Problem set-4 — Solution

1 a) $\langle (a,b),(c,d)\rangle = ac-bd$ on $\mathbb{R}^2$

Ans: No; $\langle (1,2),(1,2)\rangle = 1-4<0$.

b) $\langle A,B\rangle = \mathrm{tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$

Ans: No; $A=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$, $\langle A,A\rangle = \mathrm{tr}(A+A) = \mathrm{tr}(2A) = -4<0$.

c) $\langle f,g\rangle = \int_0^1 f'(t)\,g(t)\,dt$ on $P(\mathbb{R})$, $'$ denotes differentiation.

Ans: No; $f(t)=2t-1$, $f'(t)=2$, $\langle f,f\rangle = \int_0^1 2(2t-1)\,dt = 0$

But $f = 2t-1 \neq 0$.

d) $\langle f,g\rangle = \int_0^{1/2} f(t)\,g(t)\,dt$ on $C[0,1]$.

Ans: Yes.

2. $\dim V = n$. Let $B=\{\vec{v_1},\vec{v_2},\ldots,\vec{v_n}\}$ be a basis of V.

Let $\vec{y}\in V$. Then $\vec{y} = a_1\vec{v_1}+a_2\vec{v_2}+\cdots+a_n\vec{v_n}$

$\langle \vec{y},\vec{y}\rangle = \|\vec{y}\|^2 = a_1\langle\vec{v_1},\vec{y}\rangle + a_2\langle\vec{v_2},\vec{y}\rangle+\cdots+a_n\langle\vec{v_n},\vec{y}\rangle$

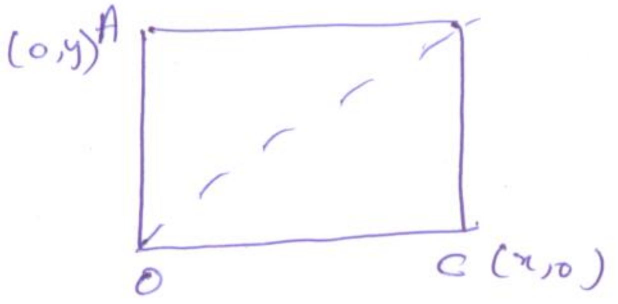
$= 0+0+\cdots+0 = 0$ $(\because \langle\vec{x},\vec{y}\rangle = 0\ \forall\, \vec{x}\in B)$

$\Rightarrow \vec{y}=\vec{0}$

3. $\vec{x},\vec{y}\in V$, $\vec{x}\perp\vec{y}$, then $\langle\vec{x},\vec{y}\rangle = 0$.

$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = \langle\vec{x}+\vec{y},\vec{x}+\vec{y}\rangle + \langle\vec{x}-\vec{y},\vec{x}-\vec{y}\rangle$, $(\because \langle\vec{x},\vec{y}\rangle = 0,\ \langle\vec{y},\vec{x}\rangle=0)$.

$= 2(\|\vec{x}\|^2 + \|\vec{y}\|)$



$OA = \|\vec{y}\|$

$OC = \|\vec{x}\|$

$AC = \|\vec{x}+\vec{y}\|$

$AC^2 = OA^2+OC^2$

$\therefore \|\vec{x}+\vec{y}\|^2 = \langle\vec{x}+\vec{y},\vec{x}+\vec{y}\rangle$

$= \|x\|^2+\|\vec{y}\|^2$.

4, 5. done in class.

6. Let $A = \{\vec{a_1}, \vec{a_2}, \dots \vec{a_n}\}$ be an orthogonal set of nonzero vectors in a IPS V. $\langle \vec{a_i}, \vec{a_j} \rangle = 0$, $i \neq j$.

Consider $\alpha_1 \vec{a_1} + \alpha_2 \vec{a_2} + \dots + \alpha_n \vec{a_n} = \vec{0}$

taking innerproducts with $\vec{a_j}$, $j = 1, \dots n$

$$\alpha_1 \langle \vec{a_1}, \vec{a_j} \rangle + \alpha_2 \langle \vec{a_2}, \vec{a_j} \rangle + \dots + \alpha_j \langle \vec{a_j}, \vec{a_j} \rangle + \dots + \alpha_n \langle \vec{a_n}, \vec{a_j} \rangle = \vec{0}.$$

This gives $\alpha_j \langle \vec{a_j}, \vec{a_j} \rangle = 0$ $\quad (\because \langle \vec{a_i}, \vec{a_j} \rangle = 0,\ j \neq i)$

$\Rightarrow \alpha_j = 0$, $j = 1, 2 \dots n$

$\Rightarrow A = \{\vec{a_1}, \vec{a_2} \dots \vec{a_n}\}$ is Linearly independent.

7. $$\text{LHS} = \left\| \sum_{i=1}^{k} a_i x_i \right\|^2$$
$$= \left\langle \sum_{i=1}^{k} a_i x_i, \sum_{j=1}^{k} a_j x_j \right\rangle$$
$$= \sum_{i=1}^{k} \langle a_i x_i, a_i x_i \rangle = \sum_{i=1}^{k} a_i \bar{a_i} \langle x_i, x_i \rangle$$
$$= \sum_{i=1}^{k} |a_i|^2 \|x_i\|^2 \quad (\because a_i \in \mathbb{R} \ \& \ \langle x_i, x_j \rangle = 0)$$

8. verify all the axioms.

