MA 2030

Linear Algebra and Numerical Analysis

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Information

Lecture notes are available at Moodle site.
It will be complete as far as material is concerned.
But you have to work out the details.
Assignments will also be available in the same page.

Come to the class in time so that class is not disturbed.
There will be problem solving sessions.
Tutors will help us.

AS & SM 1

Mark yourself present in the attendance sheet.
Do not mark others present even if they are in the class.
We assume you are honest.

Plagiarism is an offence.
Trust in yourself, you are capable.
You are here to learn and get credits for it.
The sessions will be interactive. Be prepared for it.
You must learn by doing mathematics and not just listening.

Batch Division

1. Teacher: Prof P Veeramani, Room: CRC 302
   Students: AE, BT, CE, CH, CS, EP
2. Teacher: A V Jayanthan and A Singh, Room: CRC 304
   Students: EE11
3. Teacher: Shruti Dubey, Room: CRC 305
   Students: All ME11 from ME11B043 onwards
4. Teacher: V Uma, Room: CRC 303
   Students: EE08, EE09, EE10, ME10,
   ME11 upto ME11B042, MM, NA, PH

AVJ & AS

Class Hours:
Slot-A
Monday 8-8:50
Thursday 11-11:50
Friday 10-10:50

Venue: CRC-304
Students: All EE11 students: EE11B001-132

Examination: Quiz-1, Quiz-2, and EndSem, as usual.

Please remember your Teachers' name: AVJ & AS

Vectors?

Consider vectors in the plane. Translate each to start from the origin. Identify a vector with the point on its arrow head. Is this identification truthful?

For example, let u be the vector starting at $(0,0)$ and ending at $(2,1)$. Let v be the vector starting at $(0,0)$ and ending at $(3,3)$. What is the vector $u+v$?

If $u+v$ starts at $(0,0)$ where does it end?

Well, $u = 2\hat{\imath} + \hat{\jmath},\ v = 3\hat{\imath} + 3\hat{\jmath}$.
So, $u+v = 5\hat{\imath} + 4\hat{\jmath}$.

Thus, $u+v$ ends at $(5,4)$.

Let $\mathbb{R}$ denote the set of all real numbers. Can you identify the set of all plane vectors with $\mathbb{R}^2$?

Properties of Vectors

The notion of a vector space is an abstraction of the familiar set of vectors in two or three dimensional Euclidean space.

We first recall certain 'good' properties of vectors in the real plane:
Think about the plane vectors and $\mathbb{R}^2$.

There exists a vector, namely 0, such that for all $x \in \mathbb{R}^2, \; x + 0 = x = 0 + x$.

For every $x \in \mathbb{R}^2$, there exists another vector, denoted by $-x$, such that $x + (-x) = 0 = (-x) + x$.

'Addition' distributes over 'Multiplication'.

Both addition and multiplication are 'associative'.

For all $x \in \mathbb{R}^2, \; 1 \cdot x = x$.

NOTATION

We will always use $x, \ y, \ z, \ u, \ v, \ w$ for vectors.

We will use the Greek letters $\alpha, \beta, \gamma, \ldots$ and $a, \ b, \ c$ for scalars

The symbol 0 will stand for the 'zero vector' as well as 'zero scalar'. From the context, you should know which one it represents.

$\mathbb{R}$ denotes the set of all real numbers and $\mathbb{C}$ denotes the set of all complex numbers.

$\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Whenever needed, we will specifically mention the set of scalars.

Definition

A non-empty set V with two operations $+$ (addition) and $\cdot$ (scalar multiplication) is said to be a vector space over $\mathbb{F}$ if it satisfies the following axioms:

(1) $x + y = y + x, \;\; \forall\, x, y \in V$.

(2) $(x + y) + z = x + (y + z), \;\; \forall\, x, y, z \in V$.

(3) $\exists\;\; 0 \in V$ such that $x + 0 = x \;\; \forall\, x \in V$.

(4) for each $x \in V$, $\exists\, (-x) \in V$ such that $x + (-x) = 0$.

(5) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \;\forall\, \alpha \in \mathbb{F}$ and $\forall\, x, y \in V$.

(6) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x, \; \forall\, \alpha, \beta \in \mathbb{F}$ and $\forall\, x \in V$.

(7) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x) \;\forall\, \alpha, \beta \in \mathbb{F}, \forall\, x \in V$.

(8) $1 \cdot x = x \;\forall\, x \in V$.

Examples of vector spaces

1. $V = \{0\}$ is a vector space over $\mathbb{F}$.

2. $\mathbb{R}^2 = \{(a, b) \ : \ a, \ b \in \mathbb{R}\}$ with
$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $c(x_1, x_2) = (cx_1, cx_2)$
is a vector space over $\mathbb{R}$.

We also write columns instead of rows for the elements of $\mathbb{R}^2$, i.e.,
$\mathbb{R}^2 := \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ a, b \in \mathbb{R} \right\}$

3. $\mathbb{R}^n = \{(x_1, \cdots, x_n) \ : \ x_1, \ldots, x_n \in \mathbb{R}\}$ with the usual addition and scalar multiplication is a vector space over $\mathbb{R}$.

Examples

A vector in $\mathbb{R}^n$ is also written as a column vector, i.e., in the form $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

4. $V = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; x_2 = 0\}$ is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication.

5. $V = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; 2x_1 - x_2 = 0\}$ is a vector space over $\mathbb{R}$ under the usual addition and scalar multiplication.

6. Is $V = \{(x_1, x_2) \in \mathbb{R}^2 \; : \; 3x_1 + 5x_2 = 1\}$ a vector space over $\mathbb{R}$?

Examples Contd.

7. $\mathcal{P}_n := \{a_0 + a_1 t + \cdots + a_n t^n \ : \ a_i \in \mathbb{F}\}$ with the usual polynomial addition and scalar multiplication is a vector space over $\mathbb{F}$.

8. The set $M_{m \times n}(\mathbb{F})$ of all $m \times n$ matrices with entries from $\mathbb{F}$ with the usual matrix addition and scalar multiplication is a vector space over $\mathbb{F}$.

9. $V = \mathbb{R}^2$, for $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in R$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$,
$\alpha(a_1, a_2) = (0, 0)$ if $\alpha = 0$ and
$\alpha(a_1, a_2) = (\alpha a_1, a_2/\alpha)$ if $\alpha \neq 0$.
Is V a vector space over $\mathbb{R}$?

Examples of vector spaces

9. $V = \{f : [a,\ b] \to \mathbb{R}\ :\ f \text{ is a function}\}$.

For $f, g \in V$, define $f + g$ to be the map
$(f+g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$.

For $\alpha \in \mathbb{R}$ and $f \in V$, define αf to be the map $(\alpha f)(x) = \alpha f(x)$ for all $x \in \mathbb{R}$.

What is the 'zero vector' in V?

The map f such that $f(x) = 0$ for all $x \in [a,\ b]$.

For $f \in V$, $-f$, defined by $(-f)(x) = -f(x)$ is the additive inverse.

V is a vector space over $\mathbb{R}$.

Examples Contd.

10. $V = \left\{ f : \mathbb{R} \to \mathbb{R} \ : \ \frac{d^2 f}{dx^2} + f = 0 \right\}$. Define addition and scalar multiplication as in the previous example.

For $f, g \in V$, $\frac{d^2(f+g)}{dx^2} + (f+g) = (\frac{d^2 f}{dx^2} + f) + (\frac{d^2 g}{dx^2} + g) = 0$

Similarly $\frac{d^2(\alpha f)}{dx^2} + (\alpha f) = \alpha \left[\frac{d^2 f}{dx^2} + f \right] = 0$.

$\Rightarrow$ V is closed under addition and scalar multiplication. Other properties can easily be verified. Hence V is a vector space over $\mathbb{R}$.

Basic properties of vector spaces

Let V be a vector space over $\mathbb{F}$. Let $\alpha \in \mathbb{F}$, $x, y, z \in V$.

1. The zero element is unique, i.e., if there exists θ_1, θ_2 such that $x + \theta_1 = x$ and $x + \theta_2 = x$, $\forall\, x \in V$, then $\theta_1 = \theta_2$.
   Proof. $\theta_1 = \theta_1 + \theta_2 = \theta_2$.
2. Additive inverse for each vector is unique, i.e., for $x \in V$, if there exist $\tilde{x}_1$ & $\tilde{x}_2$ such that $x + \tilde{x}_1 = 0 = x + \tilde{x}_2$, then $\tilde{x}_1 = \tilde{x}_2$.
   Proof. $\tilde{x}_1 = 0 + \tilde{x}_1 = (\tilde{x}_2 + x) + \tilde{x}_1 = \tilde{x}_2 + (x + \tilde{x}_1)$
   $= \tilde{x}_2 + 0 = \tilde{x}_2$.

   We write the additive inverse of x as $-x$ and abbreviate $x + (-y)$ to $x - y$.

Basic Properties Contd.

3. If $x + z = y + z$, then $x = y$.
   Proof. $x = x + 0 = x + z - z = y + z - z = y + 0 = y$.
   Thus, if $z + x = z + y$, then $x = y$.
4. $0 \cdot x = 0$.
   Proof: $0 + 0 \cdot x = 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$
   $\Rightarrow 0 = 0 \cdot x$.
5. $(-1) \cdot x = -x$.
   Proof. $x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x$
   $= (1 + (-1)) \cdot x = 0 \cdot x = 0 = x + (-x)$
   $\Rightarrow (-1) \cdot x = -x$.
6. $\alpha \cdot 0 = 0$.
   Proof. $\alpha \cdot 0 + 0 = \alpha \cdot 0 = \alpha \cdot (0 + 0) = \alpha \cdot 0 + \alpha \cdot 0$
   $\Rightarrow \alpha \cdot 0 = 0$.

Vector Subspaces

Subspace of a vector space is a subset which follow the 'same structure'.

We have seen that: $W = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ 2x_1 + x_2 = 0\}$ is a vector space.

$\mathcal{P}_3$ is a vector space and is a subset of the vector space $\mathcal{P}_4$

Definition: Let W be a subset of a vector space V. Then W is called a subspace of V if W is a vector space with respect to the operations of addition and scalar multiplication as in V.

Examples of subspaces

1. $\{0\} \subseteq V$ is a subspace for any vector space V.

2. $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 + x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.

More generally, if A is an $m \times n$ matrix and $x = (x_1, \cdots, x_n)$, then the set of all solutions of the equation $Ax = 0$ is a subspace of $\mathbb{R}^n$. **(Prove it!)**

3. $\mathcal{P}_n$ is a subspace of $\mathcal{P}_m$ for $n \leq m$.

4. $C[a,\ b] := \{f : [a,\ b] \to \mathbb{R} \ : \ f$ is a continuous function$\}$ is a vector subspace of $F[a,\ b] := \{f : [a,\ b] \to \mathbb{R} \ : \ f$ is a function$\}$.

5. $\mathcal{R}[a,\ b] := \{f : [a,\ b] \to \mathbb{R} \ : \ f$ is integrable $\}$ is a vector subspace of $C[a,\ b]$.

6. $C^k[a,\ b] := \{f : [a,\ b] \to \mathbb{R} \ : \ \frac{d^k f}{dx^k}$ exists $\}$ is a vector subspace of $C[a,\ b]$.

Any Shortcut?

7. $\mathcal{P}_n$ can also be seen as a subspace of $C[a,\ b]$.

Do we have to verify all 8 conditions to check whether a given subset of a vector space is a subspace?

Theorem: Let W be a non-empty subset of a vector space V. Then W is a subspace of V if and only if W is non-empty and $x + \alpha y \in W$ for all $x,\ y \in W$ and $\alpha, \in \mathbb{F}$.

Proof: If W is a subspace, then obviously the given condition is satisfied.

Conversely suppose W is a subset which satisfies the given condition.

The commutativity and associativity of addition, distributive properties and scalar multiplication with 1 are satisfied in V and hence true in W too.

Proof Contd.

Therefore, we only need to verify the existence of ‘zero vector’ and ‘additive inverse’.

$W \neq \emptyset \ \Rightarrow \ \exists\, x \in W \ \Rightarrow \ x + (-1)x = 0 \in W$. Hence the additive identity exists.

Now for $y \in W$, take $x = 0$ and $\alpha = -1$ in the given condition. We get $0 + (-1)y = -y \in W$. Hence additive inverse exists.

Therefore, W is a subspace of V. $\square$

Given two vector subspaces, what are the other possibilities of obtaining new vector subspaces from them?

Intersection of Subspaces

Take two planes passing through the origin. Is it a subspace of $\mathbb{R}^3$?

Theorem: Let V_1 and V_2 are subspace of a vector space V. Then $V_1 \cap V_2$ is a subspace of V.

Proof: V_1 & V_2 are subspaces $\Rightarrow$ $0 \in V_1 \cap V_2$. Therefore $V_1 \cap V_2 \neq \emptyset$.

Suppose $x,\ y \in V_1 \cap V_2$ and $\alpha \in \mathbb{F}$, then $x + \alpha y$ belong to both V_1 and V_2 (since they are subspaces) and hence they belong to $V_1 \cap V_2$

$\Rightarrow$ $V_1 \cap V_2$ is a subspace. $\square$

If V_1 and V_2 are subspaces, then is $V_1 \cup V_2$ a subspace?

Is the union of x-axis and y-axis a subspace of $\mathbb{R}^2$?

Is Union a Subspace?

Theorem: Let V_1 and V_2 be subspaces of a vector space. Then $V_1 \cup V_2$ is a subspace if and only if either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Proof: If $V_1 \subseteq V_2$ ($V_2 \subseteq V_1$), then $V_1 \cup V_2 = V_2$ (V_1). Then it is a subspace.

Conversely, assume that $V_1 \cup V_2$ is a subspace and $V_1 \not\subseteq V_2$. We want to show that $V_2 \subseteq V_1$.

Let $x \in V_2$. Since $V_1 \not\subseteq V_2$, $\exists\, y \in V_1 \setminus V_2$.

$\Rightarrow x, y \in V_1 \cup V_2, \Rightarrow \; x + y \in V_1 \cup V_2$, since $V_1 \cup V_2$ is a subspace.

If $x + y \in V_2$, then $y \in V_2$, which is a contradiction.

$\Rightarrow \; x + y \in V_1 \;\Rightarrow\; x \in V_1 \;\Rightarrow V_2 \subseteq V_1$. $\square$

More Subspace Examples

Check if V_0 is a subspace of V:

1. Let $V = C[-1,\ 1]$ and $V_0 = \{f \in V\ :\ f \text{ is an odd function }\}$.

Solution: The zero vector belongs to $V_0 \ \Rightarrow\ V_0 \neq \emptyset$.
If $f, g \in V_0$ and $\alpha \in \mathbb{R}$, then $(f + \alpha g)(-x) = f(-x) + \alpha g(-x) = -f(x) + \alpha(-g(x)) = -(f + \alpha g)(x) \ \Rightarrow f + \alpha g$ is an odd function.

