CHAPTER 1

GENERAL LINEAR SYSTEMS

One of the most important and frequently occurring mathematical problems in science, engineering and social sciences is finding a solution to a set of simultaneous linear equations involving several unknowns. An (m x n) system of linear equations is a system of m linear equations in n unknowns:

$$a_{11}\ x_1\ +a_{12}\ x_2 \ldots\ldots\ldots\ldots a_{1n}\ x_n\ = b_1$$

$$a_{21}\ x_1\ +a_{22}\ x_2\ + \ldots\ldots\ldots\ldots a_{2n}\ x_n\ = b_2$$

$$\vdots \qquad \vdots \qquad\qquad \vdots \qquad \vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots a_{mn} x_n = b_m$$

[redacted] (1.1)

A solution to system (1.1) is an n-tuple $(x_1, x_2, \ldots\ldots\ldots, x_n)$ of numbers that is simultaneously a solution for each equation in the system. The solution set of the system is the set of all solutions of the system. It is a subset of $\mathbb{R}^n$.

Two processes are involved in solving a general mxn system (1.1)

1. Reduction of the system (that is elimination of variables)
2. Description of the set of solutions

1.1 Reduction of System of Linear Equations

The reduction process aims at simplifying the given system by eliminating the unknowns. It is essential that the reduced system of equations have the same set of solutions as the original system.

Definition 1.1 Two systems of linear equations in n unknowns are equivalent provided that they have the same set of solutions.

Thus, the reduction procedure must yield an equivalent system of equations. That is, the reduction procedure must transform the given system by certain elementary operations into a simpler equivalent system which is then solved.

The elementary operations are of the following three types. (Here E_i denotes the i^{th} equation in the system).

(i) Interchanging two equations in the system: $E_i \rightarrow E_j$

(ii) Multiplying an equation by a non-zero number $\lambda : \lambda E_i \rightarrow E_i$

(iii) Adding to an equation a multiple of some other equation: $E_i + \lambda E_j \rightarrow E_i$

Theorem 1.2 If one system of equations is obtained from another by a finite sequence of elementary operations, then the two systems are equivalent.

Proof It is enough if one considers the effect of a single application of each elementary operation. Suppose that an elementary operation transforms the system **Ax** = **b** into the system **Bx** = **d**. If the operation is of type (i), then the two systems consist of the same equations but written in a different order. Therefore, if x solves **Ax** = **b**, then x solves the second system **Bx** = **d** and vice versa.

If the operation is of type (ii), then suppose that the i^{th} equation has been multiplied by a scalar λ with $\lambda \neq 0$. The i^{th} and j^{th} equations in Ax=b are

$$a_{i1} x_1 + a_{i2} x_2 + \ldots\ldots\ldots a_{in} x_n = b_i \tag{1.2}$$

and

$$a_{j1} x_1 + a_{j2} x_2 + \ldots\ldots\ldots a_{jn} x_n = b_j \tag{1.3}$$

and the i^{th} equation in Bx=d is

$$\lambda a_{i1} x_1 + \lambda a_{i2} x_2 + \ldots\ldots\ldots + \lambda a_{in} x_n = \lambda b_i \tag{1.4}$$

Any vector x that satisfies (1.2) also satisfies (1.4) and viceversa, because $\lambda \neq 0$.

Finally, suppose that the operation is of type (iii), let λ times the j^{th} equation be added to the i^{th} equation.Then i^{th} equation in Bx=d is

$$(a_{i1} + \lambda a_{j1})\, x_1 + \ldots\ldots\ldots (a_{in} + \lambda a_{jn})\, x_n = b_i + \lambda b_j . \tag{1.5}$$

If **Ax** = **b**, then (1.2) and (1.3) are true. ∴ (1.5) is true. Thus **Bx** = **d**. On the other hand, if **Bx** = **d**, then (1.5) and (1.3) are true. Therefore, if λ times (1.3) is subtracted from (1.5), then one gets (1.2). Hence **Ax** = **b**.

Note

(i) The three operations (i), (ii) and (iii), when applied to the rows of the augmented matrix representation of a system of equations, are called <u>elementary row operations</u>.

(ii) The process of applying elementary row operations to simplify an augmented matrix is called <u>row reduction</u>.

We adopt the following notation:

$R_i \leftrightarrow R_j$ The i^{th} and j^{th} rows are interchanged

$\lambda R_i \rightarrow R_i$ The i^{th} row is multiplied by the non zero scalar λ.

$R_i + \lambda R_j \rightarrow R_i$ add λ times the j^{th} row to the i^{th} row.

Two (mxn) matrices, A and B, are row equivalent if one can be obtained from the other by a finite sequence of elementary row operations. If $[A|\mathbf{b}]$ is the augmented matrix for a system **Ax** = **b** and if $[B|\mathbf{d}]$ is row equivalent to $[A|\mathbf{b}]$, then $[B|\mathbf{d}]$ is the augmented matrix for an equivalent system. This is because the elementary row operations for matrices exactly duplicate the elementary operations for equations.

Example 1.3

$$[A|\mathbf{b}] = \left(\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 0 \\ 1 & -1 & 3 & -1 & -2 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\cong \left(\begin{array}{cccc|c} 1 & -1 & 3 & -1 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{array}\right) R_3 - R_1 \to R_3$$

$$\cong \left(\begin{array}{cccc|c} 1 & -1 & -3 & -1 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & 2 & 4 \end{array}\right) R_3 - 2R_2 \to R_3$$

$$\cong \left(\begin{array}{cccc|c} 1 & -1 & -3 & -1 & -2 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & -4 & 4 & 4 \end{array}\right) = [B|\mathbf{d}] \qquad (1.6)$$

The system $B\mathbf{x} = \mathbf{d}$ is equivalent to system $A\mathbf{x} = \mathbf{b}$. Observe that the matrix in (1.6) has all nonzero entries appearing in a staircase-shaped region in the upper right hand portion of the matrix. The matrix in (1.6) is an example of a matrix in row-echelon form.

Definition 1.4 An (mxn) matrix is in row-echelon form if,

1. All rows that consist entirely of zeros are grouped together at the bottom of the matrix.
2. The first (counting from the left to right) nonzero entry in the $(i+1)^{st}$ row appears in a column to the right of the first nonzero entry in the i^{th} row, that is, if the first nonzero entry in the i^{th} row occurs in column j_i, then $j_1 < j_2 < \cdots$

For such a matrix, the first nonzero entry in a row is called the <u>pivot</u> for that row.

Example 1.5

Matrix $A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not in row-echelon form, because the second row (consisting of all zero entries) is not below the third row (which has a nonzero entry).

Matrix B = $\begin{pmatrix} 2 & 4 & 0 \\ 1 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ is not in row-echelon form, because the first nonzero entry in the second row does not appear in a column to the right of the first nonzero entry in the first row, $j_1 = 1 = j_2$.

Matrix C = $\begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in row-echelon form, because both the conditions of Definition 1.4 are satisfied. The pivots are −1 and 3. Matrix D = $\begin{pmatrix} 1 & 3 & 2 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ satisfies both the conditions of Definition 1.4 and is in row-echelon form. The pivots are 1,1 and 1.

Example 1.6: The matrices A, B, C are in row-echelon form where

$$A=\begin{pmatrix} 2 & 1 & 6 & 4 & -2 \\ 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix}, \quad B=\begin{pmatrix} 1 & 5 & 0 & 2 & 3 \\ 0 & 0 & 6 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C=\begin{pmatrix} 2 & 1 & 6 & 0 & 3 & 4 \\ 0 & 0 & 2 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

whereas D = $\begin{pmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 2 & 6 & 15 \\ 0 & 3 & 7 & 19 \end{pmatrix}$ is not in row-echelon form

By performing the elementary row operations $R_3 - 2R_2 \to R_3$ *and* $R_4 - 3R_2 \to R_4$ on matrix D, one obtains

$$D_1=\begin{pmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -2 & 1 \end{pmatrix}_{R_4 + R_3 \to R_4} \cong D_2 = \begin{pmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

which is in row-echelon form. Indeed, every matrix can be transformed by elementary row operations to a matrix in row-echelon form.

Theorem 1.7 Let A be an mxn matrix. Then, there is an (mxn) matrix B such that

(i) B is in row-echelon form, (ii) A is row equivalent to B.

Note If A has only zero entries, then A is already in row-echelon form. For a nonzero matrix, the reduction steps are listed below:

Reduction to row-echelon form for an mxn matrix

Step 1 Locate the first (left most) column that contains a non-zero entry.

Step 2 If necessary, interchange the first row with another row so that the first nonzero column contains a nonzero entry in the first row.

Step 3 Add appropriate multiples of the first row to each of the succeeding rows so that the first nonzero column has a nonzero entry only in the first row.

Step 4 Temporarily ignore the first row of this matrix and repeat the process on the remaining rows.

Example 1.8 This example illustrates the procedure for transformation of a matrix to row-echelon form.

$$A \cong \begin{pmatrix} 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & -2 & -4 & 5 & 7 & 14 \\ 0 & 3 & 6 & -4 & 7 & 7 \\ 0 & 0 & 0 & 2 & 4 & 9 \end{pmatrix}_{R_1 \longleftrightarrow R_2} \cong \begin{pmatrix} 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & -2 & -4 & 5 & 7 & 14 \\ 0 & 3 & 6 & -4 & 7 & 7 \\ 0 & 0 & 0 & 2 & 4 & 9 \end{pmatrix}_{\substack{R_3+2R_1 \to R_3 \\ R_4-3R_1 \to R_4}}$$

$$\cong \begin{pmatrix} 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 3 & 11 & 18 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 4 & 9 \end{pmatrix}_{\substack{R_3-3R_2 \to R_3 \\ R_4+R_2 \to R_2 \\ R_5-2R_2 \to R_5}} \cong \begin{pmatrix} 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 0 & -2 & -1 \end{pmatrix}_{\substack{R_4-2R_3 \to R_4 \\ R_5+R_3 \to R_5}}$$

$$\cong \begin{pmatrix} 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}_{R_4 \longleftrightarrow R_5} \cong \begin{pmatrix} 0 & 1 & 2 & -1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = B$$

B is row-echelon form and is row equivalent to the matrix A.

Theorem 1.2 and the fact that elementary row operations exactly duplicate elementary operations on equations imply the following Corollary.

Corollary 1.9 Any (mxn) system of linear equations is equivalent to some (mxn) system of linear equations whose Augmented matrix is in row-echelon form.

Note By multiplying each nonzero row in $[B|\mathbf{d}|$ by the reciprocal of its pivot, we can assume that each pivot in B is 1.

Example 1.10 This example illustrates the reduction of matrix A $\cong \begin{pmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

to row-echelon form making all pivots 1.

$$A = \begin{pmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{\frac{R_1}{2} \to R_1} \cong \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{\substack{R_2 - 2R_1 \to R_2 \\ R_3 - 4R_1 \to R_3}} \cong \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix}_{\substack{R_3 + R_2 \to R_3 \\ R4 + R_2 \to R_4}}$$

$$\cong \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ which is in row-echelon form, with both pivots equal to 1.

Definition 1.11 A matrix B that is in row-echelon form is in reduced row-echelon form, provided that the first nonzero element in each nonzero row is 1 and it is the only nonzero entry in its column.

Example 1.12

$$A = \begin{pmatrix} 1 & 1 & 1 & 4 & 4 \\ 2 & 3 & 4 & 9 & 16 \\ -2 & 0 & 3 & -7 & 11 \end{pmatrix}_{\substack{R_2 - 2R_1 \to R_2 \\ R_3 + 2R_1 \to R_3}} \cong \begin{pmatrix} 1 & 1 & 1 & 4 & 4 \\ 0 & 1 & 2 & 1 & 8 \\ 0 & 2 & 5 & 1 & 19 \end{pmatrix}_{\substack{R_1 - R_2 \to R_1 \\ R_3 - 2R_2 \to R_3}}$$

$$\cong \begin{pmatrix} 1 & 0 & -1 & 3 & -4 \\ 0 & 1 & 2 & 1 & 8 \\ 0 & 0 & 1 & -1 & 3 \end{pmatrix}_{\substack{R_1 + R_3 \to R_1 \\ R_2 - 2R_3 \to R_2}} \cong \begin{pmatrix} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 3 \end{pmatrix} = B$$

B is in reduced row-echelon form and A is equivalent to B.