9. For vectors x and y in an inner product space V (over $\mathbb{R}$) prove that $x-y$ and $x+y$ are orthogonal to each other iff $\|x\| = \|y\|$

Let $x-y$ and $x+y$ be orthogonal to each other $\Leftrightarrow$

$\langle x-y, x+y \rangle = 0 \Leftrightarrow \langle x, x \rangle + \langle x, y \rangle - \langle y, x \rangle - \langle y, y \rangle = 0$

$\Leftrightarrow \|x\|^2 - \|y\|^2 = 0 \Leftrightarrow \|x\| = \|y\|$

$(\because \langle x, y \rangle - \langle y, x \rangle = \langle x, y \rangle - \langle x, y \rangle = 0).$

Q. 11 a: $S^{\perp} = \{x \mid \langle x, s\rangle = 0$ for all $s \in S\}$, $S^{\perp}$ is called the orthogonal complement of the set S.

- For any nonempty set $S \subseteq V$, $0 \in S^{\perp}$.
- Let $x, y \in S^{\perp}$ and let $\alpha, \beta \in \mathbb{F}$. Then, for any $s \in S$

$$\langle \alpha x + \beta y, s\rangle = \alpha\langle x, s\rangle + \beta\langle y, s\rangle$$
$$= \alpha 0 + \beta \cdot 0 = 0$$

Hence $\alpha x + \beta y \in S^{\perp}$. Therefore $S^{\perp}$ is a subspace of V.

b. Show that $S \subseteq S^{\perp\perp}$.

Let $s \in S$ be any vector. Then $\langle x, s\rangle = 0$ for all $x \in S^{\perp}$.
As s is orthogonal to every vector in $S^{\perp}$, we have $s \in S^{\perp\perp}$. Therefore, $S \subseteq S^{\perp\perp}$.

Note:- If S is a finite dimentional subspace of V, then it can be proved that $S = S^{\perp\perp}$.

There exists subsets S of V such that $S \subsetneq S^{\perp\perp}$.
For example, let $S = \{(1,0)\} \subseteq \mathbb{R}^2$.
Then $S^{\perp} = \{(0,x) \mid x \in \mathbb{R}\}$ and $S^{\perp\perp} = \{(y,0) \mid y \in \mathbb{R}\}$.
Therefore, $S \subsetneq S^{\perp\perp}$.

c. Let V be of finite dimentional, and W be a subspace of V. Then $\dim(W) + \dim(W^{\perp}) = \dim(V)$.

Sol: Since V is of finite dimention, and W is a subspace of V, W is also of finite dimension.
Let $\dim(W) = k$ and $\dim(V) = n$. (Here $k \le n$.)
Let $\{w_1, w_2, \ldots, w_k\}$ be an orthonormal basis for W.

Let $v \in V$ be any vector. Now we can express v as

$$v = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2 + \cdots + \langle v, w_k\rangle w_k$$
$$+ v - \left(\langle v, w_1\rangle w_1 + \cdots + \langle v, w_k\rangle w_k\right)$$

Let $w = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2 + \cdots + \langle v, w_k\rangle w_k$

and $y = v - w$.

~~Since~~ $w \in W$ as W is a linear combination of vectors of the orthonormal basis $\{w_1, \cdots, w_k\}$ of W.

Now $\cancel{\langle y, w_j\rangle = \langle v,}$

$$\langle y, w_j\rangle = \langle v - w, w_j\rangle$$
$$= \langle v, w_j\rangle - \langle w, w_j\rangle$$
$$= \langle v, w_j\rangle - \langle v, w_j\rangle \langle w_j, w_j\rangle = 0.$$

Therefore, y is orthogonal to ~~the~~ every vector in $W = \mathrm{span}(\{w_1, w_2, \cdots, w_k\})$. i.e. $y \in W^{\perp}$.

From this discussion it ~~foll~~ follows that, every vector in V can be expressed as a sum of ~~vec~~ two vectors with one from W and other from $W^{\perp}$. Hence

$$V = W + W^{\perp}.$$

Next, we show that $W \cap W^{\perp} = \{0\}$. (easy to show).

From this it follows that $\dim(V) = \dim(W) + \dim(W^{\perp})$.

d(i) If W_1 and W_2 are subspaces of V, then show that $(W_1+W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

Ans Let $x \in (W_1+W_2)^{\perp}$. Then $\langle x, w\rangle = 0$ for all $w \in W_1+W_2$

$w_1 \in W_1 \Rightarrow w_1 + 0 \in W_1+W_2 \Rightarrow w_1 \in W_1+W_2$

$\Rightarrow \langle x, w_1\rangle = 0 \Rightarrow x \in W_1^{\perp}$.

Similarly, ~~one~~ $\langle x, w_2\rangle = 0 \Rightarrow x \in W_2^{\perp}$.