2. Let $V = \mathcal{P}_3$ and $V_0 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3\ :\ a_0 = 0\}$.

3. Let $V = \mathcal{P}_3$ and
$V_0 = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3\ :\ a_1 + a_2 + a_3 + a_4 = 0\}$.

Examples Contd.

4. Let V be a vector space and $v \in V$. Let $V_0 = \{\alpha v \ : \ \alpha \in \mathbb{F}\}$. Then V_0 is a vector space.

More generally, if $v_1, \ldots, v_n \in V$ and
$V_0 = \{\alpha_1 v_1 + \cdots + \alpha_n v_n \ : \ \alpha_i \in \mathbb{F} \, \forall \, i = 1, \ldots, n\}$,
then V_0 is a subspace of V.

Exercise: Non-zero subspace of $\mathbb{R}^2$ are the straight lines passing through the origin.

Assignment-1

1. Let V be a vector space over $\mathbb{F}$. Show the following:
   (a) For all $x, y, z \in V$, $x + y = z + y$ implies $x = z$.
   (b) For all $\alpha, \beta \in \mathbb{F}, x \in V, x \neq 0$, $\alpha x \neq \beta x$ iff $\alpha \neq \beta$.
   (c) V must have an infinite number of elements.
2. In each of the following a non-empty set V is given and some operations are defined. Check whether V is a vector space with these operations.
   (a) $V = \{(x_1, 0) : x_1 \in \mathbb{R}\}$ with addition and scalar multiplication as in $\mathbb{R}^3$.
   (b) $V = \{(x_1, x_2) \in \mathbb{R}^2 : 2x_1 + 3x_2 = 0\}$ with addition and scalar multiplication as in $\mathbb{R}^3$.
   (c) $V = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1\}$ with addition and scalar multiplication as in $\mathbb{R}^3$.
   (d) $V = \mathbb{R}^2$, for $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in R$, define $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$; $0\,(a_1, a_2) = (0, 0)$ and for $\alpha \neq 0, \alpha(a_1, a_2) = (\alpha a_1, a_2/\alpha)$.

Assignment-1

2.

   (e) $V = \mathbb{C}^2$, for $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in \mathbb{C}$, define
   $(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$,
   $\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$.

   (f) $V = \mathbb{R}^2$, for $(a_1, a_2), (b_1, b_2) \in V$ and $\alpha \in \mathbb{R}$ define
   $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$, $\alpha(a_1, a_2) = (a_1, 0)$.

   (g) $V = [0, \infty)$. For $x, y \in V,\ \alpha \in \mathbb{R}$, define
   $x + y = xy,\ \alpha x = |\alpha| x$.

   (h) $V = [0, \infty)$. For $x, y \in V,\ \alpha \in \mathbb{R}$, define $x + y = xy$ and
   $\alpha x = x^{\alpha}$.

3. Is $\mathcal{R}[a, b]$, the set of all real valued Riemann integrable functions on $[a, b]$, a vector space?
4. Is the set of all polynomials of degree 5 with usual addition and scalar multiplication of polynomials a vector space?
5. Let S be a non-empty set, $s \in S$. Let V be the set of all functions $f : S \to \mathbb{R}$ with $f(s) = 0$. Is V a vector space over $\mathbb{R}$ with the usual addition and scalar multiplication of functions?

Assignment-1

6. In each of the following, a vector space V and a subset W are given. Check whether W is a subspace of V.
   (a) $V = \mathbb{R}^2$; $W = \{(x_1, x_2) : x_2 = 2x_1 - 1\}$.
   (b) $V = \mathbb{R}^2$ and $V_0 =$ any straight line passing through the origin.
   (c) $V = \mathbb{R}^3$; $W = \{(x_1, x_2, x_3) : 2x_1 - x_2 - x_3 = 0\}$.
   (d) $V = C[0, 1]$; $W = \{f \in V : f \text{ is differentiable}\}$.
   (e) $V = C[-1, 1]$; $W = \{f \in V : f \text{ is an odd function}\}$.
   (f) $V = C[0, 1]$; $W = \{f \in V : f(x) \geq 0 \text{ for all } x\}$.
   (g) $V = \mathcal{P}_3(\mathbb{R})$ and $W = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 \ : \ a_0 = 0\}$.
   (h) $V = \mathcal{P}_3(\mathbb{R})$ and
   $W = \{a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_1 + a_2 + a_3 + a_4 = 0\}$.
   (i) $V = \mathcal{P}_3(\mathbb{R})$; $W = \{at + bt^2 + ct^3 : a, b, c \in \mathbb{R}\}$.
   (j) $V = \mathcal{P}_3(\mathbb{C})$;
   $W = \{a + bt + ct^2 + dt^3 : a, b, c, d \in \mathbb{C}, a + b + c + d = 0\}$.
   (k) $V = \mathcal{P}_3(\mathbb{C})$; $W = \{a + bt + ct^2 + dt^3 : a, b, c, d \text{ integers}\}$.

Assignment-1

7. Let $A \in \mathbb{R}^{n \times n}$. Let $0 \in \mathbb{R}^{n \times 1}$ be the zero vector. Is the set of all $x \in \mathbb{R}^{n \times 1}$ with $Ax = 0$ a subspace of $\mathbb{R}^{n \times 1}$?
8. Suppose U is a subspace of V and V is a subspace of W. Is U a subspace of W?
9. Let $\ell^1(\mathbb{N})$ be the set of all absolutely convergent sequences and $\ell^\infty(\mathbb{N})$ be the set of all bounded sequences with entries from $\mathbb{F}$. Show that $\ell^1(\mathbb{N})$ and $\ell^\infty(\mathbb{N})$ are vector spaces over $\mathbb{F}$.

Span

Definition: Let V be a vector space. A linear combination of vectors $v_1, \ldots, v_n \in V$ is an element of V which is in the form

$\alpha_1 v_1 + \cdots + \alpha_n v_n$ with $\alpha_j \in \mathbb{F}$, $j = 1, \ldots, n$.

If the vector $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$, for some scalars α_i, then we say that v can be expressed as a linear combination of vectors $v_1, \ldots, v_n$.

Let S be a nonempty subset of V. Then the set of all linear combinations of elements of S is called the span of S, and is denoted by span(S).

Span of the empty set is taken to be $\{0\}$.

Span Examples

1. $\text{span}\{0\} = \text{span}(\emptyset) = \{0\}$.

2. $\mathbb{C} = \text{span}\{1, i\}$ with scalars from $\mathbb{R}$.

3. Let $e_1 = (1, 0),\ e_2 = (0, 1)$. Then $\mathbb{R}^2 = \text{span}\{e_1, e_2\}$

4. More generally, if e_i denotes the vector having 1 at the i-th place and 0 everywhere else, then $\mathbb{R}^n = \text{span}\{e_1, \ldots, e_n\}$.

5. $\mathcal{P}_3 = \text{span}\{1, t, t^2, t^3\}$.

Examples Contd.

6. Let $\mathcal{P}$ denote the set of all polynomials of all degree. Then $\mathcal{P} = \text{span}\{1, t, t^2, \ldots\}$.
But $1 + t + t^2 + t^3 + \cdots \notin \mathcal{P}$.

Caution: The set S can have infinitely many elements, but a linear combination has only finitely many elements.

Theorem: Let V be a vector space and $S \subseteq V$. Then span(S) is a subspace of V and it is the smallest subspace containing S.

Proof: If $S = \emptyset$, then span(S) $= \{0\}$.
If $S \neq \emptyset,\ x, y \in$ span(S), then $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ and $y = \beta_1 y_1 + \cdots + \beta_m y_m$ for some $\alpha_i, \beta_j \in \mathbb{F}$ and $x_i, y_j \in S$.

Proof Contd.

Then for $\alpha \in \mathbb{F}$,

$$x + \alpha y = a_1 x_1 + \cdots + a_n x_n + \alpha\beta_1 y_1 + \cdots \alpha\beta_m y_m \in \text{span}(S).$$

Hence span(S) is a subspace of V.

For minimality, suppose $S \subseteq V_0$, a subspace of V.
V_0 contains all linear combination of elements of S.
That is, span(S) $\subseteq V_0$.
Hence, span(S) is the smallest subspace containing S. $\square$

Exercise: S is a subspace of V if and only if span(S) $= S$.

Exercise: If S is a subset of a vector space V, then prove that span(S) $= \bigcap\{Y \ : \ Y$ is a subspace of V containing $S\}$.

Another Exercise

Consider the system of linear equations:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\cdots & + & \cdots & + & \cdots & + & \cdots & = & \cdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m.
\end{array}
$$

Let $u_1 = (a_{11}, \ldots, a_{m1}),\ u_2 = (a_{12}, \ldots, a_{m2}), \ldots$. Show that the above system has a solution vector $x = (x_1, \ldots, x_n)$ if and only if $b = (b_1, \ldots, b_n)$ is in the span of $\{u_1, \ldots, u_n\}$.

Recall: If V_1, V_2 are subspaces of V, then so is $V_1 \cap V_2$. But $V_1 \cup V_2$ need not be a subspace of V.

Sum and Span

Let V_1 and V_2 be subspaces of a vector space V. Define

$$V_1 + V_2 = \{u + v \ : \ u \in V_1 \text{ and } v \in V_2\}.$$

Theorem: $V_1 + V_2 = \text{span}(V_1 \cup V_2)$.

Proof: Let $u \in V_1, v \in V_2$. Then $u + v \in \text{span}(V_1 \cup V_2)$.
Thus, $V_1 + V_2 \subseteq \text{span}(V_1 \cup V_2)$.
Conversely, any linear combination of elements of $V_1 \cup V_2$ is a sum of linear combinations of elements of V_1 and a linear combination of elements of V_2. Since V_1, V_2 are vector spaces, the last sum is a sum of an element of V_1 and an element of V_2.
Therefore, $\text{span}(V_1 \cup V_2) \subseteq V_1 + V_2$. $\square$

Sum can be defined for subsets instead of subspaces, but we do not require it now.

Sum?

Note: x-axis + y-axis = ?

Exercise: Suppose $V_1 \cap V_2 = \{0\}$. Then every element of $V_1 + V_2$ can be written uniquely as $x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$.
Reason: If $x_1 + x_2 = y_1 + y_2$ and $x_1, y_1 \in V_1, x_2, y_2 \in V_2$, then $x_1 - y_1 = y_2 - x_2$. Now, on is in V_1 and the other is in V_2. But the only common element is 0. hence, both are equal to 0. That is, $x_1 = y_1$ and $x_2 = y_2$.

Note: Any vector $(x, y) \in \mathbb{R}^2$ can be uniquely written as $x(1, 0) + y(0, 1)$.

Suppose we take $u = (1, 1),\ v = (-1, 2),\ w = (1, 0)$. Then

$$\begin{aligned}(2,3) &= 3(1,1) + 0(-1,2) + -1(1,0)\\ &= 1(1,1) + 1(-1,2) + 2(1,0).\end{aligned}$$

When can we ensure the uniqueness?

Assignment-2

1. Do the polynomials $t^3 - 2t^2 + 1,\ 4t^2 - t + 3$, and $3t - 2$ span $\mathcal{P}_3$? Justify your answer.
2. What is $\text{span}\{t^n : n = 0, 2, 4, 6, \ldots\}$?
3. m Let $u, v_1, v_2, \ldots, v_n$ be $n + 1$ distinct vectors in a real vector space V.
   Take $S_1 = \{v_1, v_2, \ldots, v_n\}$ and $S_2 = \{u, v_1, v_2, \ldots, v_n\}$.
   Prove that
   (a) If $\text{span}(S_1) = \text{span}(S_2)$, then $u \in \text{span}(S_1)$.
   (b) If $u \in \text{span}(S_1)$, then $\text{span}(S_1) = \text{span}(S_2)$.
4. Let S be a subset of a vector space V. Show that S is a subspace if and only if $S = \text{span}(S)$.
5. Let U be a subspace of V, $v \in V \setminus U$. Show that for every $x \in \text{span}(\{v\} \cup U)$, there exists a unique pair $(\alpha, y) \in \mathbb{F} \times U$ such that $x = \alpha v + y$.

Assignment-2

6. Show that span$\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$ is $\mathbb{R}^3$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$.
7. Let V be a vector space and A, B be subsets of V. Prove or disprove the following:
   (a) A is a subspace of V if and only if span$(A) = A$.
   (b) If $A \subseteq B$, then span$(A) \subseteq$ span(B).
   (c) span$(A \cup B) =$ span$(A) +$ span(B).
   (d) span$(A \cap B) \subseteq$ span$(A) \cap$ span(B).
8. Let U, V be subspaces of a vector space W over $\mathbb{F}$. Prove or disprove the following:
   (a) $U \cap V$ and $U + V$ are subspaces of W.
   (b) $U + V = V$ iff $U \subseteq V$.
   (c) $U \cup V$ is a subspace if and only if $U \subseteq V$ or $V \subseteq U$.
   (d) Let $U \cap V = \{0\}$. If $x \in U + V$, then there are unique $u \in U, v \in V$ such that $x = u + v$.

Assignment-2

9. Let $u_1(t) = 1$, and for $j = 2, 3, \ldots$, let $u_j(t) = 1 + t + \ldots + t^{j-1}$. Show that span$\{u_1, \ldots, u_n\}$ is $\mathcal{P}_{n-1}$, and span$\{u_1, u_2, \ldots\}$ is $\mathcal{P}$.
10. Let U, V, W be subspaces of a real vector space X.
    (a) Prove that $(U \cap V) + (U \cap W) \subseteq U \cap (V + W)$.
    (b) Give an example with appropriate U, V, W, X to show that $U \cap (V + W) \not\subseteq (U \cap V) + (U \cap W)$.

Linear Independence

Definition: A set of vectors $\{v_1, \ldots, v_n\}$ is said to be linearly dependent if one of the vectors can be written as a linear combination of others, i.e., for some $j \in \{1, \ldots, n\}$, $v_j = \alpha_1 v_1 + \cdots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \cdots + \alpha_n v_n$.

Equivalently, $\{v_1, \ldots, v_n\}$ is linearly dependent if $\exists\ j$ such that $v_j \in \text{span}\{v_i \ : \ i \neq j\}$.

Definition: A set of vectors $\{v_1, \ldots, v_n\}$ is said to be linearly independent if the set is not linearly dependent, i.e., none of the vectors can be written as a linear combination of the others.

Given a set of vectors, how do we verify these properties?

$\{v_1, \ldots, v_n\}$ is linearly dependent
$\Rightarrow \ v_j = \alpha_1 v_1 + \cdots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \cdots + \alpha_n v_n$.

$\Rightarrow \ \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ taking $\alpha_j = -1$.

How to Know?

Conversely, suppose $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ for some $\alpha_i \in \mathbb{F}$.

