DEPARTMENT OF MATHEMATICS, I.I.T. MADRAS
MA 2030 Linear Algebra and Numerical Analysis

Problems Set - 1

1. Show that a set of positive real numbers forms a vector space under the operations defined by:
$x + y = xy$ and $\alpha x = x^\alpha$.

2. In each of the following parts (a),(b),(c) , a set V is given and some operations are defined. Check whether V is a vector space with these operations. Justify your answers.
(a) $V = \mathbb{R}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
$\alpha(a_1, a_2) = (0, 0)$ if $\alpha = 0$ and $\alpha(a_1, a_2) = (\alpha a_1, a_2/\alpha)$ if $\alpha \neq 0$.
(b) $V = \mathbb{C}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{C}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$
$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$.
(c) $V = \mathbb{R}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
$\alpha(a_1, a_2) = (a_1, 0)$

3. In each of the following a vector space V and a subset W is given. Check whether W is a subspace of V.
(a) $V = \mathbb{R}^2$; $W = \{(x_1, x_2) : x_2 = 2x_1 - 1\}$
(b) $V = \mathbb{R}^3$; $W = \{(x_1, x_2, x_3) : 2x_1 - x_2 - x_3 = 0\}$
(c) $V = C([0, 1])$; $W = \{f \in V : f$ is differentiable$\}$
(d) $V = C([-1, 1])$; $W = \{f \in V : f$ is an odd function$\}$
(e) $V = C([0, 1])$; $W = \{f \in V : f(x) \geq 0$ for all $x\}$
(f) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
(g) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
(h) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2, a_3 are integers.
(i) $V = \mathbb{P}_3$; W is the set of all polynomials of the form $a_0 + a_1x + a_2x^2$.

4. Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.

5. Let V be a vector space and W, A, B be subsets of V. Prove the following statements.

(a) W is a subspace of V if and only if $span(W) = W$.
(b) If $A \subseteq B$, then $span(A) \subseteq span(B)$.
(c) $span(A \bigcup B) = span(A) + span(B)$
(d) $span(A \bigcap B) \subseteq span(A) \bigcap span(B)$

6. Let W_1 and W_2 be subspaces of a vector space V. Prove that
(a) $W_1 \bigcap W_2$ and $W_1 + W_2$ are subspaces of V.
(b) $W_1 + W_2 = W_1$ if and only if $W_2 \subseteq W_1$
(c) $W_1 \bigcup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

7. Give an example of three linearly dependent vectors in $\mathbb{R}^2$ such that none of the three is a scalar multiple of another.

8. In each of the following , a vector space V and a set A of vectors in V is given. Determine whether A is linearly dependent and if it is, express one of the vectors in A as a linear combination of the remaining vectors.
(a) $V = \mathbb{R}^3$, $A = \{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$
(b) $V = \mathbb{R}^3$, $A = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$
(c) $V = \mathbb{R}^3$, $A = \{(1, -3, -2), (-3, 1, 3), (2, 5, 7)\}$
(d) $V = \mathbb{P}_3$, $A = \{x^2 - 3x + 5, \quad x^3 + 2x^2 - x + 1, \quad x^3 + 3x^2 - 1\}$
(e) $V = \mathbb{P}_3$, $A = \{-2x^3 - 11x^2 + 3x + 2, \quad x^3 - 2x^2 + 3x + 1, \quad 2x^3 + x^2 + 3x - 2\}$
(f) $V = \mathbb{P}_3$, $A = \{6x^3 - 3x^2 + x + 2, \quad x^3 - x^2 + 2x + 3, \quad 2x^3 + x^2 - 3x + 1\}$

(g) V is the set of all matrices of order 2×2, $A = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$

(i) V is the vector space of all real valued functions defined on $\mathbb{R}$. $A = \{2, \quad sin^2 x, \quad cos^2 x\}$
(j) V is same as in (i), $A = \{1, \quad sin x, \quad sin 2x\}$.
(k) V is same as in (i), $A = \{cos 2x, \quad sin^2 x, cos^2 x\}$.
(l) $V = C([-\pi, \pi]$, $A = \{sin x, \quad sin 2x, \ldots, sin nx\}$ where n is some natural number.