Note The row echelon form of a matrix might not be unique.

Remarks

1. Every matrix in reduced row-echelon form is in row echelon form, but not conversely.
2. There is a strong connection between the reduced row-echelon form of a matrix A and the existence/uniqueness of solution to the system, **Ax** = **b**.

1.2 Description of Solution of Linear systems

In the previous section, a procedure for reducing a system of equations to a simpler but equivalent system has been given. In this section, we consider the process of describing the solution set for the system.

We consider the possible outcomes when solving a 2x2 system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \qquad (1.7)$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$ are given numbers. Each of these equations is the equation of a straight line. A solution to system (1.7) is a pair of numbers (x_1, x_2) that satisfies (1.7). That is, a simultaneous solution corresponds to a point of intersection. Thus, there are three possibilities:

1. The two lines are coincident (the same line), so that there are infinitely many solutions (Fig 1.1a). Thus, the system $x_1 - y_1 = 7$; $2x_1 - 2y_1 = 14$ has infinitely many solutions given by $(x_1, x_1 - 7)$, for any real number x_1.

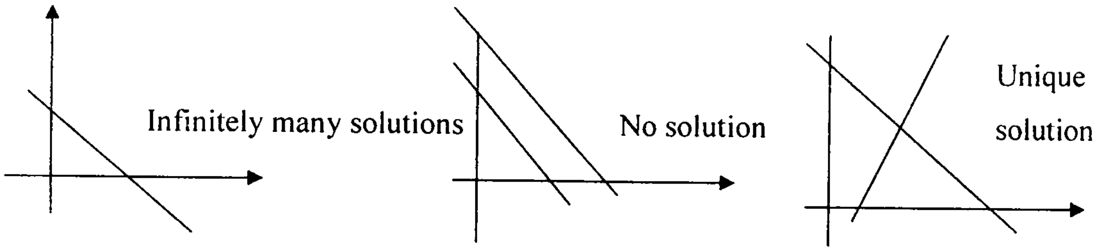

Fig 1.1 (a) lines coincide (b) lines parallel ; no point of intersection (c) one point of intersection

2. The two lines are parallel; so there are no solutions (Fig.1.1b). Thus, the system $x - y = 7$; $2x - 2y = 13$ has no solution.

3. The two lines intersect at a single point (Fig 1.1 c). Thus, the system $x - y = 7$; $x + y = 5$ has $(6, -1)$ as its unique solution.

Remark An mxn system of linear equations has either infinitely many solutions, no solution or a unique solution.

We now discuss the usefulness of row-echelon form and see how it is possible to solve the system or describe the solution set of the system by reducing the coefficient matrix to its row echelon form.

Example 1.13

$$[A|\mathbf{b}]=\begin{pmatrix}2&4&6&18\\4&5&6&24\\3&1&-2&4\end{pmatrix}_{\frac{R_1}{2}\to R_1}\cong\begin{pmatrix}1&2&3&9\\4&5&6&24\\3&1&-2&4\end{pmatrix}_{R_2-4R_1\to R_2\,;\ R_3-3R_1\to R_3}$$

$$\cong\begin{pmatrix}1&2&3&9\\0&-3&-6&-12\\0&-5&-11&-23\end{pmatrix}_{-\frac{1}{3}R_2\to R_2}\cong\begin{pmatrix}1&2&3&9\\0&1&2&4\\0&-5&-11&-23\end{pmatrix}_{\substack{R_1-2R_2\to R_1\\R_3+5R_2\to R_3}}$$

$$\cong\begin{pmatrix}1&0&-1&1\\0&1&2&4\\0&0&-1&-3\end{pmatrix}_{-R_3\to R_3}\cong\begin{pmatrix}1&0&-1&1\\0&1&2&4\\0&0&1&3\end{pmatrix}_{\substack{R_1+R_3\to R_1\\R_2-2R_3\to R_2}}$$

$$\cong\begin{pmatrix}1&0&0&4\\0&1&0&-2\\0&0&1&3\end{pmatrix}=[B|\mathbf{d}]$$

The system $A\mathbf{x}=\mathbf{b}$ has a unique solution $x_1=4, x_2=-2, x_3=3$.

Note The reduced row echelon form of the matrix has a 1 in each row and there is a unique solution.

Example 1.14 The coefficient matrix B in the system

$$[B|\mathbf{d}]=\begin{pmatrix}-5&-1&3&3\\0&3&5&8\\0&0&2&-4\end{pmatrix}$$

is in row-echelon form. The equations corresponding to this augmented matrix are

$$\begin{aligned}-5x_1-x_2+3x_3&=3\\3x_2+5x_3&=8\\2x_3&=-4\end{aligned}$$

The last equation gives $x_3=-2$. Substituting this into the second equation gives $x_2=6$.

Finally, substituting the values for x_2 and x_3 in the first equation gives $x_1=-3$.

The above procedure for finding solution of Bx=d is called <u>back substitution</u>.

Example 1.15 The linear system corresponding to the augmented matrix.

$$[B|d] = \begin{pmatrix} 1 & -3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ is } \quad \begin{aligned} x_1 - 3x_2 + 5x_4 &= 4 \\ x_3 + 2x_4 &= -7 \\ x_5 &= 1 \end{aligned}$$

Solving each equation for the variable corresponding to the pivot for that equation gives

$$x_1 = 3x_2 - 5x_4 + 4;\ x_3 = -2x_4 - 7;\ x_5 = 1$$

Note that x_2 and x_4 correspond to columns of B containing no pivot. It is possible to assign any value a to x_2 and b to x_4 and get the corresponding values for x_1, x_3 and x_5 so that all the solutions of the above system are described by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3a - 5b + 4 \\ a \\ -2b - 7 \\ b \\ 1 \end{pmatrix}$$

for any scalars a and b. x_2 and x_4 are called free variables.

Example 1.16 $[A|\mathbf{b}] = \begin{pmatrix} 2 & 4 & 6 & 18 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 30 \end{pmatrix}_{\frac{1}{2}R_1 \to R_1} \cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 30 \end{pmatrix}_{\substack{R_2 - 4R_1 \to R_2 \\ R_3 - 2R_1 \to R_3}}$

$$\cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 3 & 6 & 12 \end{pmatrix}_{-\frac{1}{3}R_2 \to R_2} \cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 12 \end{pmatrix}_{\substack{R_1 - 2R_2 \to R_1 \\ R_3 - 3R_2 \to R_3}}$$

$$\cong \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = [B|\mathbf{d}]$$

There are only two equations in three unknowns x_1, x_2, x_3 and there are an infinite number of solutions given by $(1 + x_3, 4 - 2x_3, x_3)$.

Remark The reduced row echelon form of the coefficient matrix has a row of zeros and the system has infinite number of solutions.

Example 1.17 $[A|\mathbf{b}] = \begin{pmatrix} 2 & 4 & 6 & 18 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 40 \end{pmatrix}_{\frac{1}{2}R_1 \to R_1} \cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 2 & 7 & 12 & 40 \end{pmatrix}_{\substack{R_2-4R_1 \to R_2 \\ R_3-2R_1 \to R_3}}$

$$\cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & 3 & 6 & 22 \end{pmatrix}_{\frac{-1}{3}R_2 \to R_2} \cong \begin{pmatrix} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 22 \end{pmatrix}_{\substack{R_1-2R_2 \to R_1 \\ R_3-3R_2 \to R3}}$$

$$\cong \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 10 \end{pmatrix}_{\frac{1}{10}R_3 \to R_3} \cong \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} = [B|\mathbf{d}]$$

The last equation reads as $0.x_1 + 0.x_2 - 0.x_3 = 1$, which is impossible. Therefore, the system has no solution.

Example 1.18 As the equation corresponding to the last row of the augmented matrix

$[B|\mathbf{d}] = \begin{pmatrix} 1 & -3 & 5 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is $0.x_1 + 0.x_2 + 0.x_3 = -1$, the system has no solution.

Remark The reduced row echelon form of the coefficient matrix has a row of zeros and the system has no solution.

Definition 1.19 A linear system having no solution is <u>inconsistent</u>. If it has one or more solutions, the linear system is said to be <u>consistent</u> .

Definition 1.20 Let A be an (m x n) matrix The <u>rank of A</u>, is the number of pivots in a row echelon form of A.

The general m x n system of m linear equations in n unknowns is solved by writing the system as an augmented matrix and row reducing the matrix to its reduced row echelon form. The process is continued until one of the following three situations occurs:

(i) The last nonzero equation reads $x_n = c$ for some constant c. Then, there is either a unique solution or an infinite number of solutions to the system.

(ii) The last nonzero equation reads

$$a'_{i,j} x_j + a'_{i,j+1} x_{j+1} + \dots + a'_{i,j+1+n} x_n = c$$

for some constant c where at least two of the a's are nonzero. That is, the last equation is a linear equation in two or more of the variables. Then, there are an infinite number of solutions.

(iii) The last equation reads 0 = c, where $c \neq 0$. Then, there is no solution. In this case, the system is called inconsistent. In case (i) and (ii), the system is called consistent.

Example1.21

$$[A|\mathbf{b}] = \begin{pmatrix} 1 & 3 & -5 & 1 & 4 \\ 2 & 5 & -2 & 4 & 6 \end{pmatrix}_{R_2 - 2R_1 \to R_2} \cong \begin{pmatrix} 1 & 3 & -5 & 1 & 4 \\ 0 & -1 & 8 & 2 & -2 \end{pmatrix}_{-R_2 \to R_2} \cong \begin{pmatrix} 1 & 3 & -5 & 1 & 4 \\ 0 & 1 & -8 & 2 & 2 \end{pmatrix}_{R_1 - 3R_2 \to R_1}$$

$$\cong \begin{pmatrix} 1 & 0 & 19 & -5 & -2 \\ 0 & 1 & -8 & +2 & 2 \end{pmatrix} = [B|\mathbf{d}]$$

The coefficient matrix is in reduced row echelon form (case (ii) above).

There are an infinite number of solutions. The variables x_3, x_4 are chosen arbitrarily. Then $x_2 = 2 + 8x_3 - 2x_4$ and $x_1 = -2 - 19x_3 + 5x_4$.

Theorem 1.22 Let $[A|\mathbf{b}]$ be the augmented matrix for a consistent system of linear equations in n unknowns. Further, assume that $[A|\mathbf{b}]$ is row equivalent to a matrix $[B|\mathbf{d}]$ that is in echelon form and has r nonzero rows. Then $r \leq n$, and in the solution to the given system, there are n – r variables that can be assigned arbitrary values.

Proof optional (is in Appendix A).

Corollary 1.23 Consider an (m x n) system of linear equations. If m < n, then either the system is inconsistent or it has infinitely many solutions.

Proof Consider an m x n system of linear equations where m < n. If the system is inconsistent, then there is nothing to prove. If the system is consistent, then Theorem 1.22 applies. Let the augmented matrix $[A|\mathbf{b}]$ be row equivalent to a matrix $[B|\mathbf{d}]$ that is in echelon form and has r nonzero rows. Because, the given system has m equations, the augmented matrix $[A|\mathbf{b}]$ has m rows. Therefore, the matrix $[B|\mathbf{d}]$ also has m rows. Because r is the number of nonzero rows for $[B|\mathbf{d}]$, $r \leq m$. But m < n, and therefore r < n. By theorem 1.22, there are n – r independent variables. Since n – r > 0, the system has infinitely many solutions.

1.3 Homogeneous System of Equations

The general m x n system of linear equations (1.1) is called homogeneous if all the constants $b_1, b_2 \ldots\ldots b_n$ are zero.

Note that a homogeneous system is always a consistent system, because $x_1 = x_2 = x_3 \cdots = x_n = 0$ is a solution to the system. This solution is called the trivial solution or zero solution and any other solution is called a nontrivial solution. A homogeneous system of equations, therefore, either has the trivial solution as the unique solution or it also has nontrivial (and hence infinitely many) solutions.