Suppose $\alpha_j \neq 0$ for at least one j. Then

$$v_j = \frac{\alpha_1}{-\alpha_j} v_1 + \cdots + \frac{\alpha_n}{-\alpha_j} v_n.$$

$\Rightarrow$ $\{v_1, \ldots, v_n\}$ is linearly dependent.

So we conclude:

$\{v_1, \ldots, v_n\}$ is linearly dependent if and only if $\exists\ \alpha_1, \ldots, \alpha_n \in \mathbb{F}$, not all zero, such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.

Working Rule

To show: $\{v_1, \ldots, v_n\}$ is linearly independent.

We start:
Assume $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.
Do something here, and prove that
Each $\alpha_i = 0$. We end here.

To show: $\{v_1, \ldots, v_n\}$ is linearly dependent.

We start:
We give at least one of $\alpha_1, \ldots, \alpha_n$ non-zero.
Do something here, and show that
$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$. We end here.

Determining Linear Independence

Examples: 1. $\{(1,0),(0,1)\}$ is linearly independent in $\mathbb{R}^2$.
Solution: Assume $\alpha(1,0)+\beta(0,1)=0$. Then $(\alpha,\beta)=(0,0)$.
Then $\alpha=0, \beta=0$.
Therefore, $\{(1,0),(0,1)\}$ is linearly independent.

2. $\{(1,0,0),(1,1,0),(1,1,1)\}$ is linearly independent in $\mathbb{R}^3$.
Solution: Suppose $\alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1)=(0,0,0)$.
$\Rightarrow\ \alpha+\beta+\gamma=0;\ \beta+\gamma=0;\ \gamma=0$.
$\Rightarrow\ \alpha=\beta=\gamma=0$
Therefore, the vectors are linearly independent.

Examples Contd.

3. $\{1, t, t^2\} \subseteq \mathcal{P}_2$ is linearly independent.
Solution: Let $\alpha \cdot 1 + \beta t + \gamma t^2 = 0$.
The right side is the zero polynomial. Two polynomials are equal only when the respective coefficients are equal. So, $\alpha = \beta = \gamma = 0$.
Therefore, $\{1, t, t^2\}$ is linearly independent.

4. $\{\sin x, \cos x\} \subseteq C[-\pi,\ \pi]$ is linearly independent.
Solution: Suppose $\alpha \sin x + \beta \cos x = 0$.
Putting $x = 0$ we get $\beta = 0$.
Now putting $x = \frac{\pi}{2}$, we get $\alpha = 0$.

Note: Any set containing the zero vector is linearly dependent.

Examples Contd.

Caution: $\{v_1, \ldots, v_n\}$ is linearly dependent DOES NOT IMPLY that each vector is in the span of the remaining vectors.

5. $\{(1,0),(1,1),(2,2)\}$ is linearly dependent and $(1,0) \notin \text{span}\{(1,1),(2,2)\}$.

6. $\{u, v\}$ is linearly dependent if and only if one of them is a scalar multiple of the other.

Solution: If $\{u, v\}$ is linearly dependent, then either $u \in \text{span}\{v\}$ or $v \in \text{span}\{u\}$, i.e., either $u = \alpha v$ or $v = \beta u$.

Conversely suppose one of them is a scalar multiple of the other, say $u = \alpha v$. Then $u \in \text{span}\{v\} \Rightarrow \{u, v\}$ is linearly dependent.

Linear Independence - Facts

1. Each superset of a linearly dependent set is linearly dependent.

Proof. Suppose $A \subseteq B \subseteq V$, where V is a vector space. If A is linearly dependent, then we have $\alpha_1 u_1 + \cdots + \alpha_n u_n = 0$ with $\alpha_i \neq 0$ for at least one i, for some vectors $u_1, \ldots, u_n \in A$.
However, these $u_1, \ldots, u_n$ are also vectors of B.
Hence B is linearly dependent.

2. Each subset of a linearly independent set is linearly independent.
Proof. It follows from (1).

Exercise: Suppose $\{u_1, \ldots, u_n\}$ is linearly independent and Y is a subspace of V such that $\text{span}\{u_1, \ldots, u_n\} \cap Y = \{0\}$. Prove that every vector x in the span of $\{u_1, \ldots, u_n\} \cup Y$ can be written uniquely as $x = \alpha_1 u_1 + \cdots + \alpha_n u_n + y$ with $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ and $y \in Y$.

Ordered Set

Theorem: Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly dependent ordered set in a vector space V. Then some v_k is a linear combination of the previous ones.

Proof. If v_1 is the zero vector, then $0 \in \text{span}(\emptyset)$ does the job.

Assume thus $v_1 \neq 0$. Linear dependence implies: we have scalars $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Let k be the maximum integer in $\{1, 2, \ldots, n\}$ such that $\alpha_k \neq 0$. Then

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0, \quad \alpha_k \neq 0.$$

In that case, $v_k = -\frac{1}{\alpha_k}(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{k-1} v_{k-1})$.
That is, $v_k \in \text{span}\{v_1, \ldots, v_{k-1}\}$. $\square$

A Secret

Theorem: Let $A = \{u_1, \ldots, u_m\},\ B = \{v_1, \ldots, v_n\}$ be subsets of a vector space V. Suppose that A is linearly independent and B spans V. Then $m \leq n$.

Proof. Assume that $m > n$. Then we have vectors $u_{n+1}, \ldots, u_m$ in A. Since B is a spanning set, $u_1 \in \text{span}(B)$. Thus, the set

$$B_1 = \{u_1, v_1, v_2, \ldots, v_n\}$$

is linearly dependent. Now, consider B_1 as an ordered set. We have a vector v_k such that $v_k \in \text{span}\{u_1, v_1, \ldots, v_{k-1}\}$. Remove this v_k from B_1 to obtain the set

$$C_1 = \{u_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}.$$

Notice that $\text{span}(C_1) = \text{span}(B_1) = \text{span}(B) = V$.

Proof Contd.

Add u_2 to the set C_1 to form the set

$$B_2 = \{u_2, u_1, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}.$$

Again, B_2 is linearly dependent.
Then, for some j, $v_j \in \text{span}\{u_2, u_1, v_1, \ldots, v_{j-1}\}$.
Notice that due to linear independence of $\{u_1, \ldots, u_n\}$, u_2 is not in the linear span of u_1; only a v can be in the linear span of the previous vectors.
Remove this v_j from B_2 to obtain a set C_2. Again, $\text{span}(C_2) = \text{span}(B_2) = \text{span}(B) = V$.
Continue this process of introducing a u and removing a v for n steps. Finally, v_n is removed and we end up with the set $C_n = \{u_n, u_{n-1}, \ldots, u_2, u_1\}$ which spans V.
Then $u_{n+1} \in \text{span}(C_n)$. This is a contradiction since A is linearly independent. Therefore, our assumption that $m > n$ is wrong. □

Assignment-3

1. Answer the following questions with justification:
   (a) Is every subset of a linearly independent set linearly independent?
   (a) Is every subset of a linearly dependent set linearly dependent?
   (b) Is every superset of a linearly independent set linearly independent?
   (c) Is every superset of a linearly dependent set linearly dependent?
   (d) Is union of two linearly independent sets linearly independent?
   (e) Is union of two linearly dependent sets linearly dependent?
   (f) Is intersection of two linearly independent sets linearly independent?
   (g) Is intersection of two linearly dependent sets linearly dependent?
2. Give three vectors in $\mathbb{R}^2$ such that none of the three is a scalar multiple of another.
3. Suppose S is a set of vectors and some $v \in S$ is not a linear combination of other vectors in S. Is S lin. ind.?

Assignment-3

4. In each of the following, a vector space V and $A \subseteq V$ are given. Determine whether A is linearly dependent and if it is, express one of the vectors in A as a linear combination of the remaining vectors.
   (a) $V = \mathbb{R}^3$, $A = \{(1,0,-1),(2,5,1),(0,-4,3)\}$.
   (b) $V = \mathbb{R}^3$, $A = \{(1,2,3),(4,5,6),(7,8,9)\}$.
   (c) $V = \mathbb{R}^3$, $A = \{(1,-3,-2),(-3,1,3),(2,5,7)\}$.
   (d) $V = \mathbb{P}^3$, $A = \{t^2 - 3t + 5,\ t^3 + 2t^2 - t + 1,\ t^3 + 3t^2 - 1\}$.
   (e) $V = \mathbb{P}^3$,
   $A = \{-2t^3 - 11t^2 + 3t + 2,\ t^3 - 2t^2 + 3t + 1,\ 2t^3 + t^2 + 3t - 2\}$.
   (f) $V = \mathbb{P}^3$,
   $A = \{6t^3 - 3t^2 + t + 2,\ t^3 - t^2 + 2t + 3,\ 2t^3 + t^2 - 3t + 1\}$.
   (g) $V = \mathbb{F}^{2\times 2},\ A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.
   (h) $V = \{f : \mathbb{R} \to \mathbb{R}\},\ A = \{2, \sin^2 t, \cos^2 t\}$.
   (i) $V = \{f : \mathbb{R} \to \mathbb{R}\},\ A = \{1, \sin t, \sin 2t\}$.
   (j) $V = C([-\pi,\pi]),\ A = \{\sin t, \sin 2t, \dots, \sin nt\}$ where n is some natural number.

Assignment-3

5. Show that two vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent if and only if $ad - bc \neq 0$.
6. Let $A = (a_{1j}) \in \mathbb{R}^{n \times n}$ and let $w_1, \ldots, w_n$ be the n columns of A. Let $\{u_1, \ldots, u_n\}$ be linearly independent in $\mathbb{R}^n$. Define vectors $v_1, \ldots, v_n$ by
   $v_j = a_{1j}u_1 + \ldots + a_{nj}u_n,$ for $j = 1, 2, \ldots, n$.
   Show that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent iff $\{w_1, w_2, \ldots, w_n\}$ is linearly independent.
7. Let A, B be subsets of a vector space V. Prove or disprove: $\text{span}(A) \cap \text{span}(B) = \{0\}$ iff $A \cup B$ is linearly independent.
8. Suppose V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$. Show that every $x \in V_1 + V_2$ can be written *uniquely* as $x = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$.
9. Let $p_1(t) = 1 + t + 3t^2$, $p_2(t) = 2 + 4t + t^2$, $p_3(t) = 2t + 5t^2$. Are the polynomials p_1, p_2, p_3 linearly independent?
10. Prove that in the vector space of all real valued functions, the set of functions $\{e^x,\ xe^x,\ x^3e^x\}$ is linearly independent.

Basis

Note: The set $\{e_1, \ldots, e_n\}$ is linearly independent and $\text{span}\{e_1, \ldots, e_n\} = \mathbb{R}^n$.

The set $\{1, t, \ldots, t^n\}$ is linearly independent and $\mathcal{P}_n = \text{span}\{1, t, \ldots, t^n\}$.

Definition: Let V be a vector space over $\mathbb{F}$. A subset B of V is called a basis for V if B is linearly independent and $\text{span}(B) = V$.

Basis is NOT unique: For example, both $\{1\}$ and $\{2\}$ are $\mathbb{R}$-bases of $\mathbb{R}$. In fact $\{x\}$, for any non-zero $x \in \mathbb{R}$, is an basis of $\mathbb{R}$.

Verify if $\{(1,1),(1,2)\}$ is an basis of $\mathbb{R}^2$.

Examples

Since neither of them is a scalar multiple of the other, the set is linearly independent.

1. span$\{(1,1),(1,2)\} = \mathbb{R}^2$?

Does the equation $(x,y) = \alpha(1,1) + \beta(1,2)$ has a solution in α, β?

$\Rightarrow \quad \alpha + \beta = x; \quad \alpha + 2\beta = y.$

$\Rightarrow \quad \beta = y - x$ and $\alpha = 2x - y$.

$\Rightarrow \quad$ span$\{(1,1),(1,2)\} = \mathbb{R}^2$

$\{(1,1),(1,2)\}$ is an basis of $\mathbb{R}^2$.

Examples

2. $\{e_1, \ldots, e_n\}$ is an basis of $\mathbb{R}^n$. This basis is called Standard Basis of $\mathbb{R}^n$.

Consider set $\{M_{ij} \; : \; i = 1, \ldots, m; \; j = 1, \ldots, n\}$, where M_{ij} is the matrix with (i, j)-th entry 1 and all other entries 0. Then this is a basis of $M_{m \times n}(\mathbb{F})$, called the standard basis.

Exercise: Prove that the set $\{1, 1 + t, 1 + t + t^2\}$ is a basis for $\mathcal{P}_2$. Find a similar basis for $\mathcal{P}_n$.

Exercise: If $\{p_1(t), \ldots, p_r(t)\} \subseteq \mathcal{P}$ be set of polynomials such that $\deg p_1 < \deg p_2 < \cdots < \deg p_r$, then prove that $\{p_1(t), \ldots, p_r(t)\}$ is linearly independent.

Exercise: Can you find a basis for $\mathbb{R}^2$ consisting of 3 vectors?

A Result

Theorem: Let V be a vector space and $B \subseteq V$. Then the following are equivalent:

1. B is a basis of V
2. B is a maximal linearly independent set in V,
   i.e., B is linearly independent and every proper superset of B is linearly dependent.
3. B is a minimal spanning set of V,
   i.e., $\text{span}(B) = V$ and no proper subset of B spans V.

Proof:
(1) $\Rightarrow$ (2): Since B is a basis, $\text{span}(B) = V$. If $v \in V \setminus B$, then v is a linear combination of elements of B. Then $B \cup \{v\}$ is linearly dependent. Then B is a maximal linearly independent subset of V.

Prof Contd.

(2) $\Rightarrow$ (3): Since B is linearly independent, for any $v \in B,\ v \notin \text{span}(B \setminus \{v\})$. So, no proper subset of B can span V.
If $v \in V \setminus \text{span}(B)$, then $B \cup \{v\}$ is linearly independent, which contradicts the assumption. So, $\text{span}(B) = V$.

(3) $\Rightarrow$ (1): Assume that B is a minimal spanning set of V $\Rightarrow \text{span}(B) = V$. Suppose B is linearly dependent, i.e., $\exists\ u \in B$ such that $u \in \text{span}(B \setminus \{u\})$.
$\Rightarrow\ \text{span}(B \setminus \{u\}) = V$ which contradicts the assumption that B is minimal spanning set. $\square$

Definition: A vector space V is said to be finite dimensional if there exists a finite basis for V. A vector space which is not finite dimensional is called infinite dimensional vector space.

Another Result

Example: $\mathbb{F}^n, \mathcal{P}_n, M_{m\times n}(\mathbb{F})$ are finite dimensional vector spaces over $\mathbb{F}$.

Theorem: If a vector space has a finite spanning set, then it has a finite basis.

Proof: Let $V = \text{span}(S)$ for some finite subset S of V. If S is linearly independent, then S is a basis.

Otherwise, $\exists\, u_1 \in S$ such that $u_1 \in \text{span}(S \setminus \{u_1\})$. Therefore, $\text{span}(S \setminus \{u_1\}) = V$. If $S_1 = S \setminus \{u_1\}$ is linearly independent, then S_1 is a basis.

