DEPARTMENT OF MATHEMATICS, I.I.T.MADRAS
MA 2030 Linear Algebra And Numerical Analysis
Problem Set-3

1. For the following $T : \mathbb{R}^2 \to \mathbb{R}^2$, state with reasons whether T is linear.

   (a) $T(a_1, a_2) = (1, a_2)$

   (b) $T(a_1, a_2) = (a_1, a_1^2)$

   (c) $T(a_1, a_2) = (\sin a_1, 0)$

   (d) $T(a_1, a_2) = (|a_1|, a_2)$

   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

2. Let V be a vector space with a basis $\{a_1, \dots, a_n\}$. Let W be a vector space and let $b_1, \dots, b_n \in W$. Show that there is a unique linear transformation T from V to W such that $T(a_j) = b_j$ for $j = 1, \dots, n$.

3. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?

4. Prove that there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(2,3)$?

5. Let $\mathbf{V}$ be an n-dimensional vector space over $\mathbf{R}$. Prove that there exist a linear transformation $T : \mathbf{V} \to \mathbf{R}^n$ such that T is bijective.

6. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

7. Let V, W be vector space over $\mathbf{R}$ and $T : V \to W$ be linear.

   (a) $N(T) = \{x \in V : Tx = 0\}$ is called the null space of T. Show that $N(T)$ is subspace of V.

   (b) $R(T) = \{Tx : x \in V\}$ is called the range of T. Show that $R(T)$ is a subspace of W.
   Dimension of $N(T)$ is called the nullity of T and Dimension of $R(T)$ is called the rank of T.

8. In the following prove that T is a linear transformation and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$; $T(a_1, a_2) = (a_1 - a_2, 2a_3)$

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$; $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$