Example 1.24 $\quad [A|\mathbf{b}] = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 3 & 1 & -2 & 0 \end{pmatrix}$

The reduced row echelon form of the coefficient matrix is obtained as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = [B|\mathbf{d}]$$

The system has the unique solution (0, 0, 0). It has only the trivial solution.

Example 1.25

$[A|\mathbf{b}] = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 3 & -3 & 2 & 0 \\ -1 & -11 & 6 & 0 \end{pmatrix}$ The equivalent reduced row echelon form is given by

$[B|\mathbf{d}] = \begin{pmatrix} 1 & 0 & -\frac{1}{9} & 0 \\ 0 & 1 & -\frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. There are an infinite number of solutions given by $\left(-\frac{1}{9}x_3, \frac{5}{9}x_3, x_3\right)$

Example 1.26

$$[A|\mathbf{b}] = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 4 & -2 & 7 & 0 \end{pmatrix}_{R_2 - 4R_1 \to R_2} \approx \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -6 & 11 & 0 \end{pmatrix}_{-\frac{1}{6}R_2 \to R_2} \approx \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{11}{6} & 0 \end{pmatrix}_{R_1 - R_2 \to R_1}$$

$$= \begin{pmatrix} 1 & 0 & \frac{5}{6} & 0 \\ 0 & 1 & -\frac{11}{6} & 0 \end{pmatrix} = [B|\mathbf{d}]$$

There are an infinite number of solutions given by $\left(-\frac{5}{6}x_3, \frac{11}{6}x_3, x_3\right)$. This is not surprising, since the system contains three unknowns and only two equations. In fact, if there are more unknowns than equations, the homogeneous system will always have an infinite number of solutions.

Theorem 1.27 A homogeneous m x n system of linear equations always has infinitely many nontrivial solutions when $m < n$.

Proof This follows from the Corollary 1.23, because the homogeneous system is always consistent.

Definition 1.28. An $(n \times n)$ matrix A is nonsingular if the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Furthermore, A is said to be singular if A is not nonsingular.

Theorem 1.29. Let A be an $n \times n$ matrix. The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $(n \times 1)$ column vector $\mathbf{b}$ iff A is nonsingular.

Finding the inverse of a nonsingular matrix

Theorem 1.30 If A is an $(n \times n)$ nonsingular matrix, then there is a unique $(n\times n)$ matrix B such that $AB = I$.

Proof: Because A is nonsingular, Theorem 1.29 asserts that the equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution. In particular, each of the linear systems

$$A\mathbf{x} = \mathbf{e}_1, \quad A\mathbf{x} = \mathbf{e}_2, \quad \cdots \quad A\mathbf{x} = \mathbf{e}_n$$

has a unique solution; that is, there exist unique $(n\times 1)$ vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \dots \mathbf{b}_n$ such that

$$A\mathbf{b}_1 = \mathbf{e}_1, \quad A\mathbf{b}_2 = \mathbf{e}_2, \quad \cdots \quad A\mathbf{b}_n = \mathbf{e}_n$$

If B is the $(n\times n)$ matrix

$$B = [\mathbf{b}_1, \mathbf{b}_2 \dots \mathbf{b}_n], \text{ then}$$

$$AB = [A\mathbf{b}_1, A\mathbf{b}_2, \dots A\mathbf{b}_n] = [\mathbf{e}_1, \mathbf{e}_2 \dots \mathbf{e}_n] = I$$

Theorem 1.31 If A and B are $(n\times n)$ matrices such that $AB = I$, then $BA = I$. In particular $B = A^{-1}$.

Proof: We first show that $AB = I$ implies that B is nonsingular. Let $\mathbf{x}_1$ be a solution to the equation $B\mathbf{x} = \mathbf{0}$. Then

$$\mathbf{x}_1 = I\mathbf{x}_1 = (AB)\mathbf{x}_1 = A(B\mathbf{x}_1) = A\mathbf{0} = \mathbf{0}.$$

It follows that B is nonsingular because the only solution of $B\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. By Theorem 1.30, there exists an $(n\times n)$ matrix C such that $BC = I$. Thus,

$$A = AI = A(BC) = (AB)C = IC = C.$$

i.e. $BA = BC = I$.

Note It is an immediate consequence of Theorems 1.30 and 1.31 that a nonsingular matrix A has an inverse.

Theorem 1.32 Let $A\mathbf{x}=\mathbf{b}$ be the matrix equation for a $(n\times n)$ system of linear equations, and suppose that A has an inverse. Then, the system has a unique solution given by $\mathbf{x}=A^{-1}\mathbf{b}$.

Proof:- To see that $A^{-1}\mathbf{b}$ is a solution, substitute it into $A\mathbf{x}=\mathbf{b}$ for $\mathbf{x}$. This gives

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\mathbf{b} = \mathbf{b}$$

To demonstrate that ~~the~~ solution $\mathbf{x}=A^{-1}\mathbf{b}$ is unique, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$ such that $A\mathbf{y}=\mathbf{b}$

Multiplying both sides of $A\mathbf{y}=\mathbf{b}$ by A^{-1} yields

$$A^{-1}(A\mathbf{y}) = (A^{-1}A)\mathbf{y} = I\mathbf{y} = \mathbf{y} = A^{-1}\mathbf{b}$$

Note Theorem 1.32 says, if an $(n\times n)$ matrix A has an inverse, then A is nonsingular. (follows from Theorem 1.29).

Thus, we have proved the following theorem.

Theorem 1.33 An $(n\times n)$ matrix A has an inverse, iff A is nonsingular.

Example 1.34 Let the 3×3 matrix A be given by

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -2 & -5 & 1 \\ 1 & 5 & -2 \end{pmatrix}.$$

Show that A is nonsingular, and find a (3×3) matrix B such that $AB=I$.

Solution The homogeneous system $A\mathbf{x}=\mathbf{0}$ has augmented matrix

$$[A|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ -2 & -5 & 1 & 0 \\ 1 & 5 & -2 & 0 \end{array}\right]$$

which is row equivalent to

$$[B|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

The matrix B represents a homogeneous system that has only trivial solution; so A is nonsingular and the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every (3×1) vector $\mathbf{b}$. In particular, each of the linear systems

$$A\mathbf{x} = \mathbf{e}_1 \qquad A\mathbf{x} = \mathbf{e}_2 \text{ and } A\mathbf{x} = \mathbf{e}_3$$

has a unique solution. The equation $A\mathbf{x} = \mathbf{e}_1$ has augmented matrix

$$[A|\mathbf{e}_1] = \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ -2 & -5 & 1 & 0 \\ 1 & 5 & -2 & 0 \end{array}\right]$$

and this is row equivalent to $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

Backsolving yields the unique solution $x_1 = 5$, $x_2 = -3$, $x_3 = -5$.

$\therefore \mathbf{b}_1 = \begin{pmatrix} 5 \\ -3 \\ -5 \end{pmatrix}$ is the unique solution for $A\mathbf{x} = \mathbf{e}_1$.

Similarly, solving $A\mathbf{x} = \mathbf{e}_2$ and $A\mathbf{x} = \mathbf{e}_3$, the solutions are $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

The matrix B is $B = [\mathbf{b}_1 \; \mathbf{b}_2 \; \mathbf{b}_3] = \begin{pmatrix} 5 & 1 & -2 \\ -3 & -1 & 1 \\ -5 & -2 & 1 \end{pmatrix}$

Check that $AB = I$.

B that is obtained here is A^{-1}. ~~Moreover this we require further that $BA = I$. The next theorem proves this~~

Procedure for calculating an Inverse

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$, A^{-1} is the matrix $B = [\mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3]$

where $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are the unique solutions to the three systems $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$, $A\mathbf{x} = \mathbf{e}_3$

To organize the computation of A^{-1} so that A is reduced to echelon form only once instead of three times, we form the (3×6) matrix $[A|\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]$

$$[A|\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -3 & 7 & -1 & 0 & 1 \end{array}\right] R_3 + 3R_2 \to R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + 2R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 22 & -9 & -3 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] R_1 - 2R_2 \to R_1 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & -23 & -7 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] = [I|B]$$

To see that $B = A^{-1}$, note $[A|I] = [A|\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3]$ is row equivalent to $[I|B] = [I|\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3]$ and hence the three systems $A\mathbf{x} = \mathbf{e}_1$, $A\mathbf{x} = \mathbf{e}_2$, $A\mathbf{x} = \mathbf{e}_3$ are equivalent to the systems

$I\mathbf{x} = \mathbf{b}_1$, $I\mathbf{x} = \mathbf{b}_2$, $I\mathbf{x} = \mathbf{b}_3$ respectively.

$I\mathbf{x} = \mathbf{b}_1$ gives $x_1 = 54$, $x_2 = -16$, $x_3 = -7$ $\therefore$ $A\mathbf{x} = \mathbf{e}_1$ has the unique solution

$$\mathbf{x} = \mathbf{b}_1 = \begin{pmatrix} 54 \\ -16 \\ -7 \end{pmatrix}.$$

Similarly, the solutions for $A\mathbf{x} = \mathbf{e}_2$ and $A\mathbf{x} = \mathbf{e}_3$ are

$\mathbf{b}_2 = \begin{pmatrix} -23 \\ 7 \\ 3 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix}$. $\therefore A^{-1} = B = \begin{pmatrix} 54 & -23 & -7 \\ -16 & 7 & 2 \\ -7 & 3 & 1 \end{pmatrix}$,

Assignment – 1

(13-f)

1. Using elementary row transformations determine whether the following matrix is invertible and find the inverse, if it exists.

a) $A = \begin{pmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{pmatrix}$; b) $A = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 3 & 2 \\ 2 & -2 & 7 \end{pmatrix}$

2. Solve each of the (3×3) linear systems $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$, where

$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 6 & 2 \\ 3 & 4 & -7 \end{pmatrix}$ $\mathbf{b}_1 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 3 \\ 8 \\ 7 \end{pmatrix}$

or else determine that the system is inconsistent.

CHAPTER 2

VECTOR SPACES

We all know what we mean by saying force is a vector and mass is a scalar. Mathematically, we describe such vectors by directed line segments. We also know how to add two such vectors and how to multiply such a vector by a scalar. Further, we know that addition of vectors and scalar multiplication of vectors satisfy the following:

1. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors, then $\mathbf{v}_1 + \mathbf{v}_2$ is also a vector.
2. For any three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we have $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$.
3. There exists a zero vector '$\mathbf{0}$' such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for every vector $\mathbf{v}$.
4. For any given vector $\mathbf{v}$, there is another vector, denoted by $-\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = -\mathbf{v} + \mathbf{v} = \mathbf{0}$.
5. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$ for any two vectors.
6. For any scalar α and vector $\mathbf{v}$, $\alpha \,.\, \mathbf{v} = \alpha\, \mathbf{v}$ is again a vector.
7. For vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and scalar α, we have $\alpha \,.\, (\mathbf{v}_1 + \mathbf{v}_2) = \alpha \,.\, \mathbf{v}_1 + \alpha \,.\, \mathbf{v}_2$
8. For scalars α, β and any vector $\mathbf{v}$, we have $(\alpha + \beta) \,.\, \mathbf{v} = \alpha \,.\, \mathbf{v} + \beta \,.\, \mathbf{v}$.
9. $\alpha \,.\, (\beta \mathbf{v}) = \alpha\, \beta \,.\, \mathbf{v} = \beta\alpha \,.\, \mathbf{v}$ for any two scalars α, β and vector $\mathbf{v}$.
10. $1 \,.\, \mathbf{v} = \mathbf{v}$ for any vector $\mathbf{v}$.

The first five say that the set of vectors under addition forms a commutative group.
It so happens that there are a large number of mathematical objects like matrices, functions that also satisfy similar properties. Hence it is useful to generalize the above notions and define an algebraic object known as 'a vector space over the scalars'. For us, scalars mean either the real (scalar) numbers or the complex (scalar) numbers.

2.1 Definition and Examples

Definition 2.1 A vector space over the set (field) of scalars **R** (or **C**) is a non empty set V whose elements are called vectors together with a binary operation '+' on V and a scalar multiplication

$m : \mathbf{R}\,(\mathbf{C}) \times V \rightarrow V$ satisfying the following:

1. For $\mathbf{v}_1, \mathbf{v}_2 \in V$, we have $\mathbf{v}_1 + \mathbf{v}_2 \in V$ (V is closed under +).
2. For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$, we have $(\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$ (+ is an associative operation on V).