Otherwise, one can repeat the process. The process has to stop since S is a finite set and we end up with a subset S_k of S such that S_k is linearly independent and $\text{span}(S_k) = V$. $\square$

Examples

1. Compute a basis of $V = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\}$.

Solution: $(x, y) \in V \;\Rightarrow\; y = 2x \;\Rightarrow (x, y) = (x, 2x) = x(1, 2)$
$\Rightarrow\; V = \text{span}\{(1, 2)\}$.
Since $\{(1, 2)\}$ is linearly independent, it is a basis for V.

2. Compute a basis of $V = \{(x, y, z) \in \mathbb{R}^3 \;:\; x - 2y + z = 0\}$.

Solution: $(x, y, z) \in V \;\Rightarrow\; x = 2y - z$
$\Rightarrow\; (x, y, z) = (2y - z, y, z) = y(2, 1, 0) + z(-1, 0, 1)$
$\Rightarrow\; V = \text{span}\{(2, 1, 0), (-1, 0, 1)\}$.
[**Check** whether these vectors are linearly independent.]
$\Rightarrow\; \{(2, 1, 0), (-1, 0, 1)\}$ is a basis of V.

3. Compute a basis of
$V = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 \;:\; x_1 + x_3 - x_5 = 0 \text{ and } x_2 - x_4 = 0\}$.

A Result

Note: In $\mathbb{R}$, any two real numbers are linearly dependent.

Exercise: Try to show:
(a) In $\mathbb{R}^2$, any three vectors are linearly dependent.
(b) Any 4 polynomials in $\mathcal{P}_2$ are linearly dependent.

Given a vector space, do we have an upper limit for the cardinality of linearly independent set?

Theorem: Let V be a vector space with a basis consisting of n elements. Then each basis of V has n elements.

Proof: Let A be a basis of V having n elements. Let B be any other basis having m elements. Since A is a spanning set and B is linearly independent, $m \leq n$.
Since A is linearly independent and B is a spanning set, $n \leq m$. $\square$

Consequences

Corollary 1: If V has a basis of n elements, then any set of $n+1$ or more vectors is linearly dependent.

Corollary 2: If a vector space contains an infinite linearly independent subset, then it is an infinite dimensional space.

Corollary 3: If a vector space has a basis of n vectors, then all bases for V have n vectors.

Exercise: Let $M = (a_{ij})$ be an $m \times n$ matrix with $a_{ij} \in \mathbb{F}$ and $n > m$, then there exists $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$ such that

$$a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = 0, \qquad \text{for all } i = 1, \dots, m.$$

Dimension

Definition: Suppose V is a finite dimensional vector space. Then the cardinality of a basis is said to be the dimension of V, denoted by $\dim V$.

Observation: Let V be a vector space of dimension n and A be a subset of V containing m vectors.
(a) If A is linearly independent, then $m \leq n$.
(b) If $m > n$, then A is linearly dependent.
(c) If A is linearly independent & $m = n$, then A is a basis of V.

Examples:
1. $\dim \mathbb{R} = 1$;

2. $\dim \mathbb{R}^n = n$.

3. $\mathbb{C}$ considered as an $\mathbb{R}$-vector space, $\dim \mathbb{C} = 2$.

4. Consider $\mathbb{C}^n$ as an $\mathbb{R}$-vector space. What is $\dim \mathbb{C}$? Can you produce an basis for $\mathbb{C}^n$?

Examples Contd.

5. $\dim \mathcal{P}_n = n + 1$

6. What is the dimension of $M_{m\times n}(\mathbb{R})$?

7. The set of all polynomials of all degrees, $\mathcal{P}$, is an infinite dimensional vector space.

Solution: Suppose the dimension is finite, say $\dim \mathcal{P} = n \Rightarrow$ any set of $n + 1$ vectors are linearly dependent.
$\Rightarrow$ $\{1, t, \ldots, t^n\}$ is linearly dependent, which is a contradiction.
$\Rightarrow$ the dimension can not be finite.

8. What is the dimension of $\{0\}$? **Ans:** $\dim\{0\} = 0$.

In $\mathbb{R}^2$

9. $C[a,\ b]$ is an infinite dimensional vector space.

Solution: Take the collection of functions
$\{f_n\ :\ f_n(x) = x^n\ \forall\ x \in [a,\ b];\ n = 0, 1, 2, \ldots\}$.
Then $\{f_0, f_1, \ldots, f_n\}$ is linearly independent for every n
$\Rightarrow\ C[a,\ b]$ can not have a finite basis.

Let $\{u, v\} \subseteq \mathbb{R}^2$ be linearly independent. Then can span$\{u, v\}$ be a proper subset of $\mathbb{R}^2$?
Let $\{u, v, w\} \subseteq \mathbb{R}^3$ be linearly independent. Then, can span$\{u, v, w\}$ be a proper subset of $\mathbb{R}^3$?

Note: If V is a finite dimensional vector space, then so is any subspace of V.
For, if a subspace W contains an infinite linearly independent set, then that set will remain linearly independent in V as well.
$\Rightarrow\ V$ is infinite dimensional, which contradicts our assumption.

Dimension of Subspace

Theorem: If V is a finite dimensional vector space and W is a proper subspace of V, then $\dim W < \dim V$.

Proof: Since V is finite dimensional, so is W.

Let $\{w_1, \ldots, w_m\}$ be a basis of W and $v \in V \setminus W$.

$\Rightarrow$ $\{w_1, \ldots, w_n, v\}$ is linearly independent in V. **Prove this!**

$\Rightarrow$ V contains $n+1$ vectors which are linearly independent.

If a basis of V contains n or less number of vectors, then there can not be a set with cardinality $n+1$ which is linearly independent.

$\Rightarrow$ $\dim V \geq n+1$. $\square$

Bases

Let W be a subspace of a finite dimensional vector space V. Can we say that $B_W \subseteq B_V$?

Let $W = \{(x, 0) \ : \ x \in \mathbb{R}^2$ and $V = \mathbb{R}^2$. Then $\{2, 0\}$ is a basis of W and $\{(1, 0), (0, 1)\}$ is a basis of V.

Theorem: Let W be a subspace of a finite dimensional vector space V and $B_W = \{u_1, \ldots, u_m\}$ be a basis of V. Then there exists a basis B_V of V such that $B_W \subseteq B_V$.

Proof: If $W = V$, then there is nothing to prove. Suppose W is properly contained in V.

Let $u_{m+1} \in V \setminus W$. Then $\{u_1, \ldots, u_m, u_{m+1}\}$ is linearly independent in V.

Proof Contd.

Let $W_1 = \text{span}\{u_1, \ldots, u_m, u_{m+1}\}$. If dim $V = m + 1$, then $W_1 = V$ and hence we are done.

If not, then W_1 is properly contained in V and hence there exists $u_{m+2} \in V \setminus W_1$.

$\Rightarrow \quad \{u_1, \ldots, u_{m+2}\}$ is linearly independent in V.

If dim $V = m + 2$, then we are done. Otherwise continue as before.

If dim $V = n$, then $W_{n-m} = \text{span}\{u_1, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ will be equal to V. $\square$

Sum & Intersection

Given two subspaces V_1 and V_2 of a finite dimensional vector spaces, we have two other subspaces, $\mathrm{span}(V_1 \cup V_2) = V_1 + V_2$ and $V_1 \cap V_2$.

How are the dimensions of these spaces related with the dimensions of the individual spaces?

We can easily say that $\dim(V_1 \cap V_2) \leq \dim V_i$ for each $i = 1, 2$.

If $V_1 \subseteq V_2$, then $V_1 \cap V_2 = V_1$ and hence, it can be equality in the above inequality for one of the i's.

Exercise: Prove that if $\dim(V_1 \cap V_2) = \dim V_1$, then $V_1 \subseteq V_2$.

Can we relate $\dim V_1 + V_2$ to $\dim V_1$ and $\dim V_2$?

Work Out

Let $V_1 = \{(x, y) \in \mathbb{R}^2 : 2x - y = 0\}$ and
$V_2 = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$.
What is $\dim V_1 + V_2$?

Let $V_1 = \{(x, y, z) \in \mathbb{R}^3 \; : \; x + y + z = 0\}$ and
$V_2 = \{(x, y, z) \in \mathbb{R}^3 \; : \; x - y - z = 0\}$.
What is $\dim(V_1 + V_2)$?

$\{(1, 0, -1), (1, -1, 0)\}$ is a basis of V_1 and
$\{(1, 1, 0), (1, 0, 1)\}$ is basis of V_2.

Then $V_1 + V_2 = \mathbb{R}^3$
$\Rightarrow \; \dim(V_1 + V_2) = 3 < \dim V_1 + \dim V_2$.

Are we counting something twice?

Sum & Intersection

Theorem: Let V_1 and V_2 be finite dimensional subspaces of a vector space V. Then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

Proof: $V_1 \cap V_2$ has finite dimension. Let $\{x_1, \ldots, x_n\}$ be a basis of $V_1 \cap V_2$. Notice that if $V_1 \cap V_2 = \{0\}$, we take its basis as $\emptyset$.

Note that $V_1 \cap V_2$ is a subspace of V_1 as well as V_2. We extend the basis of $V_1 \cap V_2$ to bases of V_1 and of V_2.

Let $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ be a basis of V_1.
Let $\{x_1, \ldots, x_n, w_1, \ldots, w_k\}$ be a basis of V_2.

Proof Contd.

CLAIM: $B = \{x_1, \ldots, x_n, y_1, \ldots, y_m, w_1, \ldots, w_k\}$ is a basis of $V_1 + V_2$.

Proof of the Claim: First we show that $\text{span}(B) = V_1 + V_2$.

Let $v_1 + v_2 \in V_1 + V_2$. Then

$$\begin{aligned}
v_1 &= \alpha_1 x_1 + \cdots + \alpha_n x_n + \alpha_{n+1} y_1 + \cdots + \alpha_{n+m} y_m; \\
v_2 &= \beta_1 x_1 + \cdots + \beta_n x_n + \beta_{n+1} w_1 + \cdots + \beta_{n+k} w_k. \\
\Rightarrow\ v_1 + v_2 &= \sum_{i=1}^{n} (\alpha_i + \beta_i) x_i + \sum_{j=1}^{m} \alpha_{n+j} y_j + \sum_{l=1}^{k} \beta_{n+l} w_l
\end{aligned}$$

$\Rightarrow\ V_1 + V_2 \subseteq \text{span}(B)\ \Rightarrow\ \text{span}(B) = V_1 + V_2$.

We now prove that B is linearly independent.

Proof Contd.

Suppose
$\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m + \gamma_1 w_1 + \cdots + \gamma_k w_k = 0.$

$\Rightarrow \ \alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m = -\gamma_1 w_1 - \cdots - \gamma_k w_k$

$\Rightarrow \ -\gamma_1 w_1 - \cdots - \gamma_k w_k \in V_1 \cap V_2.$

$\Rightarrow \ -\gamma_1 w_1 - \cdots - \gamma_k w_k = a_1 x_1 + \cdots + a_n x_n$

$\Rightarrow \ a_1 x_1 + \cdots + a_n x_n + \gamma_1 w_1 + \cdots + \gamma_k w_k = 0.$

$\{x_1, \ldots, x_n, w_1, \ldots, w_k\}$ is a basis of $V_2 \ \Rightarrow$ they are linearly independent
$\Rightarrow \ a_1 = \cdots = a_n = \gamma_1 = \cdots = \gamma_k = 0.$

Substituting the values of γ_i's, we get

$$\alpha_1 x_1 + \cdots + \alpha_n x_n + \beta_1 y_1 + \cdots + \beta_m y_m = 0.$$

Proof Contd.

Since $\{x_1, \dots, x_n, y_1, \dots, y_m\}$ is linearly independent, $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_m = 0$.

$\Rightarrow \{x_1, \dots, x_n, y_1, \dots, y_m, w_1, \dots, w_k\}$ is a basis for $V_1 + V_2$

Therefore

$$\begin{aligned} \dim(V_1 + V_2) &= n + m + k = (n + k) + (m + k) - k \\ &= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2). \end{aligned}$$

$\square$

Corollary: Two distinct planes through origin in $\mathbb{R}^3$ intersect on a line.

Exercise: If V_1 and V_2 are subspace of $\mathbb{R}^9$ such that $\dim V_1 = 5 = \dim V_2$, then $V_1 \cap V_2 \neq \emptyset$.

Assignment-4

1. Determine which of the following sets form bases for $\mathcal{P}_2$.
   (a) $\{-1 - t - 2t^2, 2 + t - 2t^2, 1 - 2t + 4t^2\}$.
   (b) $\{1 + 2t + t^2, 3 + t^2, t + t^2\}$.
   (c) $\{1 + 2t + 3t^2, 4 - 5t + 6t^2, 3t + t^2\}$.
2. Let $\{x, y, z\}$ be a basis for a vector space V. Is $\{x + y, y + z, z + x\}$ also a basis for V?
3. Extend the set $\{1 + t^2, 1 - t^2\}$ to a basis of $\mathcal{P}_3$.
4. Find a basis for the subspace $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ of $\mathbb{R}^3$.
5. Is $\{1 + t^n, t + t^n, \ldots, t^{n-1} + t^n, t^n\}$ a basis for $\mathcal{P}_n$?
6. Let $u_1 = 1$ and let $u_j = 1 + t + t^2 + \cdots + t^{j-1}$ for $j = 2, 3, 4, \ldots$
   Is $\text{span}\{u_1, \ldots, u_n\} = \mathcal{P}_n$? Is $\text{span}\{u_1, u_2, \ldots\} = \mathcal{P}$?
7. Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.

Assignment-4

8. Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:
   (a) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$.
   (b) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$.
   (c) $\text{span}\{(1, -1, 0, 2, 1), (2, 1, -2, 0, 0), (0, -3, 2, 4, 2), (3, 3, -4, -2, -1), (2, 4, 1, 0, 1), (5, 7, -3, -2, 0)\}$.
9. Find the dimension of the subspace $\text{span}\{1 + t^2, -1 + t + t^2, -6 + 3t, 1 + t^2 + t^3, t^3\}$ of $\mathcal{P}_3$.
10. Find a basis, and hence dimension, for each of the following subspaces of the vector space of all twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$:
    (a) $\{x \in V : x'' + x = 0\}$.
    (b) $\{x \in V : x'' - 4x' + 3x = 0\}$.
    (c) $\{x \in V : x''' - 6x'' + 11x' - 6x = 0\}$.

Assignment-4

11. Let $U = \left\{ \begin{bmatrix} a & -a \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$,

    $V = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

    (a) Prove that U and V are subspaces of $\mathbb{R}^{2\times 2}$.

    (b) Find bases, and hence dimensions, for $U \cap V$, U, V, and $U + V$.

12. Show that if V_1 and V_2 are subspace of $\mathbb{R}^9$ such that $\dim V_1 = 5 = \dim V_2$, then $V_1 \cap V_2 \neq \emptyset$.

13. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. What is $\text{span}\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$?

14. Given $a_0, a_1, \ldots, a_n \in \mathbb{R}$, let $V = \{x(t) \in C^k[0, 1] : a_n x^{(n)}(t) + \cdots + a_1 x^{(1)}(t) + a_0 x(t) = 0\}$.
    Show that V is a subspace of $C^k[0, 1]$, and find its dimension.

Assignment-4

15. Let $V = \text{span}\{(1,2,3),\, (2,1,1)\}$ and $W = \text{span}\{(1,0,1),\, (3,0,-1)\}$. Find a basis for $V \cap W$. Also, find $\dim(V + W)$.
16. Given real numbers $a_0, a_1, \ldots, a_k$, let V be the set of all solutions $x \in C^k[a,b]$ of the differential equation

$$a_0 \frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + \cdots + a_k x = 0.$$

Show that V is a vector space over $\mathbb{R}$. What is $\dim V$?
17. Consider each polynomial in $\mathcal{P}$ as a function from the set $\{0,1,2\}$ to $\mathbb{R}$. Is the set of vectors $\{t, t^2, t^3, t^4, t^5\}$ linearly independent?

Structure Preserving Maps

What are the important maps in real variables?

In $\mathbb{R}$ the important sets are the intervals.

So, what are the maps that take intervals to intervals?

In vector spaces, what are themaps that take subspaces to subspaces?

They must be those, which preserve the addition and scalar multiplication.

Definition & Examples

Definition: Let U and V be vector spaces over $\mathbb{F}$.
A function $T : U \to V$ is said to be a linear transformation (or a linear map) if

$$T(x+y) = T(x) + T(y) \quad \text{and} \quad T(\alpha x) = \alpha T(x)$$

for every $x, y \in U$ and for every $\alpha \in \mathbb{F}$.

Examples:

1. $T : V \to V,\ T(v) = 0$ for all $v \in V$.

2. $T : V \to V,\ T(v) = v$.

3. Let A denote an $m \times n$ matrix with real entries. Then for any vector $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$, Ax is a vector in $\mathbb{R}^m$.
It satisfies the properties that $A(x+y) = Ax + Ay$ and $A(\alpha x) = \alpha Ax$ for $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Examples

4. Let V be any vector space and $\alpha \in \mathbb{F}$. Then the map $T : V \to V$ defined by $T(v) = \alpha v$ is linear.

5. If f and g are two differentiable functions from an interval $[a,\ b]$ to $\mathbb{R}$, then $\frac{d}{dt}(f+g) = \frac{df}{dt} + \frac{dg}{dt}$ and $\frac{d}{dt}(\alpha f) = \alpha \frac{df}{dt}$.

6. If f and g are two real valued continuous function from $[a,\ b]$ to $\mathbb{R}$, then $\int_a^b (f+g)(t)dt = \int_a^n f(t)dt + \int_a^b g(t)dt$ and $\int_a^b (\alpha f)(t)dt = \alpha \int_a^b f(t)dt$.

7. Define $T_j : \mathbb{R}^n \to \mathbb{R}$ by $T_j(x_1, \ldots, x_n) = x_j$. Then T_j is linear. More generally, for $\alpha_1, \ldots, \alpha_n \in \mathbb{R},\ T(x_1, \ldots, x_n) = \sum_{i=1}^n \alpha_i x_i$ is a linear transformation.

Examples Contd.

8. Let V be a vector space with basis $\{u_1, \ldots, u_n\}$. Give any vector, there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$. Define a map $T : V \to \mathbb{F}$, $T_i(u) = \alpha_i$. Then T is a linear map.

Solution: Let $u, v \in V$.
Let $u = \alpha_1 u_1 + \cdots + \alpha_n u_n$ and $v = \beta_1 u_1 + \cdots + \beta_n u_n$.
$\Rightarrow \;\; u + v = \sum_{i=1}^{n} (\alpha_i + \beta_i) u_i$.
This is a representation of $u + v$ as a linear combination of $\{u_1, \ldots, u_n\}$. Since $\{u_1, \ldots, u_n\}$ is a basis, this is THE unique representation for $u + v$.
$\Rightarrow \;\; T_i(u + v) = \alpha_i + \beta_i = T_i(u) + T_i(v)$.
Similarly the other condition can be verified.

Examples Contd.

9. Let $\alpha \in [a,\ b]$. Define $T_\alpha : C[a,\ b] \to \mathbb{F}$ by

$$T_\alpha(f) = f(\alpha).$$

Verify that T_α is a linear transformation.

10. Let $T : C^1[a,\ b] \to C[a,\ b]$ be defined by

$$T(f) = f'.$$

Then T is linear.

11. Let $T : C^1[a,\ b] \to C[a,\ b]$ be defined by

$$T(f) = \alpha f + \beta f'.$$

Then verify that T is linear.

Examples Contd.

12. If T_1 and T_2 are linear transformations from V_1 to V_2, then the map $T : V_1 \to V_2$ defined by

$$T(v) = \alpha T_1(v) + \beta T_2(v)$$

is a linear trasnformation.

13. Let $A = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$. Then for any $x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$, the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $Tx = Ax$ is the rotation by an angle ϕ. We have already seen that this is a linear map.

Caution: Every map that 'looks linear' need not be linear: $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = 2x + 3$.

Properties of Linearity

Theorem: Let $T : V \to W$ be a linear transformation, then for all vectors $u, v, v_1, \ldots, v_n \in V$ and scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$:

(i) $T(0) = 0$

(ii) $T(u - v) = T(u) - T(v)$

(iii) $T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$.

Proof:

(i) $T(0) = T(0 + 0) = T(0) + T(0) \ \Rightarrow \ T(0) = 0$.

(ii) $T(u - v) = T(u + (-1)v) = T(u) + T(-1(v))$
$= T(u) + (-1)T(v) = T(u) - T(v)$.

(iii) Apply induction on n. $\square$

Examples

1. Consider the 'differential map', $D : \mathcal{P}_3 \to \mathcal{P}_2$, defined by $D(p(t)) = p'(t)$.
In this map, we know that $D(t^3) = 3t^2;\ D(t^2) = 2t;\ D(t) = 1$ and $D(1) = 0$ and use the linearity of differential operator to obtain $D(p(t))$ for any polynomial $p(t) \in \mathcal{P}_3$.

2. Suppose $T : \mathbb{R}^2 \to \mathbb{R}$ be a linear map such that $T(1,0) = 2$ and $T(0,1) = -1$, then what is $T(2,3)$? What is $T(a,b)$?

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be linear such that $T(1,0,0) = (2,3)$, $T(0,1,0) = (-1,4)$ and $T(0,0,1) = (5,-3)$. Then

$$\begin{aligned} T(3,-4,5) &= 3T(1,0,0) + (-4)T(0,1,0) + 5T(0,0,1) \\ &= 3(2,3) + (-4)(-1,4) + 5(5,-3) \\ &= (35,-22). \end{aligned}$$

A Question

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map such that $T(1,1) = (1,-1)$, $T(0,1) = (-1,1)$ and $T(2,-1) = (1,0)$. Can T be a linear transformation?

Solution: $(2,-1) = 2(1,1) - 3(0,1)$
$\Rightarrow \; T(2,-1) = 2(1,-1) - 3(-1,1) = (5,-5) \neq (1,0)$.
Hence T is not a linear map.

What are the information required to describe a linear map T?

Suppose we take a basis $\{v_1, \ldots, v_n\}$ of V; chose n vectors $w_1, \ldots, w_n \in W$ randomly. Does there exists a linear map $T : V \to W$ such that $T(v_i) = w_i$?

Action on a Basis Enough?

Theorem: Let V and W be vector spaces.
Let $B = \{v_1, \ldots, v_n\}$ be a basis for V.
Suppose $w_1, \ldots, w_n \in W$. Then,
there exists a unique linear map $T : V \to W$ such that
$T(v_1) = w_1,\ T(v_2) = w_2,\ \ldots,\ T(v_n) = w_n$.

Proof: We need to construct a map from V to W and prove that this map is linear. Secondly, we show that if two linear maps take the v_is to w_is respectively, then they are the same map.

Let $v \in V$. Then $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.

Define $T(v) = \alpha_1 w_1 + \cdots + \alpha_n w_n$. Since, $\{v_1, \ldots, v_n\}$ is a basis, given a vector v, the scalars $\alpha_1, \ldots, \alpha_n$ are unique. Therefore, this map is well-defined.

Proof Contd.

Linearity: Let $u, v \in V$. Then
$u = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $v = \beta_1 v_1 + \cdots + \beta_n v_n$
$\Rightarrow \; u + v = (\alpha_1 + \beta_1)v_1 + \cdots + (\alpha_n + \beta_n)v_n$.
Therefore

$$\begin{aligned}
T(u+v) &= (\alpha_1 + \beta_1)w_1 + \cdots + (\alpha_n + \beta_n)w_n \\
&= (\alpha_1 w_1 + \cdots + \alpha_n w_n) + (\beta_1 w_1 + \cdots + \beta_n w_n) \\
&= T(u) + T(v).
\end{aligned}$$

Similarly,

$$\begin{aligned}
T(\alpha u) &= T(\alpha\alpha_1 v_1 + \cdots + \alpha\alpha_n v_n) \\
&= \alpha\alpha_1 w_1 + \cdots + \alpha\alpha_n w_n = \alpha(\alpha_1 w_1 + \cdots + \alpha_n w_n) \\
&= \alpha T(u).
\end{aligned}$$

Therefore T is a linear transformation.

Proof Contd.

Uniqueness: Assume that $T_1(v_i) = w_i = T_2(v_i)$ for each i. Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ for some α_i's $\in \mathbb{F}$. Then

$$
\begin{aligned}
T_1(v) &= T_1(\alpha_1 v_1 + \cdots + \alpha_n v_n) \\
&= \alpha_1 T_1(v_1) + \cdots + \alpha_n T_1(v_n) \text{ (linearity)} \\
&= \alpha_1 w_1 + \cdots + \alpha_n w_n \\
&= \alpha_1 T_2(v_1) + \cdots + \alpha_n T_2(v_n) \\
&= T_2(\alpha_1 v_2 + \cdots + \alpha_n v_n) \\
&= T_2(v).
\end{aligned}
$$

$\square$

Remark: Thus a linear transformation is completely determined by its action on any basis. Notice that the vectors $w_1, \ldots, w_n$, images of the basis vectors, need not be distinct or not even be linearly independent.

Examples

1. Construct a linear map $T : \mathbb{R}^2 \to W$, where
$W = \{(x_1, x_2, x_3) \, : \, x_1 - x_2 - x_3 = 0\}$.
Describe the map completely.

Solution: Start with a basis $\{v_1 = (1, 0), \; v_2 = (0, 1)\}$ of $\mathbb{R}^2$.
Choose any two vectors in W, for example
$w_1 = (1, 1, 0)$ and $w_2 = (1, 0, 1)$.
We want $T(1, 0) = (1, 1, 0)$ and $T(0, 1) = (1, 0, 1)$.
Then define
$T(x_1, x_2) = x_1(1, 1, 0) + x_2(1, 0, 1) = (x_1 + x_2, x_1, x_2)$.
This is a linear map from $\mathbb{R}^2$ to W.

Exercise: Find another linear map from $\mathbb{R}^2$ to W.

Matrix of a Linear Transformation

Let $T : V \to W$ be a linear transformation. Fix ordered bases $B = (v_1, v_2, \ldots, v_n)$ for V and $C = (w_1, w_2, \ldots, w_m)$ for W. Now, we have scalars a_{ij} such that

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \cdots + a_{m2}w_m \\ &\vdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m \end{aligned}$$

Definition: The matrix $[T]_{B,C} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the matrix of the linear transformation T with respect to the ordered bases B and C.

Caution: Take care of the notation.
$\dim(V) = n$. $\dim(W) = m$. $T : V \to W$. $[T]_{B,C} \in \mathbb{F}^{m \times n}$.

Action on a Vector

Let $u \in U$. Then
$u = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_n u_n$ for some $\beta_i \in \mathbb{F}$.

How do the scalars look when we use all the above?

$$
\begin{aligned}
T(u) = \beta_1 T(u_1) + \cdots + \beta_n T(u_n) \\
&= \beta_1(a_{11}v_1 + a_{21}v_2 + \cdots + a_{m1}v_m) \\
&\quad +\beta_2(a_{12}v_1 + a_{22}v_2 + \cdots + a_{m2}v_m) \\
&\quad\quad \vdots \\
&\quad +\beta_n(a_{1n}v_1 + a_{2n}v_2 + \cdots + a_{mn}v_m) \\
&= (a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n)v_1 \\
&\quad +(a_{21}\beta_1 + a_{22}\beta_2 + \cdots + a_{2n}\beta_n)v_2 \\
&\quad\quad \vdots \\
&\quad +(a_{m1}\beta_1 + a_{m2}\beta_2 + \cdots + a_{mn}\beta_n)v_m
\end{aligned}
$$

What do you see if you think of co-ordinate vectors?

Matrix Multiplication

$$[u]_A = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix},\ [T(u)]_B = \begin{bmatrix} a_{11}\beta_1 + a_{12}\beta_2 + \cdots + a_{1n}\beta_n \\ \vdots \\ a_{m1}\beta_1 + a_{m2}\beta_2 + \cdots + a_{mn}\beta_n \end{bmatrix}$$

Since $[T]_{A,B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & amn \end{bmatrix}$, we see that

$[T(u)]_B = [T]_{A,B}[u]_A$.

Note: When $T : \mathbb{F}^n \to \mathbb{F}^m$ is a linear map, and E_n, E_m are the standard ordered bases for $\mathbb{F}^n, \mathbb{F}^m$,
$T([a_1, \ldots, a_n]^t) = [T]_{E_n,E_m}[a_1, \ldots, a_n]^t$.
And, the i-th column of $[T]_{E_n,E_m}$ is simply $T(e_i)$.
Moreover, If for each $u \in U$, $[T(u)]_B = M[u]_A$, then $M = [T_{A,B}]$.

Examples

1. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$.

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T(x) = Ax$ for every $x \in \mathbb{R}^3$.

Then $Ae_1 = (1, 0, -2)^t$; $Ae_2 = (-1, 1, 1)^t$ & $Ae_3 = (1, 2, 0)^t$.

Note that $(1, 0, -2)^t = 1 \cdot e_1 + 0 \cdot e_2 + -2 \cdot e_3$.

2. Let $B = \{(1, -1), (1, 0)\}$. Find $[(0, 1)]_B$.

Solution: $(0, 1) = -1(1, -1) + 1(1, 0)$.

$\Rightarrow \; [(0, 1)]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Examples Contd.

