MA 2030 Problem set 1 Solution Jan-May 2014

1. $V = \{x \in \mathbb{R} \mid x > 0\}$, $x + y = xy$ and $\alpha x = x^{\alpha}$

(i) $x > 0, y > 0 \Rightarrow x + y = xy > 0$ $\because$ product of +ve reals is positive $\therefore x + y > 0$ $\therefore x + y \in V$, closure is satisfied.

(ii), (iii) similarly verify associativity and commutativity.

(iv) $x + \frac{1}{x} = x \cdot \frac{1}{x} = 1 \Rightarrow \frac{1}{x}$ is the additive inverse of $x > 0$

(v) $\alpha x = x^{\alpha} \in V$ for $\alpha \in \mathbb{R}$.

(vi) $\alpha(x+y) = (x+y)^{\alpha} = (xy)^{\alpha} = x^{\alpha} y^{\alpha} = \alpha x + \alpha y$ (distributivity holds)

(vii) $(\alpha+\beta)x = x^{\alpha+\beta} = x^{\alpha} \cdot x^{\beta} = \alpha x + \beta x$, (distributivity holds)

(viii) $(\alpha\beta)x = x^{\alpha\beta} = (x^{\alpha})^{\beta} = (x^{\beta})^{\alpha} = \alpha(x^{\beta}) = \alpha(\beta x)$

(ix) $1 \cdot x = x^1 = x$. $\therefore V$ is a VS.

2(a) No. not a VS. since distributivity is not satisfied

$\left[(\alpha+\beta)(a_1, a_2) = \left((\alpha+\beta) a_1, \frac{a_2}{\alpha+\beta}\right) \right.$ and

$\alpha(a_1, a_2) + \beta(a_1, a_2) = \left(\alpha a_1, \frac{a_2}{\alpha}\right) + \left(\beta a_1, \frac{a_2}{\beta}\right) = \left(\alpha a_1 + \beta a_1, \frac{a_2}{\alpha} + \frac{a_2}{\beta}\right)$

$\left. = \left(\alpha a_1 + \beta a_1, a_2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right) \neq (\alpha+\beta)(a_1, a_2) \right]$

2(b) No. not a VS since associativity is not satisfied.

$[(a_1, a_2) + (b_1, b_2)] + (c_1, c_2) = (a_1 + 2b_1, a_2 + 3b_2) + (c_1 + c_2)$

$= (a_1 + 2b_1 + 2c_1, a_2 + 3b_2 + 3c_2) = \text{LHS}$

But $(a_1, a_2) + [(b_1, b_2) + (c_1, c_2)]$

$= (a_1, a_2) + (b_1 + 2c_1 + b_2 + 2c_2) = (a_1 + 2(b_1 + 2c_1), a_2 + 2(b_2 + 2c_2))$

$\neq \text{LHS}$.

2(c) No. not a VS. since distributivity is not satisfied

LHS: $(\alpha+\beta)(a_1, a_2) = (a_1, 0)$

But $\alpha(a_1, a_2) + \beta(a_1, a_2) = (a_1, 0) + (a_1, 0) = (a_1 + a_1, 0) = (2a_1, 0) \neq \text{LHS}$

3(a) $V = \mathbb{R}^2$, $W = \{(x_1, x_2) \mid x_2 = 2x_1 - 1\}$

Now $(0,0)$ is the zero element of $V = \mathbb{R}^2$

But $(0,0) \notin W$ $\because 0 = 2(0) - 1$ is impossible

$\therefore$ W is not a subspace of $V = \mathbb{R}^2$

$\boxed{\text{or}}$ Let $(x_1, x_2) \in W \Rightarrow x_2 = 2x_1 - 1$

$(y_1, y_2) \in W \Rightarrow y_2 = 2y_1 - 1$

then $(x_1, x_2) + (y_1, y_2) = (z_1, z_2)$, where $z_1 = (x_1 + y_1)$, $z_2 = (x_2 + y_2)$

Now consider $2z_1 - 1 = 2(x_1 + y_1) - 1 = 2x_1 + 2y_1 - 1 = (2x_1 - 1) + 2y_1$

$= x_2 + 2y_1$

$= x_2 + (y_2 - 1)$

$= x_2 + y_2 - 1$

$= z_2 - 1 \neq z_2$.

$\therefore$ W is not a subspace of $V = \mathbb{R}^2$.

$\boxed{\text{OR}}$ Note that by problem (4) in set-1, which we will do later, the only proper subspaces of $\mathbb{R}^2$ are straight lines passing thro' the origin. i.e $\{y = mx \mid x, y \in \mathbb{R}$, m is a real number$\}$

~~Thus~~. $x_2 = 2x_1 - 1$ is a st. line given by $x_2 - 2x_1 + 1 = 0$ not passing thro' the origin. $\therefore$ Not a VS.

3(b) $V = \mathbb{R}^3$, $W = \{(x_1, x_2, x_3) : 2x_1 - x_2 - x_3 = 0\}$

Let $v_1 = (x_1, x_2, x_3) \in W$, then $2x_1 - x_2 - x_3 = 0$

$v_2 = (y_1, y_2, y_3) \in W$, then $2y_1 - y_2 - y_3 = 0$

then $\alpha v_1 + \beta v_2 = (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) = (z_1, z_2, z_3)$ say

Now consider $2z_1 - z_2 - z_3 = 2(\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) - (\alpha x_3 + \beta y_3)$

$= \alpha(2x_1 - x_2 - x_3) + \beta(2y_1 - y_2 - y_3)$

$= \alpha(0) + \beta(0) = 0$

$\Rightarrow (z_1, z_2, z_3) \in W \Rightarrow \alpha v_1 + \beta v_2 \in W$

$\therefore$ W is a subspace of $\mathbb{R}^3$.

