1.a) Matrix of the LT w.r.t standard basis in $\mathbb{R}_3$ is $A_T = \begin{array}{c} \begin{pmatrix} 1 & -2 & 0 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array} \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$

Note $R_3 = R_1 + R_2$ $\therefore$ row vectors R_1, R_2, R_3 are not LI.

$\therefore$ row space is not equal to $\mathbb{R}^3$.

$\therefore$ T is not onto.

$\therefore$ T is not invertible.

$\boxed{\text{OR}}$ Null space of $T = \{ (x_1, x_2, x_3) \mid T(x_1, x_2, x_3) = 0 \}$

$(x_1 - x_2, -2x_1 + 3x_2 + x_3, x_2 + x_3) = (0,0,0)$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$-2x_1 + 3x_2 + x_3 = 0 \Rightarrow -2x_1 + 3x_1 + x_1 = 0$ satisfied.

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

$\therefore (x_1, x_2, x_3) = (x_1, x_1, -x_1) = x_1(1, 1, -1)$ $\therefore$ T is not invertible.

$\Rightarrow N(T) \neq \{(0,0,0)\}$ $\therefore$ T is not 1-1

$\boxed{\text{OR}}$ $|A_T| = 1(3-1) + 2(-1) = 2 - 2 = 0$ $\therefore$ T is not invertible.

b). Since $\dim \mathbb{R}^2 = 2 < \dim \mathbb{R}^3 = 3$ and $\therefore$ T is not onto.

c) Since for any $\alpha \in \mathbb{R}$, $\alpha \in P_2$ and $T(\alpha) = \alpha$. hence $\text{rank}(T) = 1$.

$\therefore$ the map T is onto and

d) $\therefore \dim V = 3 \neq \dim P_2$, $P_2 \not\cong V$.

e) $A = \{ (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_2 = 2x_1, x_4 = 4x_3, x_5 = 6x_4 \}$

$A = (x_1, x_2, x_3, x_4, x_5) = (x_1, 2x_1, x_3, 4x_3, 24x_3)$

$= x_1(1,2,0,0,0) + x_3(0,0,1,4,24)$

$A = \text{span}\{ (1,2,0,0,0), (0,0,1,4,24) \} \Rightarrow \dim(A) = 2$.

If A were N(T), then $\dim(A) + \dim R(T) = 2 + 2 = 4 \neq 5$

$\therefore$ False

$\cancel{N(T) = \{ (x_1, x_2, x_3, x_4, x_5) \mid T(x_1, x_2, x_3, x_4, x_5) = (0,0,0,0,0) \}}$

$\cancel{(x_1, 2x_1, x_3, 4x_3, 24x_3) =}$

3. $T: V \to V$, $T(A) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$; $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix}$; $T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}$; $T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}$ $[T]_B^B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}$

(2 m)

1 m

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Q.2 $(x,y,z) = (x-y)(1,0,0) + y(1,1,0) + z(0,0,1)$

$T(x,y,z) = (2x-2y+2z, 3x-2y+2z)$

Now $T(x,y,z) = (0,0) \Leftrightarrow (2x-2y+2z, 3x-2y+2z) = (0,0)$ (2)

$\Leftrightarrow$ $x = 0$ and $y = z$

Hence $N(T) = \{x(0,1,1) \mid x \in \mathbb{R}\}$.

and Nullity of T is 1. $\longrightarrow$ (1)

Alternate answer:

Some students have shown that

(*) $\{x(0,1,1) \mid x \in \mathbb{R}\} \subseteq N(T)$

and (*) By rank-nullity theorem $\dim(N(T)) = 1$.

Hence $\{x(0,1,1) \mid x \in \mathbb{R}\} = N(T)$, ~~and nulli~~ (3)

~~for such cases full marks~~

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4(a) 1. $\langle (x_1, x_2), (x_1, x_2) \rangle = x_1^2 - x_1 x_2 - x_1 x_2 + 3x_2^2$

$$= (x_1 - x_2)^2 + (\sqrt{2}\, x_2)^2 > 0$$

and $\langle (x_1, x_2), (x_1, x_2) \rangle = 0 \Leftrightarrow x_1 - x_2 = 0$ and $\sqrt{3}\, x_2 = 0 \Rightarrow x_1 = 0, x_2 = 0$.
i.e $(x_1, x_2) = (0, 0)$. $\longrightarrow \boxed{1M}$

2. $\langle X + Y, Z \rangle = \langle (x_1, x_2) + (y_1, y_2), (z_1, z_2) \rangle$

$$= \langle (x_1 + y_1, x_2 + y_2), (z_1, z_2) \rangle$$

$$= (x_1 + y_1) z_1 - (x_1 + y_1) z_2 - (x_2 + y_2) z_1 + 3(x_2 + y_2)(z_2)$$

$$= (x_1 z_1 - x_1 z_2 - x_2 z_1 + 3x_2 z_2) + (y_1 z_1 - y_1 z_2 - y_2 z_1 + 3y_2 z_2)$$

$$= \langle X, Z \rangle + \langle Y, Z \rangle.$$

$\boxed{1M}$

3) $\langle X, Y \rangle = x_1 y_1 - x_1 y_2 - x_2 y_1 + 3x_2 y_2 = y_1 x_1 - y_1 x_2 - y_2 x_1 + 3y_2 x_2$
$= \langle Y, X \rangle$.