3. There exists $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v}$ for every $\mathbf{v} \in V$. (That is, there exists an identity element in V with respect to the operation addition)

4. Given $\mathbf{v} \in V$, there exists $\mathbf{w} \in V$ such that $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} = \mathbf{0}$. Such a $\mathbf{w}$ is denoted by $-\mathbf{v}$

(That is, every element $\mathbf{v} \in V$ has an inverse element in V with respect to addition).

5. For $\mathbf{v}_1, \mathbf{v}_2 \in V$, we have $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$ (That is, addition in V is commutative). For scalar multiplication m: **R** (**C**) x V $\rightarrow$V, we write m $(\alpha, \mathbf{v}) = \alpha.\mathbf{v} = \alpha\ \mathbf{v}$.

6. For $\alpha \in$ **R** (**C**), $\mathbf{v} \in V$, $\alpha\ \mathbf{v} \in V$.

7. For $\alpha \in$ **R** (**C**), $\mathbf{v}_1, \mathbf{v}_2 \in V$, $\alpha\ (\mathbf{v}_1+\mathbf{v}_2) = \alpha\ \mathbf{v}_1 + \alpha\ \mathbf{v}_2$ (That is scalar multiplication is distributive over addition)

8. For $\alpha, \beta \in$ **R** (**C**), $\mathbf{v} \in V$, $(\alpha + \beta)\ \mathbf{v} = \alpha\ \mathbf{v} + \beta\ \mathbf{v}$

9. For $\alpha, \beta \in$ **R** (**C**), $\mathbf{v} \in V$, $\alpha . (\beta\ \mathbf{v}) = \alpha\ \beta . \mathbf{v} = \beta . (\alpha\ \mathbf{v})$.

10. $1 . \mathbf{v} = \mathbf{v}$ for $\mathbf{v} \in V$.

(Strictly speaking axioms 1 and 6 are not required).

Notation If V is a vector space under '+' and '.' , we say that (V , + , .) is a vector space. If V is a vector space over **R** (**C**), then elements of V are called vectors. We shall denote vectors by lower case Roman letters **a, b**,..., and scalars by Greek letters α , β, ... It is clear from our introduction that the set of all physical vectors in space is a Vector space under usual addition and scalar multiplication of vectors.

Are there other vector spaces? Yes. In fact, this is the reason for generalization.

Example 2.2 Take V = **R** = the set of real numbers and '+' in V to be the usual addition of real numbers. We define the scalar multiplication m: **R** x V $\rightarrow$V by m $(\alpha, \mathbf{v}) = \alpha . \mathbf{v}$ where the product in the right hand side is the usual product of the real numbers α and $\mathbf{v}$. Then, we readily see that all the ten conditions for a vector space hold good and V = **R** is a vector space over **R** with the above addition and scalar multiplication.

Example 2.3 Let V = {**0**} be a singleton set. The single element in V is denoted by **0**. We may define + in V by setting **0** + **0** = **0** and scalar multiplication by α. **0** = **0** for every scalar α. Then V becomes a vector space (check the details).

Example 2.4 Let $V = \{\mathbf{x} = (x_1, x_2) \mid x_1, x_2 \in \mathbf{R}\}$. For $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$, we define $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$. Also for $\mathbf{v} \in V$ and α, a scalar, we define $\alpha\,\mathbf{x} = (\alpha x_1, \alpha x_2)$. Then V with these operations is a vector space. We denote it by $\mathbf{R}^2$.

Example 2.5 In general, we may take $V = \{(x_1, x_2 \ldots x_n) \mid x_i \in \mathbf{R},\ 1 \le i \le n\} = \mathbf{R}^n$. We define $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$ for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ and $\mathbf{y} = (y_1, y_2 \ldots y_n) \in \mathbf{R}^n$. Also for $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$ and α a scalar, we define $\alpha\,\mathbf{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

Then $V = \mathbf{R}^n$ is a vector space over **R** for n = 1,2,3, … .

Example 2.6 Let $\mathbf{P}_n(\mathbf{R})$ be the set of all polynomials with real coefficients of degree $\le n$. Then with the usual addition of polynomials and multiplication by scalars, $\mathbf{P}_n(\mathbf{R})$ is a vector space.

If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$, then

$(f + g)(x) = (a_n + b_n)x^n + \ldots + (a_1 + b_1)x + (a_0 + b_0)$.

For $\alpha \in \mathbf{R}$, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \in \mathbf{P}_n(\mathbf{R})$.

$(\alpha f)(x) = \alpha a_n x^n + \alpha a_{n-1} x^{n-1} + \ldots + \alpha a_1 x + \alpha a_0$

(Note that for any $f \in \mathbf{P}_n(\mathbf{R})$, we may write $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ with $a_i \in \mathbf{R}$; and when $\deg f(x) = k < n$, $a_{k+1} = a_{k+2} \ldots = a_n = 0$).

Remark By $\mathbf{P}_n(X)$, we mean the set of all polynomials of degree $\le$ n and with coefficients in X. Also, it is evident that $\mathbf{P}_0 \subseteq \mathbf{P}_1 \subseteq \mathbf{P}_2 \subseteq \mathbf{P}_3 \ldots \subseteq \mathbf{P}_n \subseteq \mathbf{P}_{n+1} \subseteq \ldots$.

Example 2.7 If X = (a, b) is any open interval in **R** with a < b and **C**(X, **R**) = the set of all real valued continuous functions on X, then **C**(X, **R**) is a vector space over **R** under the following operations.

$(f + g)(x) = f(x) + g(x)$ for $f, g \in \mathbf{C}(X, \mathbf{R})$, $x \in (a, b) = X$.

$(\alpha f)(x) = \alpha\ f(x)$, for $\alpha \in \mathbf{R}$, $f \in \mathbf{C}(X, \mathbf{R})$, $x \in X$.

Example 2.8 If X=(a, b) is any open interval in **R** with a < b and $\mathbf{C}^k(X, \mathbf{R})$ = the set of all real valued functions which are k times differentiable and $\frac{d^k f}{dx^k}$ is continuous on X, then $\mathbf{C}^k(X, \mathbf{R})$ is a vector space over **R** under the operations

$(f + g)(x) = f(x) + g(x)$ for $f, g \in \mathbf{C}^k(X, \mathbf{R})$, $x \in X$.

$(\alpha f)(x) = \alpha \cdot f(x)$ for $\alpha \in \mathbf{R}$, $f \in \mathbf{C}^k(X, \mathbf{R})$, $x \in X$.

Example 2.9 Let X = (a, b), a < b. Let $F(X, \mathbf{R}) = \{f : X \to \mathbf{R} \mid f \text{ is a function from } X \text{ to } \mathbf{R}\}$.

Let $V = \{f \in F(X, \mathbf{R}), f \text{ is Riemann integrable}\}$.

Then V is a vector space over **R** under $(f + g)(x) = f(x) + g(x)$; $(\alpha f)(x) = \alpha f(x)$, $f, g \in V$, $x \in X$.

Example 2.10 Let $\mathbf{C}^\infty(X, \mathbf{R})$ be the set of all infinitely many times differentiable functions on X = (a, b). Then $\mathbf{C}^\infty(X, \mathbf{R})$ is a vector space over **R** under

$(f + g)(x) = f(x) + g(x)$ for $f, g \in \mathbf{C}^\infty(X, \mathbf{R})$, $x \in X$.

$(\alpha f)(x) = \alpha f(x)$ for $\alpha \in \mathbf{R}$, $f \in \mathbf{C}^\infty(X, \mathbf{R})$, $x \in X$.

Example 2.11 Consider the following system of linear equations in the variables $x_1, x_2, \ldots x_n$.

$$a_{11} x_1 + a_{12} x_2 + \ldots a_{1n} x_n = 0$$
$$a_{21} x_1 + a_{22} x_2 + \ldots a_{2n} x_n = 0$$
$$\text{- - - - - - - - - - - - - - - - - - - -}$$
$$a_{n1} x_1 + a_{n2} x_2 + \ldots a_{nn} x_n = 0$$

This system, definitely, has atleast one solution, namely $(x_1, x_2, x_3, \ldots, x_n) = (0, 0, \ldots 0)$.

Let $V = \{(x_1, x_2 \ldots x_n) \in \mathbf{R}^n \mid (x_1, x_2, \ldots, x_n)\}$ is a solution of the above system of equations}. It is immediately verified that if $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are any two solutions of V, then their sum $(x_1 + y_1, x_2 + y_2, \ldots x_n + y_n)$ is also a solution of the same.

Further, $\alpha \mathbf{x} = \alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$ is also a solution. This means that V = set of solutions of the above linear system of equations is a vector space.

Example 2.12 Consider an ordinary linear differential equation of the form

$$L(y) = \frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + a_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

If y_1 and y_2 are solutions of this differential equation, we observe that c_1 y_1 + c_2 y_2 is also a solution of the same for any two scalars c_1, c_2. Then the set of solutions of L (y) = 0 is a vector space.

Example 2.13 Let V = $\{(x_1, x_2) \mid x_2 = mx_1$, where m is a fixed real number$\}$. For $\mathbf{x} = (x_1, x_2) \in V$, $y = (y_1, y_2) \in V$, define $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$ and $\alpha\,\mathbf{x} = \alpha\,(x_1, x_2)$, $\alpha \in \mathbf{R}$. Then (V, +,.) is a vector space. (Observe that V is the set of points in $\mathbf{R}^2$ lying on the line y = mx, passing through the origin).

Example 2.14 Let V = $\{(x_1, x_2) \mid x_2 = 2x_1 + 3, x \in \mathbf{R}\}$. That is, V is the set of points lying on the line $x_2 = 2x_1 + 1$. V is not a vector space under addition and scalar multiplication defined in Example 2.13.

Example 2.15 Let V = $\{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}$. Then, it is readily observed that, if $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ are in V and $\mathbf{z} = \mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3) = (z_1, z_2, z_3)$, then $z_1 + 2z_2 + 3z_3 = (x_1 + y_1) + 2(x_2 + y_2) + (3x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3)$ $= 0 + 0 = 0$ which implies $\mathbf{x} + \mathbf{y} \in V$. Also, for $\alpha \in \mathbf{R}$, $\alpha\ \mathbf{x} = (\alpha x_1, \alpha x_2, \alpha x_3)$ and $\alpha x_1 + 2.\ \alpha x_2 + 3.\alpha x_3 = \alpha(x_1 + 2x_2 + 3x_3) = 0$ which implies $\alpha\mathbf{x} \in V$. Therefore (V, +, .) is a vector space. Observe that V is the set of points in $\mathbf{R}^3$ lying on a plane passing through the origin.

Example 2.16 Let V = $\mathbf{R}^{mxn}$ denote the set of matrices of order m x n with real components. Then, with the usual sum and scalar multiplication of matrices, it can be seen that $\mathbf{R}^{mxn}$ is a vector space.

Example 2.17 Let V = $\mathbf{C}^n = \{(c_1, c_2 \ldots c_n) \mid c_i$ is a complex number for i = 1,2, …n$\}$ and the set of scalars is the set of complex numbers. It can be seen that $\mathbf{C}^n$ is a vector space under the usual addition of complex numbers and multiplication of complex numbers.

This long list of examples of vector spaces is an indication of the importance and scope of the innumerable applications of this notion of linear spaces. We now present some elementary results about vector spaces. We concentrate on a single vector space V and its properties.

Let V be a vector space over **R**(**C**). Then either V = {**0**} or V must contain infinitely many vectors.

If V ≠ {**0**}, then there exists **x** ≠ **0** in V. Then {α **x** | α∈**R** (**C**)} is an infinite set. This is because of the following proposition.

Proposition 2.18 Let (V, +,·) be a vector space. Then

(i) α **0** = **0** for every real number α

(ii) 0. **x** = **0** for every **x** ∈ V

(iii) If α **x** = **0**, then α = 0 or **x** = **0** (or both)

(iv) (−1) **x** = − **x** for every vector **x** ∈ V.