3. Let $B = \{1, 1+t, 1+t^2\} \subseteq \mathcal{P}_2$. Is B a basis of $\mathcal{P}_2$? Find $[1+t+t^2]_B$.

Note here that the matrices would be different if we alter the positions of the basis vectors.i.e., the matrices w.r.t. $\{1, 1+t^2, 1+t\}$ and $\{1, 1+t, 1+t^2\}$ are different.

4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2, x_2 - x_1)$, $B_1 = \{e_1, e_2\}$ and $B_2 = \{e_1, e_2, e_3\}$. Then

$$\begin{aligned} T(e_1) = (2, 1, -1) &= 2e_1 + 1e_2 + -1e_3 \\ T(e_2) = (-1, 1, 1) &= -1e_1 + 1e_2 + 1e_3. \end{aligned}$$

Therefore $[T]_{B_1,B_2} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Note that if $A = [T]_{B_1,B_2}$, then $T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Examples Contd.

5. Let $D : \mathcal{P}_3 \to \mathcal{P}_2$ be the map given by $D(p) = p'$. Let $A = \{1, t, t^2, t^3\}$ and $B = \{1, t, t^2\}$. Then

$$[D]_{A,B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Let $B = \{1, 1+t, 1+t^2\}$. Then compute $[D]_{A,B}$

Ans: $[D]_{A,B} = \begin{bmatrix} 0 & 1 & -2 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$

6. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be the map $T(p(t)) = \int_0^t p(s)ds$. Let $A = \{1, 1+t, t+t^2\}$ and $B = \{1, t, t+t^2, t^2+t^3\}$. Then

$$[T]_{A,B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1/2 & -1/6 \\ 0 & 1/2 & 1/6 \\ 0 & 0 & 1/3 \end{bmatrix}.$$

Properties of matrices of linear maps

Theorem: Let $V, \ W$ be finite dimensional vector spaces. Let $B = (v_1, \dots, v_n)$ be an ordered basis of V and $C = (w_1, \dots, w_m)$ be an ordered basis of W. If T_1 and T_2 are linear transformations from V to W and $\alpha \in \mathbb{F}$, then

1. $[T_1 + T_2]_{B,C} = [T_1]_{B,C} + [T_2]_{B,C}$.
2. $[\alpha T_1]_{B,C} = \alpha [T_1]_{B,C}$.
3. Composition of linear maps is represented as matrix multiplication.

Proof of (3) : Suppose the ordered bases are:
$A = (u_1, \dots, u_n)$ for U; $B = (v_1, \dots, v_k)$ for V
and $C = (w_1, \dots, w_m)$ for W. Let $S : U \to V$; $T : V \to W$ be linear transformations. Let $[S]_{A,B}$ and $[T]_{B,C}$ be the matrices for S and T, resp. Let $u \in U$. It has the coordinate vector $[u]_A$.
Then, $[S(u)]_B = [S]_{A,B}[u]_A$.
$[T(S(u))]_C = [T]_{B,C}[S(u)]_B = [T]_{B,C}[S]_{A,B}[u]_A$.
That is, $[T \circ S]_{A,C} = [T]_{B,C}[S]_{A,B}$ $\square$

Assignment-5

1. In each of the following determine whether $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation:

   (a) $T(\alpha, \beta) = (1, \beta)$ (b) $T(\alpha, \beta) = (\alpha, \alpha^2)$

   (c) $T(\alpha, \beta) = (\sin \alpha, 0)$ (d) $T(\alpha, \beta) = (|\alpha|, \beta)$

   (e) $T(\alpha, \beta) = (\alpha + 1, \beta)$ (f) $T(\alpha, \beta) = (2\alpha + \beta, \alpha + \beta^2)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $T(1, 0) = (1, 4)$ and $T(1, 1) = (2, 5)$. What is $T(2, 3)$? Is T one-one?

3. In each of the following, determine whether T is a linear transformation:

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(1, 1) = (1, 0, 2)$ and $T(2, 3) = (1, -1, 4)$.

   (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1, 0, 3) = (1, 1)$ and $T(-2, 0, -6) = (2, 1)$.

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1, 1, 0) = (0, 0)$, $T(0, 1, 1) = (1, 1)$ and $T(1, 0, 1) = (1, 0)$.

Assignment-5

3.
   (d) $T : C^1[0,1] \to \mathbb{R}$ with $T(u) = \int_0^1 (u(t))^2\,dt$.
   (e) $T : C^1[0,1] \to \mathbb{R}^2$ with $T(u) = (\int_0^1 u(t)\,dt, u'(0))$.
   (f) $T : \mathcal{P}_3 \to \mathbb{R}$ with $T(a + bt^2) = 0$ for any $a, b \in \mathbb{R}$.
   (g) $T : \mathcal{P}_n(\mathbb{R}) \xrightarrow{onto} \mathbb{R}$ with $T(p(x)) = p(\alpha)$, for a fixed $\alpha \in \mathbb{R}$.
4. Let U, V be vector spaces with $\{u_1, \ldots, u_n\}$ a basis for U. Let $v_1, \ldots, v_n \in V$. Show that
   (a) There exists a unique linear transformation $T : U \to V$ with $T(u_i) = v_i$ for $i = 1, 2, \ldots, n$.
   (b) This T is one-one iff $\{v_1, \ldots, v_n\}$ is linearly independent.
   (c) This T is onto iff $\text{span}\{v_1, \ldots, v_n\} = V$.

Assignment-5

5. Construct an isomorphism between the spaces $\mathbb{F}^{n+1}$ and $\mathcal{P}_n(\mathbb{F})$.
6. Let V_1 and V_2 be finite dimensional vector spaces and $T : V_1 \to V_2$ be a linear transformation. Give reasons for the following:
   (a) rank $T \leq \dim V_1$.
   (b) T is onto implies $\dim V_2 \leq \dim V_1$.
   (c) T is one-one implies $\dim V_1 \leq \dim V_2$.
   (d) $\dim V_1 > \dim V_2$ implies T is not one-one.
   (e) $\dim V_1 < \dim V_2$ implies T is not onto.
   (f) Suppose $\dim V_1 = \dim V_2$. Then, T is one-one if and only T is onto.

Assignment-5

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(\alpha, \beta, \gamma) = (\beta + \gamma, \gamma + \alpha, \alpha + \beta)$. Find $[T]_{E_1,E_2}$ where
   (a) $E_1 = \{(1, 0, 0), (0, 0, 1), (0, 1, 0)\}$,
   $E_2 = \{(0, 0, 1), (1, 0, 0), (0, 1, 0)\}$.
   (b) $E_1 = \{(1, 1, -1), (-1, 1, 1), (1, -1, 1)\}$,
   $E_2 = \{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$.
8. Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f) = f(2)$. Compute $[T]$ using the standard bases of the spaces.
9. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(a, b) = (a - b, a, 2b + b)$. Suppose B be the standard basis for $\mathbb{R}^2$, $C = \{(1, 2), (2, 3)\}$, and $D = \{(1, 1, 0), (0, 1, 1), (2, 2, 3)\}$. Compute $[T]_{B,D}$ and $[T]_{C,D}$.
10. Let $T : \mathcal{P}_2 \to \mathcal{P}_3$ be defined by $T(a + bt + ct^2) = at + bt^2 + ct^3$. If $E_1 = \{1 + t, 1 - t, t^2\}$ and $E_2 = \{1, 1 + t, 1 + t + t^2, t^3\}$, then what is $[T]_{E_1,E_2}$?

Assignment-5

11. Let $E_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, $E_2 = \{1,\ t,\ t^2\}$ and $E_3 = \{1\}$.

    (a) Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T(A) = A^t$. Compute $[T]_{E_1,E_1}$.

    (b) Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^{2\times 2}$ by $T(f) = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f'(3) \end{bmatrix}$. Compute $[T]_{E_2,E_1}$.

    (c) Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}$ by $T(A) = tr(A)$. Compute $[T]_{E_1,E_3}$.

Assignment-5

12. Given bases $E_1 = \{1 + t,\ 1 - t,\ t^2\}$ and $E_2 = \{1,\ 1 + t,\ 1 + t + t^2,\ t^3\}$ for $\mathcal{P}_2(\mathbb{R})$ and $\mathcal{P}_3(\mathbb{R})$, respectively, and the linear transformation $S : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ with $S(p(t)) = t\,p(t)$, find the matrix $[\,S\,]_{E_1,E_2}$.
13. Let $E_1 = \{u_1, \ldots, u_n\}$ and $E_2 = \{v_1, \ldots, v_m\}$ be bases of V_1 and V_2, respectively. Let $T : V_1 :\to V_2$ be a linear transformation. Show that T is one-one iff columns of $[T]_{E_1,E_2}$ are linearly independent iff $det[T]_{E_1,E_2} \neq 0$.

Kernel & Range

Definition: Let V and W be vector spaces and $T : V \to W$ be a linear transformation.

1. Kernel of $T = N(T) = \{v \in V \ : \ T(v) = 0\}$.
2. Range of
   $T = R(T) = \{w \in W \ : \ w = T(v) \text{ for some } v \in V\}$.

Kernel is also called the Null Space, and Range as Range Space.

Theorem: Let $T : V \to W$ be a linear transformation. Then

1. $N(T) \neq \emptyset, \ R(T) \neq \emptyset$.
2. $N(T)$ is a subspaces of V and $R(T)$ is a subspace of W.

Proof: $T(0) = 0$. This proves (1).
Let $u, \ v \in N(T)$ and $\alpha \in \mathbb{F}$. Then
$T(u + \alpha v) = T(u) + \alpha T(v) = 0 \ \Rightarrow \ u + \alpha v \in N(T)$
Therefore, $N(T)$ is a subspace of V.
If $x, \ y \in R(T)$ and $\alpha \in \mathbb{F}$, then there exist $u, v \in V$ such that $T(u) = x$ and $T(v) = y$. Then,
$T(u + \alpha v) = T(u) + \alpha T(v) = x + \alpha y \ \Rightarrow \ x + \alpha y \in R(T)$
Therefore, $R(T)$ is a subspace of W.

An Example

Since these are vector spaces,
$N(T)$ is also called the null space of T and
$R(T)$ is also called the range space of T.

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
$T(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_3)$. Find a basis for $N(T)$ and a basis for $R(T)$.

Solution: $T(x_1, x_2, x_3) = (0, 0) \;\Rightarrow\; x_2 = -x_1$ and $x_3 = x_1$.
Therefore
$N(T) = \{(x_1, x_2, x_3) \;:\; x_1 = -x_2 = x_3\} = \text{span}\{(1, -1, 1)\}$.

$T(0, a, -b) = (0 + a, 0 - (-b)) = (a, b)$. Thus, $R(T) = \mathbb{R}^2$.

Exercise: Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 0)$. Find $R(T)$ and $N(T)$.

Rank & Nullity

Definition: Let $T : V \to W$ be a linear transformation.
Nullity of $T = \text{null}(T) = \dim N(T)$.
Rank of $T = \text{rank}(T) = \dim R(T)$.

Theorem: Let V, W be vector spaces. Let $\{v_1, \dots, v_n\}$ be a basis fof V. Let $T : V \to W$ be a linear transformation. Then

1. T is one-one if and only if $N(T) = \{0\}$ iff $\text{null}(T) = 0$.
2. T is one-one iff $\{T(v_1), \dots, T(v_n)\}$ is linearly independent.
3. T is onto iff $\text{span}(\{T(v_1), \dots, T(v_n)\}) = W$.

Proof: (1) Assume that T is one-one. Let $T(x) = 0$. Then
$T(x) = T(0) \Rightarrow x = 0 \Rightarrow N(T) = \{0\}$.

Conversely, suppose $N(T) = \{0\}$. Let $T(x) = T(y)$.
Then $T(x) - T(y) = 0 \Rightarrow T(x - y) = 0 \Rightarrow x - y \in N(T)$.
Since $N(T) = \{0\}$, $x = y \Rightarrow T$ is one-one.

Proof Contd.

(2) Let T be one-one. Then $N(T) = \{0\}$.
Suppose $\alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) = 0$.
Then $T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = 0$.
As $N(T) = \{0\},\ \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.
Since $\{v_1, \ldots, v_n\}$ is linearly independent, each $\alpha_i = 0$.
Thus, $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.

Conversely, suppose $\{T(v_1), \ldots, T(v_n)\}$ is linearly independent.
Let $T(x) = T(y)$. We have $x = \sum(\alpha_i v_i),\ y = \sum(\beta_i v_i)$ for some $\alpha_i, \beta_i \in \mathbb{F}$.
Then $(\alpha_1 - b_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n = 0$.
Thus, each $\alpha_i = \beta_i \quad \Rightarrow \quad x = y$.
That is, T is one-one. $\square$

Proof Contd.

(3) We show that $R(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}$.
For this, let $w \in R(T)$. Then
$w = T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$.
Since T is linear, $w = \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n)$.
That is, $w \in \text{span}\{T(v_1), \ldots, T(v_n)\}$.

Conversely, if $z \in \text{span}\{T(v_1), \ldots, T(v_n)\}$, then
$z = \beta_1 T(v_1) + \cdots + \beta_n T(v_n)$, for some $\beta_1, \ldots, \beta_n \in \mathbb{F}$.
As T is linear, $z = T(\beta_1 v_1 + \cdots + \beta_n v_n) \in R(T)$.

Hence, $R(T) = \text{span}\{T(v_1), \ldots, T(v_n)\}$.
Now, the statement follows from the observation that
T is an onto map iff $R(T) = W$. $\square$

Caution: The vectors $T(v_1), \ldots, T(v_n)$ need not be linearly independent even though $\{v_1, \ldots, v_n\}$ is a basis of V.

An Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (x_1 - x_2, 2x_1 + x_2)$. Show that T is bijective.

Solution: Suppose $T(x_1, x_2) = (0, 0)$. Then $x_1 = x_2$ and $2x_1 = -x_2$. This implies that $x_1 = x_2 = 0$.
Therefore $N(T) = \{0\}$. Hence T is one-one.

$\{(1, 0), (0, 1)\}$ is a basis of $\mathbb{R}^2$.
$T(1, 0) = (1, 2)$ and $T(0, 1) = (-1, 1)$.
Now, $\text{span}(\{(1, 2), (-1, 1)\}) = \mathbb{R}^2$.
So, T is an onto map.
Therefore T is a bijective linear transformation.

Note: Since $\{(1, 2), (-1, 1)\}$ is a basis of $\mathbb{R}^2$, T is bijective.

Definition: A bijective linear transformation is called an isomorphism. If there exists an isomorphism from one vector space to the other, we say that the spaces are isomorphic to each other.

Isomorphism

Theorem: Let $T : V \to W$ be an isomorphism. Then $T^{-1} : W \to V$ is also an isomorphism. Moreover with bases A, B for V, W, resp., we have $[T^{-1}]_{B,A} = ([T]_{A,B})^{-1}$.

Proof: Let $w_1, w_2 \in W$. Then there exists $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$.