$\therefore (W_1+W_2)^{\perp} \subseteq (W_1^{\perp} \cap W_2^{\perp})$ --- (1).

Let $x \in W_1^{\perp} \cap W_2^{\perp}$ be any vector. ~~and let $x = w_1 + w_2$~~.

~~where with $w_1 \in W_1$~~

Let $w = w_1 + w_2 \in W_1 + W_2$ with $w_1 \in W_1$ and $w_2 \in W_2$.

Now $\langle x, w\rangle = \langle x, w_1 + w_2\rangle = \langle x, w_1\rangle + \langle x, w_2\rangle$

$= 0 + 0 = 0$ ($\because x \in W_1^{\perp}$ and $x \in W_2^{\perp}$).

$\Rightarrow x \in (W_1 + W_2)^{\perp}$.

$\Rightarrow W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$ --- (2).

From (1) and (2) we have $(W_1+W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$.

d(ii) If V is a finite dimentional vector space then $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

Ans:- Let S and T be two subsets of an I.P.S. V with $S \subseteq T$, then $T^{\perp} \subseteq S^{\perp}$.

Let $x \in T^{\perp}$. Then $\langle x, t\rangle = 0$ for all $t \in T$.

Since $S \subseteq T$, we have $\langle x, s\rangle = 0$ for all $s \in S$.

$\Rightarrow x \in S^{\perp}$. Therefore, $T^{\perp} \subseteq S^{\perp}$.

~~By [illegible]~~

Let $x \in W_1^{\perp} + W_2^{\perp}$. Then $x = y + z$ for some $y \in W_1^{\perp}$ and $z \in W_2^{\perp}$.

For any $w \in (W_1^{\perp} \cap W_2^{\perp})$ we have

$$\langle x, w \rangle = \langle y+z, w \rangle = \langle y, w \rangle + \langle z, w \rangle = 0 + 0 = 0 .$$

$\Rightarrow x \in (W_1 + W_2)^{\perp}$

Hence $W_1^{\perp} + W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$.

Next, we show that $(W_1 \cap W_2)^{\perp} \subseteq (W_1^{\perp} + W_2^{\perp})$.

~~Since V is a finite dimen~~ Let $\dim(V) = n$, $\dim(W_1) = k_1$, $\dim W_2 = k_2$ and $\dim(W_1 \cap W_2) = k$.

Let $\{v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_{k_1}, u_1, u_2, \cdots, u_{k_2-k}, \cancel{v_{k_1+1}, \cdots, v_{k_2}}, v_{k_2+1}, v_n\}$ be an orthogonal basis for V, with

$\{v_1, \cdots, v_k\}$ basis for $W_1 \cap W_2$

$\{v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k_1}\}$ basis for W_1

$\{v_1, \cdots, v_k, u_1, u_2, \cdots, u_{k_2-k}\}$ basis for W_2

Since V is a finite dimensional ~~ve~~ I.P.S, we can write $V = (W_1 \cap W_2) + (W_1 \cap W_2)^{\perp}$.

Let $x \in (W_1 \cap W_2)^{\perp}$ be any vector. ~~and~~ Then x can be written as

$$x = \sum_{i=1}^{k_2-k} \alpha_i u_i + \sum_{j=k_1+1}^{n} \alpha_j v_j$$

$$= u + v$$

Clearly $u \in W_2^{\perp}$ and $v \in W_1^{\perp}$.

Hence $x \in W_1^{\perp} + W_2^{\perp}$. Hence $(W_1 \cap W_2)^{\perp} \subseteq (W_1^{\perp} + W_2^{\perp})$. D.

12. done in class.

13. $\langle x, x\rangle = \langle T(x), T(x)\rangle \Rightarrow \|x\|^2 = \|Tx\|^2 \cdot \forall x \in V.$

$\|x\|^2 = \langle x, x\rangle = 0 \Rightarrow \langle Tx, Tx\rangle = 0 \Rightarrow \|Tx\|^2 = 0$

$\Rightarrow T$ is 1-1.

$\langle x+y, z\rangle = \langle T(x+y), T(z)\rangle$

$= \langle Tx + Ty, Tz\rangle$

$= \langle Tx, Tz\rangle + \langle Ty, Tz\rangle$

$= \langle x, z\rangle + \langle y, z\rangle$ is satisfied

$\langle x, y\rangle \cancel{\langle y, x\rangle \Rightarrow} = \langle Tx, Ty\rangle = \overline{\langle Ty, Tx\rangle} = \overline{\langle y, x\rangle}$ is satisfied.

14. $T: V \to V$

Let $\vec{x} \in \ker T$.

then $T\vec{x} = \vec{0}$.

Now $\langle T\vec{x}, T\vec{x}\rangle = \langle \vec{0}, \vec{0}\rangle = \|T(x)\|^2 = \|x\|^2 = \vec{0}$

$\Rightarrow \vec{x} = \vec{0}$

$\Rightarrow \ker T = \{\vec{0}\}$

$\Rightarrow T$ is 1-1.