Proof (i) By (3) of Definition 2.1, **0** + **0** = **0** ; and from (7) of Definition 2.1

$$\alpha\,(\mathbf{0} + \mathbf{0}) = \alpha\,\mathbf{0} + \alpha\mathbf{0} = \alpha\mathbf{0} \qquad (2.1)$$

Adding −α **0** to both sides of the equation (2.1) and using (2) of Definition 2.1, we obtain

$[\alpha\mathbf{0} + \alpha\mathbf{0}] + (-\,\alpha\mathbf{0}) = \alpha\mathbf{0} + (-\,\alpha\mathbf{0})$; $\alpha\mathbf{0} + [\alpha\mathbf{0} + (-\,\alpha\mathbf{0})] = \mathbf{0}$; $\alpha\mathbf{0} + \mathbf{0} = \mathbf{0}$; $\alpha\mathbf{0} = \mathbf{0}$.

(ii) 0 + 0 = 0; 0**x** = (0+0)**x** = 0**x** + 0**x** (by (7) of Definition 2.1;

$0\mathbf{x} + (-\,0\mathbf{x}) = 0\mathbf{x} + [\,0\mathbf{x} + (-\,0\mathbf{x})\,]$; $\mathbf{0} = 0\mathbf{x} + \mathbf{0} = 0\mathbf{x}$

(iii) Let α**x** = **0**. If α ≠ 0, multiply both sides by $\frac{1}{\alpha}$ to obtain $(\frac{1}{\alpha})\,(\alpha\,\mathbf{x}\,) = \frac{1}{\alpha}\,(\mathbf{0}) = \mathbf{0}$ (by (i)),

but $(\frac{1}{\alpha})\,(\alpha\mathbf{x}\,) = 1\mathbf{x} = \mathbf{x}$, (by (9) of Definition 2.1) so **x** = **0**.

(iv) 1+(−1) =0; $\mathbf{0} = 0\mathbf{x} = [\,1+(-1)\,]\,\mathbf{x} = 1\,\mathbf{x} + (-1)\,\mathbf{x} = \mathbf{x} + (-1)\mathbf{x}$ (by (ii)). Add −**x** to both sides

$\mathbf{0} + (-\mathbf{x}) = \mathbf{x} + (-1)\,\mathbf{x} + (-\mathbf{x}) = \mathbf{x} + (-\mathbf{x}) + (-1)\,\mathbf{x} = \mathbf{0} + (-1)\mathbf{x} = (-1)\mathbf{x}$. Thus, $-\,\mathbf{x} = (-1)\,\mathbf{x}$

2.2 Subspaces

From Example 2.4, section 2.1, we know that $\mathbf{R}^2 = \{(x_1, x_2)\} \mid x_1 , x_2 \in\mathbf{R}\}$ is a vector space. Also, from Example 2.13, section 2.1, $V = \{(x_1, x_2) \mid x_2 = mx_1\}$ is a vector space and $V \subseteq \mathbf{R}^2$. That is, $\mathbf{R}^2$ has a subset which is a vector space. In fact, all vector spaces have subsets which are also vector spaces.

Definition 2.19 Let H be a nonempty subset of a vector space V and suppose that H is itself a vector space under the operations of addition and scalar multiplication defined on V. Then H is a subspace of V.

The following result makes it easy to determine whether a subset of V is indeed a subspace of V.

Theorem 2.20 A nonempty subset H of the vector space V is a subspace of V if the following conditions hold:

(i) If $\mathbf{x} \in H$ and $\mathbf{y} \in H$, then $\mathbf{x} + \mathbf{y} \in H$; (ii) If $\mathbf{x} \in H$, then $\alpha\,\mathbf{x} \in H\ \forall$ scalar α.

Proof In order to show that H is a vector space, we need to prove that all the axioms (1) to (10) hold good under the operations of addition and scalar multiplication defined in V. Axioms (1) and (6) hold by the given hypothesis in H. Since vectors in H are also in V, the axioms (2), (5), (7), (9) and (10) hold good. Let $\mathbf{x} \in H$, then $0\mathbf{x} \in H$ by (ii) of the given hypothesis. But $0\mathbf{x} = \mathbf{0}$ by Proposition 2.18. Therefore, $\mathbf{0} \in H$, and therefore (3) holds good. Also, $(-1)\,\mathbf{x} \in H\ \forall\ \mathbf{x} \in H$ (by (ii) of the given hypothesis). Now $-\mathbf{x} = (-1)\,\mathbf{x} \in H$ (by Proposition 2.18). Therefore, axiom (4) holds good. Therefore, H is a vector space.

Remarks (i) Thus to test whether a nonempty subset H of V is a vector space, it is only necessary to verify that $\mathbf{x} + \mathbf{y}$ and $\alpha\mathbf{x} \in H$ where $\mathbf{x}, \mathbf{y} \in H$ and α is a scalar.

(ii) Every subspace of a vector space V contains **0**. (This helps in verifying whether a particular subset of V is not a vector space, that is if a subset does not contain **0**, then it is not a subspace).

Example 2.21 For any vector space V, the subset $\{\mathbf{0}\}$ consisting of the zero vector alone is a subspace since $\mathbf{0} + \mathbf{0} = \mathbf{0}$ and $\alpha\mathbf{0} = \mathbf{0}\ \forall$ real number α. It is called a **trivial subspace**.

Example 2.22 V is a subspace of itself for every vector space V.

Remark Examples 2.21 and 2.22 show that every vector space V contains two subspaces $\{\mathbf{0}\}$ and V (unless of course $V = \{\mathbf{0}\}$). We shall find other subspaces. These are called **proper subspaces**.

Example 2.23 $H=\{(x_1, x_2) \mid x_2 = mx_1\}$ is a subspace of $\mathbf{R}^2$. In fact, we shall show later that the set of vectors lying on straight lines passing through the origin are the only proper subspaces of $\mathbf{R}^2$.

Example 2.24 $H = \{(x_1, x_2, x_3) \mid x_1 = at, x_2 = bt, x_3 = ct,\ \ a, b, c, t \text{ are real}\}$. H consists of vectors in $\mathbf{R}^3$ lying on straight lines passing through the origin. H is a subspace of $\mathbf{R}^3$ since, for $\mathbf{x} = (at_1, bt_1, ct_1) \in H$, $\mathbf{y} = (at_2, bt_2, ct_2) \in H$,

$\mathbf{x} + \mathbf{y} = (a(t_1 + t_2), b(t_1 + t_2), c(t_1 + t_2)) \in H$ and $(a(\alpha t_1), b(\alpha t_2), c(\alpha t_3)) \in H$.

Example 2.25 Let $H = \{ (x_1, x_2, x_3) \mid ax + by + cz = 0,\ \ a,b,c \text{ real}\}$

H is a subspace of $\mathbf{R}^3$. Note that H consists of vectors lying on planes passing through the origin.

Remarks 1. We shall show later that the set of vectors lying on either straight lines or planes passing through the origin are the only proper subspaces of $\mathbf{R}^3$.

2. Not every vector space has proper subspaces.

Example 2.26 Let H be a subspace of **R**. (**R** is a vector space over itself; that is **R** is a vector space with the scalars taken to be the reals). If $H \neq \{\mathbf{0}\}$, then H contains a nonzero real number say α. Then $(1/\alpha)\,\alpha = 1 \in H$. $\beta 1 = \beta \in H$ for every real β. $\Rightarrow \mathbf{R} \subseteq H$. But $H \subseteq \mathbf{R}$. Therefore, **R** has no proper subspace.

Example 2.27 If $\mathbf{P}_n$ denotes the vector space of polynomials of degree $\leq n$, and if $0 \leq m < n$, then $\mathbf{P}_m$ is a proper subspace of $\mathbf{P}_n$.

Example 2.28 $\mathbf{P}_n\,[0,1]$ denotes the set of polynomials defined over $[0,1]$ of degree $\leq n$. $\mathbf{P}_n\,[0,1] \subseteq \mathbf{C}\,[0,1]$ since every polynomial is continuous. $\mathbf{P}_n$ is a vector space for every integer n and therefore each $\mathbf{P}_n$ is a subspace of $\mathbf{C}\,[0,1]$.

Example 2.29 Let $\mathbf{C}^1[0,1]$ denote the set of functions with continuous first derivatives defined on $[0,1]$. Since every differentiable function is continuous $\mathbf{C}^1\,[0,1] \subseteq \mathbf{C}[0,1]$. The sum and scalar multiple of differentiable functions are differentiable, so $\mathbf{C}^1\,[0,1]$ is a subspace of $\mathbf{C}[0,1]$. It is a proper subspace since not every continuous function is differentiable.

Example 2.30 If $f \in \mathbf{C}[0,1]$, then $\int_0^1 f(x)\,dx$ exists. Let $H = \{f \in \mathbf{C}[0,1] \mid \int_0^1 f(x)\,dx = 0\}$. If $f, g \in H$, then $\int_0^1 [f(x) + g(x)]\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = 0 + 0 = 0$ and $\int_0^1 \alpha f(x)\,dx = \alpha \int_0^1 f(x)\,dx = \alpha 0 = 0\ \forall$ real number α. Therefore, it is a proper subspace of $\mathbf{C}[0,1]$.

2.3 Linear independence, Basis and Dimension

Definition 2.31 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ be vectors in a vector space V. Then, any expression of the form $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n$ where $\alpha_1, \alpha_2, \ldots \alpha_n$ are scalars is called a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$.

Example 2.32

In R^3, $\begin{pmatrix}-6\\6\\8\end{pmatrix}$ is a linear combination of, $\begin{pmatrix}-2\\4\\2\end{pmatrix}$ and, $\begin{pmatrix}-2\\1\\3\end{pmatrix}$ since, $\begin{pmatrix}-6\\6\\8\end{pmatrix} = 1\begin{pmatrix}-2\\4\\2\end{pmatrix} + 2\begin{pmatrix}-2\\1\\3\end{pmatrix}$

Example 2.33 In $\mathbf{P}_n$, every polynomial can be written as a linear combination of the monomials $1, x, x^2, \dots x^n$.

Definition 2.34 Let $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ be n vectors in a vector space V. The vectors are said to be <u>**linearly dependent**</u>, if there exists n scalars $\alpha_1, \alpha_2, \dots \alpha_n$ <u>not all zero</u> such that $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0}$. If the only linear combination of these vectors that equals the zero vector is the <u>trivial linear combination</u> (with $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$), then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n$ are said to be <u>**linearly independent**</u>.

Note

1. The empty set is linearly independent.
2. If one of the vectors is the zero vector, then the set is linearly dependent.

The following result helps us to determine whether a set of two vectors is linearly dependent or independent.

Theorem 2.35 Two vectors are linearly dependent iff one is a scalar multiple of the other.

Proof Let $\mathbf{y} = \alpha\, \mathbf{x}$ for some scalar $\alpha \neq 0$. Then $\alpha\, \mathbf{x} - \mathbf{y} = \mathbf{0}$ and $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Conversely, suppose $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent. Then, by definition, there exists constant α_1 and α_2, not both zero, such that $\alpha_1 \mathbf{x} + \alpha_2 \mathbf{y} = \mathbf{0}$. If $\alpha_1 \neq 0$, divide by α_1, $\mathbf{x} + (\alpha_2/\alpha_1)\, \mathbf{y} = \mathbf{0}$ or $\mathbf{x} = -(\alpha_2/\alpha_1)\, \mathbf{y}$ and therefore $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$. If $\alpha_1 = 0$, then $\alpha_2 \neq 0$ and hence $\mathbf{y} = \mathbf{0} = 0\mathbf{x}$. The following result is useful in some cases.

