1. Answer the following questions with justification:
   (a) Is every subset of a linearly independent set linearly independent?
   (a) Is every subset of a linearly dependent set linearly dependent?
   (b) Is every superset of a linearly independent set linearly independent?
   (c) Is every superset of a linearly dependent set linearly dependent?
   (d) Is union of two linearly independent sets linearly independent?
   (e) Is union of two linearly dependent sets linearly dependent?
   (f) Is intersection of two linearly independent sets linearly independent?
   (g) Is intersection of two linearly dependent sets linearly dependent?
2. Give three vectors in $\mathbb{R}^2$ such that none of the three is a scalar multiple of another.
3. Suppose S is a set of vectors and some $v \in S$ is not a linear combination of other vectors in S. Is S lin. ind.?

4. In each of the following, a vector space V and $A \subseteq V$ are given. Determine whether A is linearly dependent and if it is, express one of the vectors in A as a linear combination of the remaining vectors.

(a) $V = \mathbb{R}^3$, $A = \{(1, 0, -1), (2, 5, 1), (0, -4, 3)\}$.

(b) $V = \mathbb{R}^3$, $A = \{(1, 2, 3), (4, 5, 6), (7, 8, 9)\}$.

(c) $V = \mathbb{R}^3$, $A = \{(1, -3, -2), (-3, 1, 3), (2, 5, 7)\}$.

(d) $V = \mathbb{P}^3$, $A = \{t^2 - 3t + 5,\ t^3 + 2t^2 - t + 1,\ t^3 + 3t^2 - 1\}$.

(e) $V = \mathbb{P}^3$,
$A = \{-2t^3 - 11t^2 + 3t + 2,\ t^3 - 2t^2 + 3t + 1,\ 2t^3 + t^2 + 3t - 2\}$.

(f) $V = \mathbb{P}^3$,
$A = \{6t^3 - 3t^2 + t + 2,\ t^3 - t^2 + 2t + 3,\ 2t^3 + t^2 - 3t + 1\}$.

(g) $V = \mathbb{F}^{2\times 2}$, $A = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$.

(h) $V = \{f : \mathbb{R} \to \mathbb{R}\}$, $A = \{2, \sin^2 t, \cos^2 t\}$.

(i) $V = \{f : \mathbb{R} \to \mathbb{R}\}$, $A = \{1, \sin t, \sin 2t\}$.

(j) $V = C([-\pi, \pi])$, $A = \{\sin t, \sin 2t, \dots, \sin nt\}$ where n is some natural number.

5. Show that two vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent if and only if $ad - bc \neq 0$.
6. Let $A = (a_{1j}) \in \mathbb{R}^{n\times n}$ and let $w_1, \ldots, w_n$ be the n columns of A. Let $\{u_1, \ldots, u_n\}$ be linearly independent in $\mathbb{R}^n$. Define vectors $v_1, \ldots, v_n$ by
$v_j = a_{1j}u_1 + \ldots + a_{nj}u_n$, for $j = 1, 2, \ldots, n$.
Show that $\{v_1, v_2, \ldots, v_n\}$ is linearly independent iff $\{w_1, w_2, \ldots, w_n\}$ is linearly independent.
7. Let A, B be subsets of a vector space V. Prove or disprove: $\text{span}(A) \cap \text{span}(B) = \{0\}$ iff $A \cup B$ is linearly independent.
8. Suppose V_1 and V_2 are subspaces of a vector space V such that $V_1 \cap V_2 = \{0\}$. Show that every $x \in V_1 + V_2$ can be written *uniquely* as $x = x_1 + x_2$ with $x_1 \in V_1$ and $x_2 \in V_2$.
9. Let $p_1(t) = 1 + t + 3t^2$, $p_2(t) = 2 + 4t + t^2$, $p_3(t) = 2t + 5t^2$. Are the polynomials p_1, p_2, p_3 linearly independent?
10. Prove that in the vector space of all real valued functions, the set of functions $\{e^x,\ xe^x,\ x^3e^x\}$ is linearly independent.