Therefore $T(v_1 + v_2) = w_1 + w_2$
$\Rightarrow \; T^{-1}(w_1 + w_2) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$.
Similarly,
$T^{-1}(\alpha w_1) = T^{-1}(\alpha T(v_1))$
$= T^{-1}(T(\alpha v_1)) = \alpha v_1 = \alpha T^{-1}(w_1)$.
The other statement is proved by taking composition of T and T^{-1}. $\square$

Rank-Nullity Theorem

Theorem: Let V be a finite dimensional vector space and $T : V \to W$ be a linear transformation. Then

$$\text{rank}(T) + \text{null}(T) = \dim(V).$$

Proof: We need to show that $\dim R(T) + \dim N(T) = \dim V$. For this, we will produce a basis of V consisting of a basis of $N(T)$ and inverse images of a basis of $R(T)$.

Let $\{v_1, \ldots, v_k\}$ be a basis of $N(T)$, for some $k \geq 0$. Note that if T is one-one, then $k = 0$ which means the set is empty.

Also, if T is the zero map, then $N(T) = V$ and $R(T) = \{0\}$. This implies that $k = n$ and hence the theorem is true.

Proof Contd.

Assume that T is not the zero map. Extend the basis of $N(T)$ to a basis of V, say $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$.

If we prove that $\{T(v_{k+1}), \ldots, T(v_n)\}$ is a basis of $R(T)$, then the theorem is proved.

Let $w \in R(T)$. Then there exists a $v \in V$ such that $T(v) = w$. Let $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Then

$$\begin{aligned} T(v) &= \alpha_1 T(v_1) + \cdots + \alpha_k T(v_k) + \alpha_{k+1} T(v_{k+1}) + \cdots + \alpha_n T(v_n) \\ &= \alpha_{k+1} T(v_{k+1}) + \cdots + \alpha_n T(v_n) \\ &\in \text{Span}(\{T(v_{k+1}), \ldots, T(v_n)\}). \end{aligned}$$

Therefore $R(T) = \text{Span}(\{T(v_{k+1}), \ldots, T(v_n)\})$.

Proof Contd.

Suppose $\beta_{k+1} T(v_{k+1}) + \cdots + \beta_n T(v_n) = 0$.

$\Rightarrow\ T(\beta_{k+1} v_{k+1} + \cdots + \beta_n v_n) = 0$.

$\Rightarrow\ \beta_{k+1} v_{k+1} + \cdots + \beta_n v_n \in N(T)$.

$\Rightarrow\ \beta_1 v_1 + \cdots + \beta_k v_k - \beta_{k+1} v_{k+1} - \cdots - \beta_n v_n = 0$.

$\Rightarrow\ \beta_i = 0$ for all $i = 1, \ldots, n$

$\Rightarrow\ \{T(v_{k+1}), \ldots, T(v_n)\}$ is linearly independent.

$\Rightarrow$ dim $R(T) = n - k =$ dim $V -$ dim $N(T)$. $\square$

Some interesting consequences

1. There does not exist a one-one map $T : \mathbb{R}^2 \to \mathbb{R}$ (more generally from $\mathbb{R}^m$ to $\mathbb{R}^n$ for any $m > n$).
2. There does not exists an onto map $T : \mathbb{R} \to \mathbb{R}^2$ (more generally from $\mathbb{R}^m$ to $\mathbb{R}^n$ for any $m < n$).
3. If $\dim V = \dim W$ and $T : V \to W$, then T is one-one if and only if T is onto.

Definition: Two vector spaces V and W are isomorphic if there exists an isomorphism $T : V \to W$.

Examples:

1. $T : \mathcal{P}_n \to \mathbb{R}^{n+1}$ defined by
$T(a_0 + a_1 t + \cdots + a_n t^n) = (a_0, \ldots, a_n)$ is an isomorphism.

2. $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by
$T(a_1, \ldots, a_n) = (a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_n)$ is an isomorphism.

Isomorphism & Dimension

We have seen that if $T : V \to W$ is an isomorphism, then $\dim V = \dim W$.

Is the converse true? i.e., if $\dim V = \dim W$, are they isomorphic?

Let $B_1 = \{v_1, \dots, v_n\}$ be a basis of V and $B_2 = \{w_1, \dots, w_n\}$ be a basis of W.
Define $T : V \to W$ by
$T(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 w_1 + \cdots + \alpha_n w_n$.

Since B_1 and B_2 spans V and W respectively, T is onto.
Since B_1 and B_2 are linearly independent, T is one-one.
Therefore T is an isomorphism.

Theorem: V and W are isomorphic if and only if $\dim V = \dim W$.

Canonical Basis Isomorphism

Definition: Let V, W be $\mathbb{F}-$vector spaces. A one-one linear transformation from V onto W is called an isomorphism.

If $\dim(V) = n$, then there exists an isomorphism from V to $\mathbb{F}^n$. Given an ordered basis $(v_1, v_2, \ldots, v_n)$ of V, each $v \in V$ is associated with its coordinate vector in $\mathbb{F}^n$.

Clue: $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

There can be many isomorphisms. But we have one special isomorphism. It maps v to its co-ordinate vector.

The map $\phi_{v_1,\ldots,v_n} : V \to \mathbb{F}^n$ with $\phi(v) = [v]_{e_1,\ldots,e_n}$

is called the canonical basis isomorphism between V and $\mathbb{F}^n$.

The Diagram

Let $B = (v_1, \ldots, v_n)$ be an ordered basis of V.
Let $C = (w_1, \ldots, w_m)$ an ordered basis of W.
Have standard bases for $\mathbb{F}^n$ and $\mathbb{F}^m$.
The canonical basis isomorphisms are:
$\phi_{v_1,\ldots,v_n}$ from V to $\mathbb{F}^n$ and
$\phi_{w_1,\ldots,w_m}$ from W to $\mathbb{F}^m$.
Let $T : V \to W$ be a linear transformation and
$[T]_{B,C}$ be its matrix representation. Then

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_{v_1,\ldots,v_n}"', "\simeq"] & W \arrow[d, "\simeq"', "\phi_{w_1,\ldots,w_m}"] \\ F^n \arrow[r, "{[T]_{B,C}}"'] & F^m \end{tikzcd}$$

This means $T = \phi^{-1}_{w_1\ldots w_m} \circ [T]_{B,C} \circ \phi_{v_1,\ldots,v_n}$.

Linear Functionals

Definition: Let V be a vector space over $\mathbb{F}$. A linear transformation $T : V \to \mathbb{F}$ is called a linear functional.
A linear transformation from V to V is called a linear operator.

Examples of Functionals:

1. $f_1 : \mathbb{R}^n \to \mathbb{R}$ defined by $f_1(x_1, \ldots, x_n) = x_1$ is a linear functional.

2. $f_j : \mathbb{R}^n \to \mathbb{R}$ defined by $f_j(x_1, \ldots, x_n) = x_j$ is a linear functional.

3. Let $(v_1, \ldots, v_n)$ be an ordered basis for a vector space V. Define $f_j : V \to \mathbb{F}$ by $f_j(v) = \alpha_j$ when $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$. Verify that f_j is a linear functional. These functionals are called co-ordinate functionals with respect to the given basis.

4. Let $\tau \in [a, b]$ be fixed. Let $f_\tau : C[a, b] \to \mathbb{F}$ be defined by $f_\tau(x) = x(\tau),$ for $x \in C[a, b]$. Then f_τ is a linear functional.

Examples Contd.

5. Fix points $\tau_1, \ldots, \tau_n$ in $[a, b]$, and scalars $\alpha_1, \ldots \alpha_n$ in $\mathbb{F}$. Let $f : C[a, b] \to \mathbb{F}$ be defined by
$f(x) = \sum_{i=1}^{n} \alpha_i x(\tau_i)$ for $x \in C[a, b]$.
Then f is a linear functional.

6. Define $T : C[a, b] \to \mathbb{F}$ by $f(x) = \int_a^b x(t)\, dt,$ for $x \in C[a, b]$.
Then f is a linear functional.

Theorem: The set of all linear transformations from a vector space V to a vector space W is a vector space with usual addition of functions and multiplication of a function with a scalar.

Definition: The space of all linear transformations from a vector space V to a vector space W is denoted by $\mathcal{L}(V, W)$. The space of linear functionals on V is denoted by V', and is called the dual space of V.

Dimension of $\mathcal{L}(V, W)$

Theorem: If $\dim V = n$, $\dim W = m$, then $\dim \mathcal{L}(V, W) = mn$.
Therefore, $\dim V' = n$.

Due to the canonical basis isomorphisms, (Remember the diagram?) $\mathcal{L}(V, W)$ is isomorphic to $\mathbb{F}^{m \times n}$. $\square$

Alternative proof: Fix two ordered bases:
$\{v_1, \ldots, v_n\}$ for V and $\{w_1, \ldots, w_m\}$ for W.
Define linear transformations
$T_{ij} : V \to W$ for $i = 1, \ldots, n,\ j = 1, \ldots, m$ by
$T_{ij}(v_i) = w_j,\ T_{ij}(\text{any other } v_k) = 0$.
Now, show that the set of all these T_{ij} is a basis of $\mathcal{L}(V, W)$. $\square$

In particular, a basis for V' now looks like the set of functionals
$f_i : V \to \mathbb{F}$, where $f_i(v_i) = 1,\ f_i(\text{any other } v_k) = 0$.
This basis is called the dual basis.

How does a matrix of a linear functional look like?
And how are the co-ordinate vectors of linear functionals in the dual basis?

System of Linear Equations

Definition: Let $A \in \mathbb{F}^{m\times n}, b \in \mathbb{F}^m$. Then

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

is called a system of linear equations for the unknowns $x_1, \ldots, x_n$ with coefficients in $\mathbb{F}$.
If the b_i are all zero, the system is said to be homogeneous.

We consider the matrix A as a linear transformation $A : \mathbb{F}^n \to \mathbb{F}^m$ and the system is written as $Ax = b$.

The solution set of the system $Ax = b$ is $Sol(A, b) = \{x \in \mathbb{F}^n : Ax = b\}$.

The system $Ax = b$ is *solvable* if $Sol(A, b) \neq \emptyset$.

Augmented Matrix

Theorem: $Ax = b$ is solvable iff $\text{rank}(A) = \text{rank}[A|b]$.

Proof: $Ax = b$ is solvable iff $b = Ax$ for some $x \in \mathbb{F}^n$ iff $b \in R(A)$.

Let $b \in R(A)$. As $R(A) = \mathit{span}(Ae_i) = \mathit{span}$ of columns of A,

$\text{rank}(A)$ does not change if b is added to the set of column vectors of A. Then $\text{rank}(A) = \text{rank}[A|b]$.

Conversely, suppose $\text{rank}(A) = \text{rank}[A|b]$. The columns of $[A|b]$ generate the subspace, call it U, containing columns of A and b. $R(A)$ is a subspace of this space U.
But $R(A)$ and U has now the same dimension.

Hence, $U = R(A)$. That is, $b \in R(A)$. $\square$

Translates of $N(A)$

Theorem: Let $x_0 \in \mathbb{F}^n$ be a solution of $Ax = b$. Then $Sol(A, b) = x_0 + N(A) = \{x_0 + x : x \in N(A)\}$.

Proof: If $x \in N(A)$, then $A(x_0 + x) = Ax_0 + Ax = Ax_0 = b$. That is, $x_0 + x \in Sol(A, b)$.

If $v \in Sol(A, b)$ and $Ax_0 = b$, then $A(v - x_0) = Av - Ax_0 = b - b = 0$.

That is, $v - x_0 \in N(A)$.
Or that $v - x_0 \in N(A)$. Then, $v \in x_0 + N(A)$. $\square$

Corollaries

1. If x_0 is a solution of $Ax = b$ and $\{v_1, \ldots, v_r\}$ is a basis for $N(A)$, then
$Sol(A, b) = \{x_0 + \lambda_1 v_1 + \cdots + \lambda_r v_r : \lambda_i \in \mathbb{F}\}$.
Here, $r = \text{null}(A) = n - \text{rank}(A)$.
2. A solvable system $Ax = b$ is uniquely solvable
iff $N(A) = 0$
iff $\text{rank}(A) = n$.
3. If A ia a square matrix, then
$Ax = b$ is uniquely solvable iff $det(A) \neq 0$.

Assignment-6

1. In the following, prove that T is a linear transformation. Determine rank T and null T by finding bases for $R(T)$ and $N(T)$.
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2;\; T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3;\; T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.
   (c) $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{2\times 2}$;
   $$T\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{bmatrix}.$$
   (d) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R});\; T(f(x)) = xf(x) + f'(x)$.
   (e) $T : \mathbb{R}^{n\times n} \to \mathbb{R};\; T(A) = tr(A)$.

Assignment-6

2. Give an example for each of the following:
   (a) A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $N(T) = R(T)$.
   (b) Distinct linear transformations T, U with $N(T) = N(U)$ and $R(T) = R(U)$.
3. Let V be a non-trivial real vector space. Let $T : V \to \mathbb{R}$ be a non-zero linear map. Prove or disprove:
   T is onto if and only if null $T = \dim V - 1$.
4. Let U, V be finite dimensional real vector spaces, and $T : U \to V$ linear. Prove or disprove:
   (a) rank $T \leq \dim U$.
   (b) If T is onto, then $\dim V \leq \dim U$.
   (c) If T is one-one, then $\dim U \leq \dim V$.
   (d) If $\dim U = \dim V$, then "T is one-one iff T is onto".

Assignment-6

5. Let V be the vector space of real valued functions on $\mathbb{R}$ which have derivatives of all orders. Let $T : V \to V$ be the differential operator: $Tx = x'$. What is $N(T)$?
6. Let $T : V \to V$ be a linear operator such that $T^2 = T$. Let I denote the identity operator. Prove that $R(T) = N(I - T)$ and $N(T) = R(I - T)$.
7. Find bases for the null space $N(T)$ and the range space $R(T)$ of the linear transformation T in each of the following:
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ with $T(x_1, x_2) = (x_1 - x_2, 2x_2)$,
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(x_1, x_2) = (x_1 + x_2, 0, 2x_3 - x_2)$,
   (c) $T : \mathbb{R}^{n \times n} \to \mathbb{R}$ with $T(A) = tr(A)$.
8. Let the linear transformation $A : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ be defined by $A(p(t)) = t\,p(t) + \dfrac{dp(t)}{dt}$. Find $N(A)$ and $R(A)$.
9. Let $B : V \to W$ be a linear transformation, where V and W are real vector spaces with dim $W <$ dim $V < 2013$. Show that B cannot be one-one.

