ and $t = 1$ are problematic. hence, we consider the integrals

$$\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt, \qquad \int_{1/2}^1 t^{x-1}(1-t)^{y-1}\, dt.$$

We note that if $0 < t \leq 1/2$, then $(1-t)^{y-1} \leq 2^{1-y}$ so that $t^{x-1}(1-t)^{y-1} \leq 2^{1-y}t^{x-1}$. Since $\int_0^{1/2} t^{x-1}\, dt$ converges it follows that $\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt$ converges. To deal with the second integral, consider the change of variable $u = 1 - t$. Then

$$\int_{1/2}^1 t^{x-1}(1-t)^{y-1}\, dt = \int_0^{1/2} u^{y-1}(1-u)^{x-1}\, du$$

which converges by the above argument. Hence,

$$\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{1-y}\,dt, \qquad x>0,\, y>0$$

converges for every $x>0$, $y>0$.

4.5 Additional Exercises

1. Does $\displaystyle\int_1^\infty \sin\left(\frac{1}{x^2}\right) dx$ converge?

   [*Hint*: Note that $\left|\sin\left(\frac{1}{x^2}\right)\right| \le \frac{1}{x^2}$.]

2. Does $\displaystyle\int_2^\infty \frac{\cos x}{x(\log x)^2} dx$ converge?

   [*Hint*: Observe $\left|\frac{\cos x}{x(\log x)^2}\right| \le \frac{1}{x(\log x)^2}$ and use the change of variable $t = \log x$.]

3. Does $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2} dx$ converge?

   [*Hint*: Observe $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ for $x \ge 1$ and $\frac{\sin^2 x}{x^2}$, $0 < x \le 1$ has a continuous extension on $[0,1]$.]

4. Does $\displaystyle\int_0^1 \frac{\sin x}{x^2} dx$ converge?

   [*Hint*: Observe $\frac{\sin x}{x^2} = \left(\frac{\sin x}{x}\right)\frac{1}{x} \ge \left(\frac{\sin 1}{1}\right)$.]

5. Does $\displaystyle\int_{a_0}^\infty f(x)dx$ exists implies $\displaystyle\int_a^b f(x)dx \to 0$ as $a, b \to \infty$.

   [*Hint*: Note that $\int_a^b f(x)dx = \int_{a_0}^b f(x)dx - \int_{a_0}^a f(x)dx \to 0$ as $a, b \to \infty$.]

6. Does $\displaystyle\int_0^\infty e^{-x^2}\, dx$ converge?

   [*Hint*: Note that e^{-x^2} is continuous on $[0,1]$, and $e^{-x^2} \le \frac{1}{x^2}$ for $1 \le x \le \infty$.]

7. Does $\displaystyle\int_2^\infty \frac{\sin(\log x)}{x} dx$ converge?

   [*Hint*: Use the change of variable $t = \log x$, and the fact that $\int_{\log 2}^\infty \sin t\, dt$ diverges.]

8. Does $\displaystyle\int_0^1 \ln x dx$ converge?

   [*Hint*: Use the change of variable $t = \log x$.]