Theorem 2.36 In a vector space V, the nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \dots \mathbf{v}_n$ are linearly dependent iff at least one of them can be written as a linear combination of the vectors that precede it. That is, for some k, $1 < k \leq n$, there are scalars $\alpha_1, \alpha_2, \dots \alpha_{k-1}$ such that

$$\mathbf{v}_k = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1} \qquad (2.2)$$

Proof Suppose (2.2) holds. Then $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_1 + \dots + \alpha_{k-1} \mathbf{v}_{k-1} - \mathbf{v}_k + 0\, \mathbf{v}_{k+1} + \dots + 0\, \mathbf{v}_n = \mathbf{0}$ implying that the vectors are linearly dependent. Conversely, suppose that the vectors are linearly dependent so that there exists n scalars $\alpha_1, \alpha_2, \dots \alpha_n$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n = \mathbf{0} \qquad (2.3).$$

Let k be the largest integer such that $\alpha_k \neq 0$ (k may be equal to n). Note that $k \geq 2$ for if $k = 1$, then (2.3) becomes $\alpha_1 \mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\,\mathbf{v}_n = \mathbf{0}$. Since $\alpha_1 \neq 0$, $\mathbf{v}_1 = \mathbf{0}$ which contradicts the fact that $\mathbf{v}_1$ is a non zero vector. Thus $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_1 + \ldots + \alpha_k \mathbf{v}_k = \mathbf{0}$ with $\alpha_k \neq 0$. Dividing by α_k gives $\mathbf{v}_k = -(\alpha_1/\alpha_k)\mathbf{v}_1 - (\alpha_2/\alpha_k)\mathbf{v}_2 \ldots - (\alpha_{k-1}/\alpha_k)\mathbf{v}_{k-1} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \ldots + \beta_{k-1} \mathbf{v}_{k-1}$.

Example 2.37 The vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and , $\begin{pmatrix}-4\\3\end{pmatrix}$, and $\begin{pmatrix}3\\1\end{pmatrix}$ in $\mathbf{R}^2$ are linearly dependent. We need to find constants α_1, α_2 such that $\begin{pmatrix}3\\1\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\2\end{pmatrix} + \alpha_2 \begin{pmatrix}-4\\3\end{pmatrix}$ which gives

$\alpha_1 - 4\alpha_2 = 3$; $2\alpha_1 + 3\alpha_2 = 1$. On solving, we get $\alpha_1 = 13/11$, $\alpha_2 = -5/11$.

Thus $\begin{pmatrix}3\\1\end{pmatrix} = \frac{13}{11}\begin{pmatrix}1\\2\end{pmatrix} - \frac{5}{11}\begin{pmatrix}-4\\3\end{pmatrix}$. Therefore the vectors are linearly dependent.

Example 2.38 The monomials $1, x, x^2, \ldots, x^n$ are linearly independent in $\mathbf{P}_n$. If they are dependent, then there is an integer k and constants $\alpha_0, \alpha_1 \ldots \alpha_{k-1}$ such that $x^k = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots \alpha_{k-1} x^{k-1}$ which is not possible since the degree of the polynomial on the left is k while the degree of the polynomial on the right is $\leq k-1$. Therefore, the monomials must be independent.

Definition 2.39 The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ in a vector space V are said to **span** V if every vector in V can be written as a linear combination of them. That is $\forall\, \mathbf{v} \in V$, there are scalars $\alpha_1, \alpha_2 \ldots \alpha_n$ such that $\mathbf{v}_1 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots \alpha_n \mathbf{v}_n$.

Example 2.40 The vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ span $\mathbf{R}^3$.

Example 2.41 The monomials $1, x, x^2, \ldots x^n$ span $\mathbf{P}_n$.

Definition 2.42 A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a **basis** for V, if (1) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is linearly independent; (2) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ spans V.

Example 2.43 In $\mathbf{R}^n$, define $\mathbf{e}_1 = (1,0,0,0 \cdots 0)$, $\mathbf{e}_2 = (0,1,0,0 \cdots 0)$, $\ldots$ $\mathbf{e}_n$ $(0,0,0,0 \cdots 1)$. If $\mathbf{0} = (0,0,0,0 \cdots 0) = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \ldots + \alpha_n \mathbf{e}_n = (\alpha_1, \alpha_2, \cdots \alpha_n)$, then $\alpha_1 = \alpha_2 \ldots = \alpha_n = 0$. Therefore, $\mathbf{e}_1, \mathbf{e}_2 \ldots \mathbf{e}_n$ are linearly independent. If $\mathbf{x} = (x_1, x_2, \cdots x_n) \in \mathbf{R}^n$, then $\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \ldots + x_n \mathbf{e}_n$. Therefore, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ span $\mathbf{R}^n$. Therefore $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$ called a standard basis for $\mathbf{R}^n$.

Example 2.44 The vectors $(1,2)$ and $(-1,1)$ in $\mathbf{R}^2$ are linearly independent because neither one is a multiple of the other. Further, any vector $\mathbf{x} = (x_1, x_2)$ in $\mathbf{R}^2$ can be expressed as

$\mathbf{x} = (x_1, x_2) = \left(\frac{x_1 + x_2}{3}\right)(1,2) + \left(\frac{-2x_1 + x_2}{3}\right)(-1,1)$. Therefore, $\{(1,2),(-1,1)\}$ is basis for $\mathbf{R}^2$.

Example 2.45 Since the monomials $1, x, x^2, \ldots x^n$ are linearly independent in $\mathbf{P}_n$ and since they span $\mathbf{P}_n$, $\{1, x, x^2, \ldots x^n\}$ is a basis for $\mathbf{P}_n$.

Remarks In $\mathbf{R}^2$, $\{(1,0),(0,1)\}$ and $\{(1,2),(-1,1)\}$ are bases. In fact, $\mathbf{R}^2$ has an infinite number of bases. In fact in what follows we prove the following two results.

1. A set of m vectors in $\mathbf{R}^n$ is linearly dependent if $m > n$.

2. Any set of n linearly independent vectors in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$.

Theorem 2.46 Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $B = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be subsets of a vector space V such that A is linearly independent and B spans V. Then $m \leq n$.

Proof Consider the set $\{\mathbf{a}_m, \mathbf{b}_1, \ldots \mathbf{b}_n\}$. This is linearly dependent because $\mathbf{a}_m$ can be expressed as a linear combination of $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Hence by Theorem 2.36, one of the elements, say $\mathbf{b}_j$ can be expressed as a linear combination of $\mathbf{a}_m, \mathbf{b}_1, \ldots, \mathbf{b}_{j-1}$. We drop this element and consider the set $B_1 = \{\mathbf{a}_m, \mathbf{b}_1, \ldots \mathbf{b}_n\} \setminus \{\mathbf{b}_j\}$. Note that B_1 has n elements and B_1 also spans V. Next we consider the set obtained by writing $\mathbf{a}_{m-1}$ and then elements of B_1, $\{\mathbf{a}_{m-1}, \mathbf{a}_m, \mathbf{b}_1, \ldots\}$ and repeat the process to obtain the set B_2. Repeating the same process m times, we obtain the set B_m. Since at each stage, we add one element from A and drop one element, the number of elements in B_m is the same as the number of elements in B, namely n. Also no $\mathbf{a}_i$ gets dropped at any stage because A is linearly independent. Thus $A \subseteq B_m$. Hence $m \leq n$.

Corollary 2.47 If $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ and $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ are bases of a vector space V, then $m = n$.

Proof Since A is linearly independent and B spans V, by Theorem 2.46, $m \leq n$. Similarly $n \leq m$.

Definition 2.48 Suppose that the nontrivial vector space V has a finite basis. Then, the dimension of V is the number of vectors in a basis. We write, dim V = n. If V does not have a finite basis, V is said to be **infinite – dimensional**. If $V = \{\mathbf{0}\}$, then V is said to be **zero – dimensional**.

Example 2.49 Since $\{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$, dim $\mathbf{R}^n = n$.

Example 2.50 Since $\{1, x, x^2, \ldots x^n\}$ is a basis for $\mathbf{P}_n$, dim $\mathbf{P}_n = n + 1$.

Example 2.51 C[0, 1] is an infinite – dimensional vector space because the functions 1, x , x^2 , ... are all in C[0, 1] and form an infinite, linearly independent set. This means that C[0,1] can not have a finite basis.

Theorem 2.52 Suppose V is a vector space of dimension n. A is a subset of V containing m vectors.

(a) If A is linearly independent, then $m \le n$.

(b) If $m > n$, then A is linearly dependent.

(c) If $m = n$ and A is linearly independent, then A is a basis of V.

Proof (a) Consider a basis B of V and apply Theorem 2.46 to sets A, B.

(b) Follows from (a).

(c) We need to show that A also spans V. If not, there exists $\mathbf{x} \in V$ such that **x** can not be written as a linear combination of $\mathbf{a}_1,\ldots,\mathbf{a}_m$. Then by Theorem 2.46, $\{\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{x}\}$ is linearly independent set with $m+1 = n+1$ vectors. This contradicts (a).

Example 2.53 $H = \{(x_1, x_2, x_3) \mid \alpha x_1 + \beta x_2 + \gamma x_3 = 0, \alpha, \beta, \gamma \text{ are real}\}$ is a subspace of $\mathbf{R}^3$. We show that dim H =2. At least one of α, β or γ is nonzero. Suppose $\gamma \neq 0$. Then, for $(x_1, x_2, x_3) \in H$,

$x_3 = -\frac{\alpha}{\gamma}x_1 - \frac{\beta}{\gamma}x_2$. Therefore, $(x_1,x_2,x_3) = \left(x_1, x_2, -\frac{\alpha}{\gamma}x_1 - \frac{\beta}{\gamma}x_2\right) = x_1\left(1,0,-\frac{\alpha}{\gamma}\right) + x_1\left(0,1,-\frac{\beta}{\gamma}\right)$.

Thus, the vector $\left(1,0,-\frac{\alpha}{\gamma}\right)$ and $\left(0,1,-\frac{\beta}{\gamma}\right)$ span H and the vectors are linearly independent and they form a basis for H. Therefore, dim H = 2.

Example 2.54 In this example, we show that if H is a two-dimensional subspace of $\mathbf{R}^3$, then all the vectors in H lie on the same plane passing through the origin. Let $\mathbf{v}_1 = (a_1, b_1, c_1)$ and $\mathbf{v}_2 = (a_2, b_2, c_2)$ be a basis for H. If $\mathbf{x} = (x_1, x_2, x_3) \in H$, then there exists scalars (real numbers) s and t such that $(x_1, x_2, x_3) = s\,(a_1, b_1, c_1) + t\,(a_2, b_2, c_2)$ which implies $x_1 = sa_1 + ta_2$, $x_2 = sb_1 + tb_2$, $x_3 = sc_1 + tc_2$. Let $\mathbf{y} = (\alpha, \beta, \gamma) = \mathbf{v}_1 \times \mathbf{v}_2$. Now, $\mathbf{y}.\ \mathbf{v}_1 = 0$ and $\mathbf{y}.\ \mathbf{v}_2 = 0$.

Consider $\alpha x_1 + \beta x_2 + \gamma x_3 = \alpha (sa_1 + t a_2) + \beta (sb_1 + tb_2) + \gamma (sc_1 + tc_2)$

$$= s(\alpha a_1 + \beta b_1 + \gamma c_1) + t(\alpha a_2 + \beta b_2 + \gamma c_2) = 0.$$

Thus, if $(x_1, x_2, x_3) \in H$, then $\alpha x_1 + \beta x_2 + \gamma x_3 = 0$ which shows that H is a plane passing through the origin with normal $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$.

Remark Examples 2.53 and 2.54 show that the only proper subspaces of $\mathbf{R}^3$ of dimension 2 are the planes passing through the origin. We now focus our attention on another way of finding subspaces of vector spaces.

Definition 2.55 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ be a set of n vectors in a vector space V. The **span** of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is the set of linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$.

span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\} = \{\mathbf{v} \mid \mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n$. where $\alpha_1, \alpha_2, \ldots \alpha_n$ are scalars$\}$.

Theorem 2.56 Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ is a subspace of V.

Proof Let $\mathbf{v}, \mathbf{w} \in$ span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$. Then $\mathbf{v} = \alpha_1\mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n$; $\mathbf{w} = \beta_1\mathbf{v}_1 + \ldots + \beta_n \mathbf{v}_n$. Now $\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + \ldots (\alpha_n + \beta_n)\mathbf{v}_n = \gamma_1 \mathbf{v}_1 + \ldots + \gamma_n \mathbf{v}_n \in$ span$\{\mathbf{v}_1 \ldots \mathbf{v}_n\}$. This implies $\mathbf{v} + \mathbf{w} \in$ span $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$. Similarly $\alpha \mathbf{v} = \alpha(\alpha_1 \mathbf{v}_1 + \ldots + \alpha_n \mathbf{v}_n) = \alpha\alpha_1 \mathbf{v}_1 + \ldots + \alpha\alpha_n \mathbf{v}_n = \gamma_1\mathbf{v}_1 + \ldots + \gamma_n\mathbf{v}_n \in$ span $\{\mathbf{v}_1, \ldots \mathbf{v}_n\}$

Theorem 2.57 Let H be a subspace of a finite dimensional vector space V. Then, H is finite dimensional and dim H $\leq$ dim V.