$\therefore$ $\mathbb{R}^2$ is an IPS w.r.t the mentioned IP.

b) $A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{matrix} \vec{v_1} \\ \vec{v_2} \end{matrix} \right\}$.

$\vec{u_1} = \dfrac{\vec{v_1}}{\|\vec{v_1}\|}$, $\|v_1\|^2 = \langle (1,0), (1,0) \rangle = 1 - 0 - 0 + 3(0)(0) = 1$.
$\Rightarrow \|\vec{v_1}\| = 1$ $\longrightarrow \boxed{1M}$

$\therefore \vec{u_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\vec{v_2}' = \vec{v_2} - \langle v_2, \vec{u_1} \rangle \vec{u_1} = (2,3) - \langle (2,3), (1,0) \rangle (1,0)$
$= (2,3) - (2 - 0 - 3 + 0)(1,0)$
$= (2,3) - (-1)(1,0)$
$= (2,3) + (1,0) = (3,3)$ $\longrightarrow \boxed{1M}$

$\therefore \|\vec{v_2}'\|^2 = \langle (3,3), (3,3) \rangle = \cancel{9} - \cancel{9} - 9 + 27 = 18 \Rightarrow \|\vec{v_2}'\| = 3\sqrt{2}$.

$\therefore \vec{u_2} = \dfrac{\vec{v_2}'}{\|\vec{v_2}'\|} = \dfrac{(3,3)}{3\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\longrightarrow \boxed{1M}$

$\therefore$ an orthonormal basis for $\mathbb{R}^2$ w.r.t the mentioned IP is

$Q = \{ ~~(\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}})~~ \}$. $\{(1,0), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

i) $Q = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$. If we use the mentioned IP and factorize A then $A = QR \Rightarrow R = Q^t A = \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 \\ 0 & 3\sqrt{2} \end{pmatrix} \longrightarrow \boxed{2M}$

where entries in R are obtained using the given IP. $\langle (1,0), (1,0) \rangle = 1$, $\langle (1,0), (2,3) \rangle = -1$, $\langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (1,0) \rangle = 0$; $\langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, ...$

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$\boxed{OR}$ if we use the given inner product, then

$Q = \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Then the entries in R are directly obtained from Q3(b) as

$$R = \begin{pmatrix} \langle \vec{v_1}, \vec{u_1} \rangle, & \langle \vec{v_2}, \vec{u_1} \rangle \\ 0 & \langle \vec{v_2}, \vec{u_2} \rangle \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 3\sqrt{2} \end{pmatrix} \longrightarrow \boxed{2M}$$

since

$\langle (\vec{v_1}, \vec{v_1}), (\vec{u_1}, \vec{u_1}) \rangle = \langle (1,0), (1,0) \rangle = 1$

$\langle (\vec{v_2}, \vec{u_2}), (\vec{v_2}, \vec{u_2}) \rangle = \langle (2,3), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \rangle = \cancel{\frac{2}{\sqrt{2}}} - \cancel{\frac{2}{\sqrt{3}}} - \frac{3}{\sqrt{2}} + \frac{9}{\sqrt{2}} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$

$\boxed{OR}$. if we use the given inner product and

$A = QR \therefore R = Q^{-1} R$, here the matrix multiplication in the usual one.

$|Q| = \frac{1}{\sqrt{2}}$

$$Q^{-1} = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$R = Q^{-1} R = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 3\sqrt{2} \end{pmatrix} \longrightarrow \boxed{2M}$$

Note that $Q^t \neq Q^{-1}$

$\boxed{OR}$ in one uses the usual dot product as IP on $\mathbb{R}^2$.

then $A = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\} = \{ \vec{v_1}, \vec{v_2} \}$.

$||\vec{v_1}||^2 = (1,0) \cdot (1,0) = 1$.

$\vec{u_1} = \frac{\vec{v_1}}{||\vec{v_1}||} = \frac{(1,0)}{1} = (1,0)$.

$\vec{v_2}' = \vec{v_2} - (v_2 \cdot u_1) u_1 = (2,3) - [(2,3) \cdot (1,0)](1,0)$
$= (2,3) - 2(1,0) = (2,3) - (2,0) = (0,3)$.

$||\vec{v_2}'||^2 = (0,3) \cdot (0 \cdot 3) = 9$.

$\vec{u_2} = \frac{\vec{v_2}'}{||\vec{v_2}'||} = \frac{(0,3)}{3} = (0,1)$. $\therefore$ an orthonormal basis for $\mathbb{R}^2$ w.r.t usual dot product as IP is

$\Rightarrow Q = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. i.e. $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$\therefore R = Q^t A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \longrightarrow \boxed{2M}$$

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