3(c) $V = C[0,1]$, $W = \{f \in V : f$ is differentiable$\}$.

Let $f, g \in W$, then $\alpha f + \beta g \in W$ ($\because$ sum of differentiable functions is differentiable and αf, βg are differentiable)

Yes. W is a subspace of V.

4. Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.

Let H be any proper subspace of $\mathbb{R}^2$.

Then $H \neq \{\vec{0}\}$ and $H \neq \mathbb{R}^2$.

$\hookrightarrow \{(0,0)\}$

Note: since H is a subspace $\vec{0} = (0,0) \in H$.

Since $H \neq \{(0,0)\} = \{\vec{0}\}$, $\exists$ a $\vec{v} = (x,y) \neq (0,0) \in H$

Since H is a subspace, $\alpha\vec{v} \in H$.

i.e. $L = (\alpha x, \alpha y) \in H$. for all real α.

L is a straight line through the origin and $\vec{v}$.

This shows that if H is a subspace of $\mathbb{R}^2$, then H consists of points lying on straight lines of the form L passing through the origin.

i.e. $L \subseteq H \longrightarrow (1)$

Conversely, if we show that $H \subseteq L \to (2)$, then

$(1) + (2) \Rightarrow H \equiv L$.

In order to show that $H \subseteq L$, we need to show that if an element is in H, then it is also in L or, we need to show if $\vec{w} \notin L$, then $\vec{w} \notin H$.

If possible, let $\vec{w} \notin L$ but $\vec{w} \in H$.

$\because \vec{v} \in H$, $\vec{w} \in H$, $\alpha\vec{v} + \beta\vec{w} \in H$ $\because$ H is a subspace.

so that $\{\alpha\vec{v} + \beta\vec{w} \mid \alpha, \beta \text{ are scalars}\} \subseteq H$.

But $\{\alpha\vec{v} + \beta\vec{w} \mid \alpha, \beta \text{ are scalars}\} = \mathbb{R}^2$.

$\therefore$ $\mathbb{R}^2 \subseteq H$ But $H \subset \mathbb{R}^2$ (given in the problem)

$\therefore$ $H \equiv \mathbb{R}^2$ but this is a contradiction to the fact that H is a proper subspace of $\mathbb{R}^2$. $\therefore$ our assumption that $\vec{w} \notin L$ but $\vec{w} \in H$ is wrong $\therefore$ if $\vec{w} \notin L$, then $\vec{w} \notin H$

$\therefore$ $H \subseteq L \longrightarrow (2)$ $\qquad (1) + (2) \Rightarrow H = L$.

In each of the following, a vector space V and a set A of vectors in V is given. Determine whether A is linearly dependent and if it is express one of the vectors in A as a linear combination of the remaining vectors.

(a) $V = \mathbb{R}^3$. $A = \{(1,0,-1), (2,5,1), (0,-4,3)\}$.

$$\alpha(1,0,-1) + \beta(2,5,1) + \gamma(0,-4,3) = (0,0,0)$$

$\Rightarrow$

$$\alpha + 2\beta = 0$$

$$5\beta - 4\gamma = 0 \Rightarrow \beta = \frac{4\gamma}{5}$$

$$-\alpha + \beta + 3\gamma = 0.$$

$$-\alpha + \frac{4\gamma}{5} + 3\gamma = 0 \Rightarrow -\alpha + \frac{19\gamma}{5} = 0 \Rightarrow \alpha = \frac{19\gamma}{5}$$

$$\frac{19\gamma}{5} + 2\left(\frac{4\gamma}{5}\right) = 0 \Rightarrow \frac{27\gamma}{5} = 0 \Rightarrow \gamma = 0 \Rightarrow \beta = 0 \Rightarrow \alpha = 0$$

$\Rightarrow$ A is linearly independent.

$\boxed{\text{Or}}$ check the value of determinant

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 5 & 1 \\ 0 & -4 & 3 \end{vmatrix} = 1(15+4) - 1(-12) = 19 + 12 \neq 0$$

$\therefore$ linearly independent.

$\boxed{\text{You can do (b) \& (c) similarly}}$.

(l) $V = C[-\pi, \pi]$

$A = \{\sin x, \sin 2x, \ldots \sin nx\}$

zero element of $C[-\pi,\pi]$ is 0_{func}

Consider $a_1 \sin x + a_2 \sin 2x + \cdots + a_n \sin nx = 0$

Now, multiply by $\sin x$ integrate w.r.t x bet $-\pi$ & π

then $\int_{-\pi}^{\pi} a_1 \sin^2 x\, dx + a_2 \int_{-\pi}^{\pi} \sin x \sin 2x\, dx + a_n \int_{-\pi}^{\pi} \sin x \sin nx\, dx = 0$

$\Rightarrow \quad a_1 \pi + a_2(0) + \cdots + a_n(0) = 0 \Rightarrow \boxed{a_1 = 0}$

IIIly multiply by $\sin kx$, $k = 1, 2, 3, \ldots n$ & integrate w.r.t x bet $-\pi$ & π, You can show as above that $a_1 = 0$, $a_2 = 0$, $\ldots$ $a_n = 0$

$\Rightarrow$ A is linearly independent.