Proof: Let dim V = n. Let B be a basis for H. B is linearly independent in H and therefore linearly independent in V, since H is a subspace of V. Since dim V = n and a set of (n+1) or more vectors is linearly dependent, number of elements in B is $\leq$ n. Therefore, H is finite dimensional.

In what follows, we use the above theorem to find all proper subspaces of $\mathbf{R}^3$. Let H be a subspace of $\mathbf{R}^3$. Then dim H = 0, 1, 2 or 3. If dim H = 0 , then H = $\{\mathbf{0}\}$. If dim H = 3, then let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis of H. Then H = span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbf{R}^3$. (Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of $\mathbf{R}^3$ by Theorem 2.52). This implies that H $\equiv \mathbf{R}^3$. Therefore, H is not a proper subspace of $\mathbf{R}^3$. If dim H = 2, then we have already seen that H is the set of vectors lying on the planes passing through the origin.

We now see what is H if dim H =1. In fact, we will show that the only proper subspaces of $\mathbf{R}^3$ of dimension 1 are the set of vectors lying on straight lines passing through the origin.

Example 2.58 Suppose that H is a subspace of $\mathbf{R}^3$ with dim H = 1. Then, a basis of H has one vector, say $\mathbf{x} = (x_1, x_2, x_3)$ and therefore any other vector in H, say, $\mathbf{y} = (y_1, y_2, y_3)$ can be expressed as $\mathbf{y} = (y_1, y_2, y_3) = t\,(x_1, x_2, x_3) = (tx_1, tx_2, tx_3)$, t is a real number which implies that H consists of vectors lying on straight lines passing through the origin in $\mathbf{R}^3$. That is,

$H = \{ (y_1, y_2, y_3) \mid y_1 = at,\ y_2 = bt,\ y_3 = ct,\ t \text{ real} \}$

Example 2.59 Let $H = \{(x_1, x_2, x_3) \mid x_1 = at,\ x_2 = bt,\ x_3 = ct,\ t \text{ real} \}$. Then any $\mathbf{y} \in H$ is such that $\mathbf{y} = (y_1, y_2, y_3) = t\,(x_1, x_2, x_3)$. This implies that H is one-dimensional.

Remark Examples 2.58 and 2.59 show that if H is a subspace of $\mathbf{R}^3$ of dim 1, then H is the set of vectors lying on straight lines through the origin in $\mathbf{R}^3$. Thus, the only proper subspace of $\mathbf{R}^3$ of dimension one, are the straight lines through the origin in $\mathbf{R}^3$.

We have seen that n linearly independent vectors in $\mathbf{R}^n$ constitute a basis for $\mathbf{R}^n$. This fact holds in any finite dimensional vector space.

Theorem 2.60 Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m$ are linearly independent vectors in a vector space V of dimension n and m < n . Then, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ can be enlarged to a basis for V. That is, there exists vectors $\mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \ldots \mathbf{v}_n$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m, \mathbf{v}_{m+1}, \ldots \mathbf{v}_n\}$ is a basis for V.

Proof Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ be a basis for V. Then, there exists scalars such that

$$\begin{aligned}
\mathbf{v}_1 &= a_{11}\,\mathbf{u}_1 + a_{12}\,\mathbf{u}_2 + \ldots + a_{1n}\,\mathbf{u}_n \\
\mathbf{v}_2 &= a_{21}\,\mathbf{u}_1 + a_{22}\,\mathbf{u}_2 + \ldots + a_{2n}\,\mathbf{u}_n \\
&\ \vdots \\
\mathbf{v}_m &= a_{m1}\,\mathbf{u}_1 + a_{m2}\,\mathbf{u}_2 + \ldots + a_{mn}\,\mathbf{u}_n
\end{aligned}$$

Let $\mathbf{a}_i = (a_{i1}, a_{i2}, \ldots a_{in})$, i = 1, 2, ... m. The $\mathbf{a}_i$'s are m linearly independent vectors in $\mathbf{R}^n$ (otherwise the $\mathbf{v}_i$'s would not be independent. We now show this here. If possible, let $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ be linearly dependent. Then, there exists scalars $\alpha_1, \alpha_2, \ldots \alpha_m$ (not all zero) such that

$$\alpha_1 (a_{11}, a_{12}, \ldots, a_{1n}) + \alpha_2 (a_{21}, a_{22}, \ldots, a_{2n}) + \ldots + \alpha_m (a_{m1}, a_{m2}, \ldots, a_{mn}) = \mathbf{0} \qquad (2.4)$$

Now consider $\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \ldots + \alpha_m \mathbf{v}_m$. Then

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \ldots + \alpha_m \mathbf{v}_m = \alpha_1 (a_{11}\,\mathbf{u}_1 + a_{12}\,\mathbf{u}_2 + \ldots + a_{1n}\,\mathbf{u}_n) + \alpha_2 (a_{21}\,\mathbf{u}_1 + a_{22}\,\mathbf{u}_2 + \ldots + a_{2n}\,\mathbf{u}_n)$$
$$+ \ldots + \alpha_m (a_{m1}\,\mathbf{u}_1 + a_{m2}\,\mathbf{u}_2 + \ldots + a_{mn}\,\mathbf{u}_n)$$

$$= \mathbf{u}_1 (\alpha_1 a_{11} + \alpha_2 a_{21} + \ldots + \alpha_m a_{m1}) + \mathbf{u}_2 (\alpha_1 a_{12} + \alpha_2 a_{22} + \ldots + \alpha_m a_{m2}) + \ldots + \mathbf{u}_n (\alpha_1 a_{1n} + \alpha_2 a_{2n} \ldots + \alpha_m a_{mn})$$

$$= \mathbf{0} \text{ (by (2.4))}$$

This implies $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is linearly dependent which is a contradiction to the fact that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is linearly independent. Expand the $\mathbf{a}_i$'s into a basis $\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_m, \mathbf{a}_{m+1}, \ldots \mathbf{a}_n$ for $\mathbf{R}^n$ by adding $(n - m)$ linearly independent vectors to the set. Then, if $\mathbf{a}_k = (a_{k1}, a_{k2}, \ldots, a_{kn})$ for $m < k \leq n$, define $\mathbf{v}_k = a_{k1} \mathbf{u}_1 + a_{k2} \mathbf{u}_2 + \ldots + a_{kn} \mathbf{u}_n$, for $k = m+1, m+2, \ldots n$.

Since $\det \begin{pmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \ldots & & \\ a_{n1} & a_{n2} & a_{nn} \end{pmatrix} \neq 0$, the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ forms a basis for V, since it consists of n linearly independent vectors in V with dim V= n.

Theorem 2.61 Let H and K be subspaces of V. Define $H + K = \{\mathbf{h} + \mathbf{k} \mid \mathbf{h} \in H, \mathbf{k} \in K\}$. Then (i) H + K is a subspace of V (ii) If $H \cap K = \{\mathbf{0}\}$, then dim (H+K) = dim H + dim K.

Proof (i) Let $\mathbf{x}, \mathbf{y} \in H + K$ then $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1$; $\mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$, $\mathbf{u}_1, \mathbf{u}_2 \in H$, $\mathbf{v}_1, \mathbf{v}_2 \in K$. Consider $\mathbf{x} + \mathbf{y} = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2) \in H + K$ (Since H is a subspace $\mathbf{u}_1 + \mathbf{u}_2 \in H$ and K is a subspace, $\mathbf{v}_1 + \mathbf{v}_2 \in K$). Similarly, $\alpha\, \mathbf{u} = \alpha\, (\mathbf{u}_1 + \mathbf{v}_1) = \alpha\, \mathbf{u}_1 + \alpha\, \mathbf{v}_1 \in H + K$. Therefore, H + K is a subspace.

(b) Let $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_n\}$ be a basis for H. Let $\{\mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m\}$ be a basis for K. Clearly, $B = \{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m\}$ spans H + K. We now show that B is a basis for H + K. Suppose $\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_n \mathbf{u}_n + \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \ldots + \beta_m \mathbf{v}_m = \mathbf{0}$, where not all of the coefficients are zero. Let $\mathbf{h} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \ldots + \alpha_n \mathbf{u}_n$ and $\mathbf{k} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \ldots + \beta_m \mathbf{v}_m$. Then, neither **h**, nor **k** is the zero vector. Also, $\mathbf{h} \in H$ and $\mathbf{k} \in K$. But then $\mathbf{h} + \mathbf{k} = \mathbf{0}$ or $\mathbf{h} = -\mathbf{k} \in K$. Thus, $\mathbf{0} \neq \mathbf{h} \in H \cap K$ contradicting the fact that $H \cap K = \{\mathbf{0}\}$. Therefore, all α_i's and β_j's are zero. This implies $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m\}$ is linearly independent. Therefore, $B = \{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_n, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m\}$ is a basis for H+K. Therefore, dim (H+K) = n + m = dim H + dim K.

Theorem 2.62 Let V be vector space and W_1 and W_2 be subspaces of V. Then $\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$.

Proof Now $W_1 \cap W_2 \subseteq W_i$, $i = 1, 2$. Let $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_k\}$be a basis for $W_1 \cap W_2$. Extend this to a basis $\{\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m\}$ of W_1 and to a basis. $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_k, \mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n\}$ of W_2. Then $\dim W_1 + \dim W_2 - \dim (W_1 \cap W_2) = k + m + k + n - k = m + n + k$.

We now claim that $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n\}$ is a basis for $W_1 + W_2$. Let $\mathbf{x} \in W_1 + W_2$. Then $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$, with $\mathbf{w}_1 \in W_1$ and $\mathbf{w}_2 \in W_2$ Let $\mathbf{w}_1 = \sum_{i=1}^{k} \alpha_i \mathbf{u}_i + \sum_{j=1}^{m} \beta_j \mathbf{v}_j$,

$\mathbf{w}_2 = \sum_{i=1}^{k} \gamma_i \mathbf{u}_i + \sum_{j=1}^{n} \delta_j \mathbf{w}_j$. Therefore, $\mathbf{x} = \sum_{i=1}^{k} (\alpha_i + \gamma_i)\mathbf{u}_i + \sum_{j=1}^{m} \beta_j \mathbf{v}_j + \sum_{j=1}^{n} \delta_j \mathbf{w}_j$. This implies $\{\mathbf{u}_1, \mathbf{u}_2 \ldots \mathbf{u}_k, \mathbf{v}_1, \mathbf{v}_2 \ldots \mathbf{v}_m, \mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n\}$ spans $W_1 + W_2$ .We now show that B is linearly independent. Assume

$$\Sigma \alpha_i \mathbf{u}_i + \Sigma \beta_j \mathbf{v}_j + \Sigma \gamma_r \mathbf{w}_r = \mathbf{0} \tag{2.5}$$

We need to show that α_i, $i=1 \ldots k$, β_j, $j=1, \ldots m$, γ_r, $r=1, \ldots n$, are all zero. From (2.5), we have $\sum \alpha_i \mathbf{u}_i + \sum \beta_j \mathbf{v}_j = -\sum \gamma_r \mathbf{w}_r$. The expression on the right hand side is an element of W_2 and the left hand side is an element of W_1. Therefore, $-\sum_{r=1}^{n} \gamma_r \mathbf{w}_r \in W_1 \cap W_2$ and we can write

$\sum_{j=1}^{k} \alpha_j \mathbf{u}_j = -\sum_{r=1}^{n} \gamma_r \mathbf{w}_r$ so that $\sum_{j=1}^{k} \alpha_j \mathbf{u}_j + \sum_{r=1}^{n} \gamma_r \mathbf{w}_r = \mathbf{0}$. But $\{\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{w}_1, \ldots \mathbf{w}_n\}$ is a basis for W_2 and therefore is linearly independent. This implies $\alpha_1 = \alpha_2 = \cdots = \alpha_k = \gamma_1 = \gamma_2 = \cdots = \gamma_n = 0$. In particular $\sum_{i=1}^{k} \alpha_i \mathbf{u}_i + \sum_{j=1}^{m} \beta_j \mathbf{v}_j = -\sum_{r=1}^{n} \gamma_r \mathbf{w}_r = \mathbf{0}$. Since $\{\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{v}_1, \ldots \mathbf{v}_m\}$ is a basis for W_1, it is linearly independent. Therefore, $\alpha_i = 0$, $\beta_j = 0$. That is α_i, β_j γ_r are all zero which implies that $\{\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{v}_1, \ldots \mathbf{v}_m, \mathbf{w}_1, \ldots \mathbf{w}_n\}$ is linearly independent. Therefore, $\{\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{v}_1, \ldots \mathbf{v}_m\, \mathbf{w}_1, \ldots \mathbf{w}_n\}$ is a basis for $W_1 + W_2$. Therefore, $\dim W_1 + W_2 = k + m + n$. Thus $\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2)$.