Assignment-6

10. Let $M = (a_{ij})$ be an $m \times n$ matrix with $a_{ij} \in \mathbb{F}$ and $n > m$. Show that there exists $(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}^n$ such that $a_{i1}\alpha_1 + a_{i2}\alpha_2 + \cdots + a_{in}\alpha_n = 0, \;$ for all $i = 1, \ldots, m$. Interpret the result for linear systems.
11. Let $A \in \mathbb{F}^{m \times n}$ have columns $A_1, \ldots, A_n$. Let $b \in \mathbb{F}^m$. Show the following:
    (a) The equation $Ax = 0$ has a non-zero solution if and only if $A_1, \ldots, A_n$ are linearly dependent.
    (b) The equation $Ax = b$ has at least one solution if and only if $b \in \text{span}\{A_1, \ldots, A_n\}$.
    (c) The equation $Ax = b$ has at most one solution if and only if $A_1, \ldots, A_n$ are linearly independent.
    (d) The equation $Ax = b$ has a unique solution if and only if $\text{rank}\, A = \text{rank}[A|b] = n =$ number of unknowns.

Assignment-6

12. Consider the system of linear equations: $x_1 - x_2 + 2x_3 - 3x_4 = 7$, $4x_1 + 3x_3 + x_4 = 9$, $2x_1 - 5x_2 + x_3 = -2$, $3x_1 - x_2 - x_3 + 2x_4 = -2$. By determining ranks, decide whether the system has a solution.
13. Prove: If U is a subspace of $\mathbb{F}^n$ and $x \in \mathbb{F}^n$, then there exists a system of linear equations having n equations and n unknowns, with coefficients in $\mathbb{F}$, whose solution set equals $x + U$.

Geometry in Vector Spaces?

To study geometrical problems, in which lengths and angles play a role, we need additional structures in a vector space. For example, in $\mathbb{R}^2$ and $\mathbb{R}^3$, we have scalar product of vectors.

In $\mathbb{R}^2$, we have length as $\|x\| = x \cdot x$ and $\cos(\text{angle}(x, y)) = \frac{x \cdot y}{\|x\| \, \|y\|}$.

Let V be a vector space over $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$. We define *inner product* in a vector space V by accepting some of the fundamental properties of the scalar product in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Inner Product

Definition: An inner product on a vector space V is a map $(x, y) \rightsquigarrow \langle x, y \rangle$ which associates a pair of vectors in V to a scalar $\langle x, y \rangle$ satisfying

(a) $\langle x, x \rangle \geq 0$ for each $x \in V$.
(b) $\langle x, x \rangle = 0$ iff $x = 0$.
(c) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ for all $x, y, z \in V$.
(d) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$ for each $\alpha \in \mathbb{F}$ and for all $x, y \in V$.
(e) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in V$.

Examples:

1. The scalar product on $\mathbb{R}^2$ and also on $\mathbb{R}^3$ are inner products.

2. For $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{F}^n$,

$$\langle x, y \rangle = \sum_{j=1}^{n} x_j \overline{y_j}$$

This inner product is called the standard inner product on $\mathbb{F}^n$.

Examples

3. Let V be a vector space. Let $B = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for V. Let $x = \sum_{i=1}^{n} \alpha_i u_i$ and $y = \sum_{i=1}^{n} \beta_i u_i$.
Define $\langle x, y \rangle_B = \sum_{i=1}^{n} \alpha_i \overline{\beta_i}$.
This is an inner product on V.

4. Let V be a vector space with $dim(V) = n$. Let $T : V \to \mathbb{F}^n$ be a bijective linear transformation. Then

$$\langle x, y \rangle_T = \langle Tx, Ty \rangle$$

is an inner product on V. Here, $\langle \cdot \rangle$ on the right hand side denotes an inner product on $\mathbb{F}^n$.

Examples

5. Let $t_1, t_2, \ldots, t_{n+1}$ be distinct real numbers. For any $p, q \in \mathcal{P}_n$, define $\langle p, q\rangle = \sum_{i=1}^{n+1} p(t_i)\overline{q(t_i)}$.
This is an inner product on $\mathcal{P}_n$.

6. For $f, g \in C[a, b]$, take $\langle f, g\rangle = \int_a^b f(t)\overline{g(t)}\, dt$. This is an inner product on $C[a, b]$.

7. In all the above examples, consider $\mathbb{R}$ as the underlying scalar field and remove the overline from the definition of inner products. Then the resulting function $\langle \cdot \rangle$ is an inner product on the corresponding vector space.

Some Properties of Inner Products

A vector space with an inner product on it is called an **inner product space** (ips).

Theorem: Let V be an ips. For all $x, y, z \in V$ and for all $\alpha \in \mathbb{F}$,
$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle, \qquad \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle.$

Proof: $\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle}$

$$= \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle.$$

$$\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \overline{\alpha \langle y, x \rangle}$$

$$= \overline{\alpha}\, \overline{\langle y, x \rangle} = \overline{\alpha} \langle x, y \rangle.$$

$\square$

Length or Norm

Definition: Let V be an ips. For any $x \in V$, the length of x, also called the norm of x is $\|x\| = \sqrt{\langle x, x \rangle}$.

For any $x \in V$, $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.

Theorem: Let $x, y \in V$, an ips. The parallelogram law holds: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

Proof:
$\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$.
Complete the proof. $\square$

Cauchy-Schwartz Inequality

Theorem: Let $x, y \in V$, an ips. Then $|\langle x, y\rangle| \leq \|x\|\,\|y\|$.
Further, $|\langle x, y\rangle| = \|x\|\,\|y\|$ iff $\{x, y\}$ is linearly dependent.

Proof: If $y = 0$, then obvious. Assume $y \neq 0$. Set $\alpha = \frac{\langle x,y\rangle}{\langle y,y\rangle}$.
Now, $0 \leq \|x - \alpha y\|^2 = \langle x - \alpha y, x - \alpha y\rangle$

$= \langle x, x\rangle - \langle x, \alpha y\rangle - \alpha\langle y, x\rangle + \alpha\langle y, \alpha y\rangle$

$= \langle x, x\rangle - \bar{\alpha}\langle x, y\rangle - \alpha[\langle y, x\rangle - \bar{\alpha}\langle y, y\rangle]$ $[\bar{\alpha} = \frac{\langle y,x\rangle}{\langle y,y\rangle}]$

$$= \|x\|^2 - \frac{\langle y, x\rangle}{\langle y, y\rangle}\langle x, y\rangle = \|x\|^2 - \frac{|\langle x, y\rangle|^2}{\|y\|^2}$$

$\Rightarrow |\langle x, y\rangle| \leq \|x\|\,\|y\|$.

Next, equality holds iff $y = 0$ or $x = \frac{\langle x,y\rangle}{\langle y,y\rangle}\, y$. $\square$

Triangle Inequality and Angle

Theorem: *(Trinagle Inequality)* For all x, y in an ips,

$$\|x + y\| \le \|x\| + \|y\|.$$

Proof: $\|x + y\|^2 = \langle x + y, x + y\rangle$
$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2$
$\le \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2 \le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$
$= (\|x\| + \|y\|)^2.$ $\square$

Definition: Let $x, y \in V$, an ips. The acute angle between x and y is denoted by $\theta(x, y)$, and is defined by
$\cos\theta(x, y) = \dfrac{|\langle x, y\rangle|}{\|x\|\,\|y\|}.$

Orthogonality

Definition: Let $x, y \in V$, an ips. The vector x is orthogonal to y, i.e., $x \perp y$ iff $\langle x, y \rangle = 0$.

If $x \perp y$, then clearly, $y \perp x$.

Examples:

1. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Then $e_i \perp e_j$ whenever $i \neq j$.

2. In $C[0, 2\pi]$, define $\langle f, g \rangle = \int_0^{2\pi} f(t)g(t)dt$. Since $\int_0^{2\pi} \cos mt \, \sin nt \, dt = 0$ for $m \neq n$,

$$\cos mt \perp \sin nt, \text{ whenever } m \neq n.$$

It follows that (a) If $x \perp y$, then $y \perp x$. (b) $0 \perp x$ for every x.

Pythagoras

Theorem: *(Pythagoras)* Let V be an ips. Let $x, y \in V$.
(a) If $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.
(b) Suppose V is a real vector space.
If $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then $x \perp y$.

Proof: (a) $\langle x + y, x + y \rangle = \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$.
Since $x \perp y$, both $\langle x, y \rangle = 0 = \langle y, x \rangle$.
(b) Let V be a real vector space. Then $\langle x, y \rangle = \langle y, x \rangle$.
If $\|x + y\|^2 = \|x\|^2 + \|y\|^2$, then $\langle x, y \rangle = 0$. $\square$

Example: Take $V = \mathbb{C}$, a complex ips with $\langle x, y \rangle = x\overline{y}$, as usual. Now,
$\|1 + i\|^2 = (1 + i)\overline{(1 + i)} = 1 + 1 = \|1\|^2 + \|i\|^2$.
But $\langle 1, i \rangle = 1 \times (-i) = -i \neq 0$.

Orthogonal Set

Definition: Let V be an ips, $S \subseteq V$, and $x \in V$.
(a) $x \perp S$ iff for each $y \in S,\ x \perp y$.
(b) $S^\perp = \{x \in V : x \perp S\}$.
(c) S is called an orthogonal set when
$x, y \in S,\ x \neq y$ implies $x \perp y$.

Example: Let V be an ips. Then
(a) $V^\perp = \{0\}$.
(b) $\{0\}^\perp = V$.
(c) If S is a superset of some basis for V, then $S^\perp = \{0\}$.

For (c), let $x \in V$. Then, $x = \sum_{i=1}^{n} \alpha_i u_i$ for some n, some $\alpha_i \in \mathbb{F}$ and for some $u_i \in S$. If $y \perp S$, then $y \perp x$ as well. That is, $y \perp V$.

Orthonormal Sets

Definition: Let V be an ips. A set $S \subseteq V$ is called an orthonormal set if S is orthogonal and $\|x\| = 1$ for each $x \in S$. In addition, if V is finite dimensional, then an orthonormal set S is called an orthonormal basis provided S is also a basis for V.

Example: (a) The standard basis of $\mathbb{R}^n$ is an orthonormal basis of it.

(b) The set of functions $\{\cos mt : m \in \mathbb{N}\}$ in the real ips $C[0, 2\pi]$ with inner product as $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)\,dt$ is an orthogonal set. But $\int_0^{2\pi} \cos^2 t\,dt \neq 1$. Hence, it is not an orthonormal set.

(c) However, $\{(\cos mt)/\sqrt{\pi} : m \in \mathbb{N}\}$ is an orthonormal set in $C[0, 2\pi]$.

Linear Independence

Theorem: Every orthogonal set of non-zero vectors is linearly independent. Every orthonormal set is linearly independent.

Proof. Let S be an orthogonal set in an ips V.

For $n \in \mathbb{N},\ v_i \in S,\ \alpha_i \in \mathbb{F}$, suppose $\sum_{i=1}^{n} \alpha_i v_i = 0$.

Then for each $j \in \{1, \ldots, n\}$,

$$0 = \langle \sum_{i=1}^{n} \alpha_i v_i, v_j \rangle = \sum_{i=1}^{n} \alpha_i \langle v_i, v_j \rangle = \alpha_j \langle v_j, v_j \rangle.$$

Since $v_j \neq 0,\ \alpha_j = 0$. $\square$

A Result

Theorem: Let $S = \{u_1, u_2, \ldots, u_n\}$ be an orthonormal set in an ips V. Let $x \in span(S)$. Then

$$x = \sum_{j=1}^{n} \langle x, u_j \rangle u_j \quad \text{and} \quad \|x\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2.$$

Proof: $x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$. Then $\langle x, u_j \rangle = \alpha_j$.

Next, $\|x\|^2 = \langle \sum_j \alpha_j u_j, \sum_i \alpha_i u_i \rangle = \sum_j \sum_i \alpha_j \overline{\alpha_i} \langle u_j, u_i \rangle$

$$= \sum_j \alpha_j \overline{\alpha_j} = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2. \qquad \square$$

Corollary: *(Fourior Expansion and Parseval's Identity)*
Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for an ips V. Let $x \in V$. Then

$$x = \sum_{j=1}^{n} \langle x, v_j \rangle v_j \quad \text{and} \quad \|x\|^2 = \sum_{j=1}^{n} |\langle x, v_j \rangle|^2.$$

One More Corollary

Theorem: *(Bessel's Inequality)* Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal set in an ips V. Let $x \in V$. Then

$$\sum_{j=1}^{n} |\langle x, u_j \rangle|^2 \leq \|x\|^2.$$

Proof: Let $y = \sum_{j=1}^{n} \langle x, u_j \rangle \, u_j$ Then $\langle x, u_i \rangle = \langle y, u_i \rangle$.

That is, $x - y \perp u_i$, for each i. So, $x - y \perp y$.

By Pythagoras' theorem, $\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2$.

As $y \in span\{u_1, \ldots, u_n\}$, by Parseval's identitty,

$\|y\|^2 = \sum_{j=1}^{n} |\langle x, u_j \rangle|^2$. $\square$

Assignment-7

1. Check whether each of the following is an inner product on the given vector spaces.
   (a) $\langle x, y\rangle = x_1y_1$ for $x = (x_1, x_2),\ y = (y_1, y_2)$ on $V = \mathbb{R}^2$.
   (b) $\langle x, y\rangle = x_1y_1$ for $x = (x_1, x_2),\ y = (y_1, y_2)$ on $V = \mathbb{C}^2$.
   (c) $\langle f, g\rangle = \int_0^1 f'(t)g(t)\, dt$ on $V = \mathcal{P}$
2. Let B be a basis for a finite dimensional inner product space. Prove that if $\langle x, y\rangle = 0$ for all $x \in B$, then $y = 0$.
3. Let $A = (a_{ij}) \in \mathbb{R}^{2\times 2}$. For $x, y \in \mathbb{R}^{2\times 1}$, let $f_A(x, y) = y^tAx$. Show that f_A is an inner product on $\mathbb{R}^{2\times 1}$ if and only if $a_{12} = a_{21},\ a_{11} > 0,\ a_{22} > 0$, and $a_{11}a_{22} - a_{12}a_{21} > 0$.

Assignment-7

4. Let V be an inner product space, and let $x, y \in V$. Show the following:
   (a) $\|x\| \geq 0$.
   (b) $x = 0$ iff $\|x\| = 0$.
   (c) $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in \mathbb{F}$.
   (d) $\|x + \alpha y\| = \|x - \alpha y\|$ for all $\alpha \in \mathbb{F}$ iff $\langle x, y\rangle = 0$.
   (e) If $\|x + y\| = \|x\| + \|y\|$, then either $y = 0$ or $x = \alpha y$, for some $\alpha \in \mathbb{F}$.
5. Let V be an inner product space over $\mathbb{C}$. Prove that for all $x, y \in V$, $\mathrm{Re}\langle \mathrm{ix}, \mathrm{y}\rangle = -\mathrm{Im}\langle \mathrm{x}, \mathrm{y}\rangle$.
6. Let V_1 and V_2 be inner product spaces. Let $T : V_1 \to V_2$ be a linear transformation. Prove that
   for all $(x, y) \in V_1 \times V_2$,
   $\langle Tx, Ty\rangle = \langle x, y\rangle$ if and only if $\|Tx\| = \|x\|$.
   [Notice that both the inner products are denoted by $\langle\cdot, \cdot\rangle$.]