DEPARTMENT OF MATHEMATICS, I.I.T. MADRAS

MA 2030 Linear Algebra and Numerical Analysis

Problems Set - 2

1. Determine which of the following sets form bases for $\mathbb{P}_2$.
   (a) $\{-1 - x - 2x^2, \quad 2 + x - 2x^2, \quad 1 - 2x + 4x^2\}$
   (b) $\{1 + 2x + x^2, \quad 3 + x^2, \quad x + x^2\}$
   (c) $\{1 + 2x + 3x^2, \quad 4 - 5x + 6x^2, \quad 3x + x^2\}$

2. Do the polynomials $x^3 - 2x^2 + 1$, $4x^2 - x + 3$ and $3x - 2$ span $\mathbb{P}_3$? Justify your answer.

3. Suppose that V is a vector space with a basis $\{a, b, c\}$. Show that $\{a + b, b + c, c + a\}$ is also a basis for V.

4. Show that the set of all solutions of the system

$$x_1 - 2x_2 + x_3 = 0 \quad , \quad 2x_1 - 3x_2 + x_3 = 0$$

   is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

5. Suppose $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ are subsets of a vector space V such that A is linearly independent and $span(B) = V$. Show that $n \geq m$. Using this, show that any two bases of V have the same number of elements.

6. Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:
   (a) $W_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$
   (b) $W_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$
   (c) $W_3 = span(\{(1, -1, 0, 2, 1), (2, 1, -2, 0, 0), (0, -3, 2, 4, 2), (3, 3, -4, -2, -1), (2, 4, 1, 0, 1), (5, 7, -3, -2, 0)\})$

7. For each of the following matrix A, find a basis and dimension of the following subspaces: row space of A, column space of A, null space of $A := \{x : Ax = 0\}$, Range of $A := \{y : Ax = y \quad \text{for some} \quad x\}$.
   (a) $A = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 2 & 4 & -1 & 3 & 2 & -1 \end{bmatrix}$

8. Find a basis and dimension of the subspace $span(\{1 + x^2, -1 + x + x^2, -6 + 3x, 1 + x^2 + x^3, x^3\})$ of $\mathbb{P}_3$.

9. Find a basis and dimension of each of the following subspaces of the vector space V of all thrice differentiable functions:
(a) $W_1 = \{x \in V : x'' + x = 0\}$
(b) $W_2 = \{x \in V : x'' - 4x' + 3x = 0\}$
(c) $W_3 = \{x \in V : x''' - 6x'' + 11x' - 6x = 0\}$

10. Show that every linearly independent set in a finite dimensional vector space can be extended to a basis. Using this show that if W is a subspace of V, then $Dim(W) \leq Dim(V)$.

11. Extend the set $\{1 + x^2, 1 - x^2\}$ to a basis of $\mathbb{P}_3$

12. Let W be a proper subspace of $\mathbb{R}^3$. Show that W must be a line passing through the origin or a plane passing through the origin.

13. Let V be a vector space of dimension n. Show that
(a) every subset of V containing more than n vectors is linearly dependent.
(b) no subset of V containing less than n vectors can span V.

14. Let V be a vector space of dimension n and A be a subset of V containing n vectors. Show that
(a) if A is linearly independent, then A is a basis of V,
(b)if $span(A) = V$, then A is a basis of V.

15. If W_1 and W_2 are subspaces of a vector space V and $W_1 + W_2$ is finite dimensional, then show that
$Dim(W_1 + W_1) = Dim(W_1) + Dim(W_2) - Dim(W_1 \bigcap W_2)$
Guess and prove a similar formula for three subspaces.

16. Let V be the vector space of all 2×2 matrices with real entries. Let W_1 be the set of all matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of all matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.
(a) Prove that W_1 and W_2 are subspaces of V.
(b) Find dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \bigcap W_2$.

17. Find dimensions of $W_1 + W_2$ and $W_1 \bigcap W_2$ for the subspaces W_1, W_2 in problems 6 and 9.