1. Determine which of the following sets form bases for $\mathcal{P}_2$.
   (a) $\{-1 - t - 2t^2, 2 + t - 2t^2, 1 - 2t + 4t^2\}$.
   (b) $\{1 + 2t + t^2, 3 + t^2, t + t^2\}$.
   (c) $\{1 + 2t + 3t^2, 4 - 5t + 6t^2, 3t + t^2\}$.
2. Let $\{x, y, z\}$ be a basis for a vector space V. Is $\{x + y, y + z, z + x\}$ also a basis for V?
3. Extend the set $\{1 + t^2, 1 - t^2\}$ to a basis of $\mathcal{P}_3$.
4. Find a basis for the subspace $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ of $\mathbb{R}^3$.
5. Is $\{1 + t^n, t + t^n, \ldots, t^{n-1} + t^n, t^n\}$ a basis for $\mathcal{P}_n$?
6. Let $u_1 = 1$ and let $u_j = 1 + t + t^2 + \cdots + t^{j-1}$ for $j = 2, 3, 4, \ldots$
   Is $\text{span}\{u_1, \ldots, u_n\} = \mathcal{P}_n$? Is $\text{span}\{u_1, u_2, \ldots\} = \mathcal{P}$?
7. Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.

8. Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:

   (a) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$.
   (b) $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$.
   (c) $\operatorname{span}\{(1, -1, 0, 2, 1), (2, 1, -2, 0, 0), (0, -3, 2, 4, 2), (3, 3, -4, -2, -1), (2, 4, 1, 0, 1), (5, 7, -3, -2, 0)\}$.

9. Find the dimension of the subspace $\operatorname{span}\{1 + t^2, -1 + t + t^2, -6 + 3t, 1 + t^2 + t^3, t^3\}$ of $\mathcal{P}_3$.

10. Find a basis, and hence dimension, for each of the following subspaces of the vector space of all twice differentiable functions from $\mathbb{R}$ to $\mathbb{R}$:

   (a) $\{x \in V : x'' + x = 0\}$.
   (b) $\{x \in V : x'' - 4x' + 3x = 0\}$.
   (c) $\{x \in V : x''' - 6x'' + 11x' - 6x = 0\}$.

11. Let $U = \left\{ \begin{bmatrix} a & -a \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$,

$V = \left\{ \begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$.

    (a) Prove that U and V are subspaces of $\mathbb{R}^{2\times 2}$.
    (b) Find bases, and hence dimensions, for $U \cap V$, U, V, and $U + V$.

12. Show that if V_1 and V_2 are subspace of $\mathbb{R}^9$ such that $\dim V_1 = 5 = \dim V_2$, then $V_1 \cap V_2 \neq \emptyset$.

13. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. What is $\operatorname{span}\{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$?

14. Given $a_0, a_1, \ldots, a_n \in \mathbb{R}$, let $V = \{x(t) \in C^k[0,1] : a_n x^{(n)}(t) + \cdots + a_1 x^{(1)}(t) + a_0 x(t) = 0\}$.
    Show that V is a subspace of $C^k[0,1]$, and find its dimension.

15. Let $V = \text{span}\{(1,2,3), (2,1,1)\}$ and $W = \text{span}\{(1,0,1), (3,0,-1)\}$. Find a basis for $V \cap W$. Also, find $\dim(V+W)$.
16. Given real numbers $a_0, a_1, \ldots, a_k$, let V be the set of all solutions $x \in C^k[a,b]$ of the differential equation

$$a_0 \frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + \cdots + a_k x = 0.$$

    Show that V is a vector space over $\mathbb{R}$. What is $\dim V$?
17. Consider each polynomial in $\mathcal{P}$ as a function from the set $\{0,1,2\}$ to $\mathbb{R}$. Is the set of vectors $\{t, t^2, t^3, t^4, t^5\}$ linearly independent?