Assignment-9

1. Let $A \in \mathbb{F}^{n \times n}$. Prove that rank $A < n$ if and only if $det(A) = 0$.
2. Find the characteristic equation, the eigenvalues and the associated eigenvectors for the matrices given below.

   (a) $\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} -2 & 0 & 3 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

   (e) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$.

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3. Find the eigenvalues and associated eigenvectors of the differentiation operator $d/dt : \mathcal{P}_3 \to \mathcal{P}_3$.
4. Prove: The eigenvalues of a triangular matrix (upper or lower) are the entries on the diagonal.
5. Can any non-zero vector in any nontrivial vector space be an eigenvector of some linear transformation?
6. Given a scalar λ, can any non-zero vector in any nontrivial vector space be an eigenvector associated with the eigenvalue λ of some linear transformation?
7. Let T_1 and T_2 be linear operators on V, λ is an eigenvalue of T_1 and μ is an eigenvalue of T_2. Is it necessary that $\lambda\mu$ is an eigenvalue of $T_1 T_2$? Why? What is wrong with the following statement?

   $\lambda\mu$ an eigenvalue of $T_1 T_2$ because,
   if $T_1 x = \lambda x$ and $T_2 x = \mu x$, then
   $T_1 T_2 x = T_1(\mu x) = \mu T_1 x = \mu\lambda x = \lambda\mu x$.

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8. Let $T : V \to V$ be a linear transformation. Prove the following:
   (a) If T is a bijection and $0 \neq \lambda \in \mathbb{F}$, then λ is an eigenvalue of T if and only if $1/\lambda$ is an eigenvalue of T^{-1}.
   (b) If λ is an eigenvalue of T then λ^k is an eigenvalue of T^k.
   (c) If λ is an eigenvalue of T and $\alpha \in \mathbb{F}$, then $\lambda + \alpha$ is an eigenvalue of $T + \alpha I$.
   (d) If $p(t) = a_0 + a_1 t + \ldots + a_k t^k$ for some $a_0, a_1, \ldots, a_k$ in $\mathbb{F}$, and if λ is an eigenvalue of T then $p(\lambda)$ is an eigenvalue of $p(T) := a_0 I + a_1 T + \ldots + a_k T^k$.
9. Let A be an $n \times n$ matrix and α be a scalar such that each row (or each column) sums to α. Show that α is an eigenvalue of A.