Remark Note that Theorem 2.61 (ii) is a special case of Theorem 2.62.

1. Show that '0' element or additive identity of a VS is unique

Suppose $0_1, 0_2$ are both additive identities.

Then $0_1 + 0_2 = 0_1$ since 0_2 is an identity
$0_2 + 0_1 = 0_2$ since 0_1 is an identity } $\therefore 0_1 = 0_2$ $\therefore$ unique.

2. Show that in a VS, each element has a unique additive inverse.

Let $V_1 \in V$ has two additive inverses, say V_2 and V_3.

Then $V_1 + V_2 = 0$ ($\because V_2$ is additive inverse of V_1)
$V_1 + V_3 = 0$ ($\because V_3$ is additive inverse of V_1) where '0' is the zero element of V.

Now $V_2 = V_2 + 0 = V_2 + (V_1 + V_3) = (V_2 + V_1) + V_3 = (V_1 + V_2) + V_3 = 0 + V_3 = V_3$.

$\Rightarrow \boxed{V_2 = V_3}$ $\therefore$ additive inverse of an element in a VS V is unique.

3. Let $x, y \in V$. Prove that There exists a unique $z \ni x + z = y$.

Take $z = -x + y$, then $x + z = y$

Suppose $x + z = y$, then $-x + (x + z) = -x + y$

ie $(-x + x) + z = -x + y$

$0 + z = -x + y$

$\Rightarrow z = -x + y$.

Thus $z = -x + y$ is the unique solution of $x + z = y$.

4 Prove Two vectors are linearly dependent iff one is a scalar multiple of the other.

Proof:- Let $\vec{V_2} = \alpha \vec{V_1}$ for some scalar $\alpha \neq 0$.

Then $\vec{V_2} - \alpha \vec{V_1} = \vec{0} \Rightarrow \exists$ scalars $1, -\alpha$ not both zero $\ni$

$\vec{V_2} - \alpha \vec{V_1} = \vec{0} \Rightarrow \vec{V_2}$ and $\vec{V_1}$ are linearly dependent.

Conversely, suppose $\vec{V_1}$ and $\vec{V_2}$ are dependent

Then, by defn. $\exists$ constants α_1, α_2, not both zero such that

$\alpha_1 \vec{V_1} + \alpha_2 \vec{V_2} = \vec{0}$. If $\alpha_1 \neq 0$, divide by α_1

then $\vec{V_1} + \frac{\alpha_2}{\alpha_1} \vec{V_2} = \vec{0} \Rightarrow \vec{V_1} = -\frac{\alpha_2}{\alpha_1} \vec{V_2}$ $\therefore \vec{V_1}$ and $\vec{V_2}$ are linearly dependent.

If $\alpha_1 = 0$, then $\alpha_2 \neq 0$ and we have $\alpha_2 \vec{V_2} = \vec{0} \Rightarrow \vec{V_2} = \vec{0} = 0\vec{V_1}$.

$\therefore \vec{V_1}$ and $\vec{V_2}$ are linearly dependent.

Show that the set of vectors lying on [illegible] through origin are the only proper [illegible]ce of $\mathbb{R}^2$.

Let H be a proper subspace of $\mathbb{R}^2$

$H \neq \{\bar{0}\}$, $H \neq \mathbb{R}^2$

$H \neq \{\bar{0}\}$ ... a nonzero vector, say $\bar{v} = (a,b) \neq (0,0) \in H$.

H being a subspace $\alpha \bar{v} \in H$ for real α.

Let $L = \{\alpha \bar{v} \mid \alpha \in \mathbb{R}\} \subset H$

But L is a set of vectors lying on a st line thro' origin and $\bar{v}$.

We now show that $H \subset L$

ie to show if $\bar{w} \notin L$ then $\bar{w} \notin H$.

If possible let $\bar{w} \notin L$ but $\bar{w} \in H$

Then $\bar{v}, \bar{w} \in H \Rightarrow \alpha\bar{v} + \beta\bar{w} \in H$

$\therefore \mathbb{R}^2 = \{\alpha\bar{v} + \beta\bar{w} \mid \alpha, \beta \text{ reals}\} \subset H$

if $\bar{w} \notin L$ then $\bar{w} \notin H$ $\therefore H \subset L$; $\mathbb{R}^2 = H$ ≠ to that $H \neq \mathbb{R}^2$

Theorem Let V be a vector space and $B \subset V$. Then, the following are equivalent.

1. B is a basis of V
2. B is a maximal linearly independent set in V

   i.e. B is linearly independent and $B \cup \{u\}$ is linearly dependent for any $u \in V$.
3. B is a minimal spanning set of V.

   i.e. $span(B) = V$ and no proper subset of B can span V.

Proof:- $1 \Rightarrow 2$.

$\because$ B is a basis, $span(B) = V$.

If $v \in V$, then v is a linear combination of elements of B $\Rightarrow B \cup \{v\}$ is LD

$\Rightarrow B$ is a maximal LI subset of V.

$2 \Rightarrow 3$. $\because$ B is LI, for any $v \in B$,

$v \notin spa(B \setminus \{v\}) \Rightarrow$ no subset of B can span V.

If $v \in V \setminus span(B)$ then $B \cup \{v\}$ is LI which contradicts the assumption $\Rightarrow span(B) = V$.

(29d) ~~B2~~

(3) $\Rightarrow$ (1).

Assume that B is a minimal spanning set of V

$\Rightarrow$ span$(B) = V$.

Suppose B is LD, ~~then~~ ie $\exists u \in B$

such that $u \in$ span$(B \setminus \{u\})$

$\Rightarrow$ span$(B \setminus \{u\}) = V$ which contradicts

the assumption that B is minimal spanning set.

Th If a vector space has a finite spanning set, then it has a finite basis

Proof:– Let $V = \text{span}\, S$ for some $S \subset V$ with $|S| < \infty$. If S is LI, then S is a basis.

Otherwise, $\exists\, u_1 \in S \Rightarrow u_1 \in \text{span}(S \setminus \{u_1\})$

$\therefore \text{span}(S \setminus \{u_1\}) = V$.

If $S_1 = S \setminus \{u_1\}$ is LI, then S_1 is a basis

otherwise, one can repeat the process. The process has to stop since S is a finite set and we end up with a subset S_k of S such that S_k is LI and $\text{span}\, S_k = V$.

Examples Compute a basis of

$$V = \{(x,y) \in \mathbb{R}^2 : 2x - y = 0\}$$

$(x,y) \in V \Rightarrow y = 2x \Rightarrow (x,y) = (x, 2x) = x(1,2)$

$\Rightarrow V = \text{span}(\{(1,2)\})$

$\because \{(1,2)\}$ is LI, it is a basis for V.

(29f)

2. Compute basis of

$V = \{(x,y,z) \in \mathbb{R}^3 \mid x - 2y + z = 0\}$

$(x,y,z) \in V \Rightarrow x = 2y - z.$

$\Rightarrow (x,y,z) = (2y - z, y, z) = y(2,1,0) + z(-1,0,1)$

$\Rightarrow V = \text{span}(\{(2,1,0), (-1,0,1)\})$

<u>Check</u> $\{(2,1,0), (-1,0,1)\}$ is LI.

Then $\{(2,1,0), (-1,0,1)\}$ is a basis of V.

3. Compute a basis of

$V = \{(x_1, x_2 \ldots x_5) \in \mathbb{R}^5 : x_1 + x_3 - x_5 = 0$ and $x_2 - x_4 = 0\}$

span $(\{0\}) = \{0\}$

Convention: span of the empty set is taken to be $\{0\}$.

1. $\mathbb{C} = \text{span}(\{1, i\})$ with scalars from $\mathbb{R}$.

2. $e_1 = (1,0)$ $\quad \mathbb{R}^2 = \text{span}(\{e_1, e_2\})$
   $e_2 = (0,1)$

3. $\mathbb{R}^n = \text{span}(\{e_1, e_2 \ldots e_n\})$; $e_i = (0, \ldots \underset{i\text{th place}}{1}, 0, 0 . 0)$

4. $P_3 = \text{span}(\{1, x, x^2, x^3\})$

5. P denote the set of all polynomials of all degree.
   $P = \text{span}(\{1, x, x^2, \ldots\})$

Caution The set S can have infinitely many elements, but a linear combination has only finitely many elements.

Theorem Let $(V, +, \cdot)$ be a vector space. Let $S \subseteq V$. Then span S is a subspace of V and it is the smallest one containing S.

Proof:- If S is empty, then span $S = \{0\}$

If S is non-empty, $x, y \in \text{span } S$,

then $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$
$y = \beta_1 y_1 + \cdots \beta_m y_m$ for some $\alpha_i, \beta_j \in F$
and $x_j, y_j \in S$.

(29h) (52)

Then for $\alpha \in \mathbb{F}$

$$x+\alpha y = a_1 x_1 + \dots + a_n x_n + \alpha\beta y_1 + \dots + \alpha\beta_m y_m \in \text{span } S$$

$\therefore$ span S is a subspace of V.

If V_0 is a subspace containing S, then V_0 contains all linear combination of elements of S.

$\Rightarrow$ span $S \subseteq V_0$ $\Rightarrow$ span S is the smallest subspace containing S.

Exercise 1. S is a subspace of V iff span S = S.

2. If S is a subset of a vector space V, then prove that span $S = \bigcap \{Y : Y$ is a subspace of V containing $S\}$.

3. Consider the system of linear eqs

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$\dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

Let $u_1 = (a_{11} \dots a_{m1})$, $u_2 = (a_{12} \dots a_{m2})$ ...

Show that the system has a solution vector $x = (x_1, x_2 \dots x_n)$ iff $b = (b_1, b_2 \dots b_n)$ is in the span of $\{u_1, u_2 \dots u_n\}$.

Let V_1 and V_2 be subspaces of a vector space V.

Define

$$V_1 + V_2 = \{u + v : u \in V_1 \ \& \ v \in V_2\}$$

<u>Theorem</u> $V_1 + V_2 = span(V_1 \cup V_2)$

<u>Proof:</u> Let $u + v \in V_1 + V_2$ Then

Clearly $u + v \in span(V_1 \cup V_2)$

Note that $V_1 \cup V_2 \subseteq V_1 + V_2$.

Let $v \in span(V_1 \cup V_2)$

Then $\exists\ u_1, u_2 \cdots u_n \in V_1$

$v_1, v_2 \cdots v_m \in V_2$ such that

$$u = \alpha_1 u_1 + \cdots + \alpha_n u_n + \beta_1 v_1 + \cdots \beta_m v_m .$$

$\because\ \alpha_1 u_1 + \cdots \alpha_n u_n \in V_1$ & $\beta_1 v_1 + \cdots \beta_m v_m \in V_2$,

$v \in V_1 + V_2$

$\Rightarrow V_1 + V_2$ is a subspace of V

<u>Note</u> x-axis + y-axis = ??

<u>Exercise</u> Suppose $V_1 \cap V_2 = \{0\}$

Then every element of $V_1 + V_2$ can be written uniquely as $x_1 + x_2$ with $x_1 \in V_1$ & $x_2 \in V_2$.

(29;)

Show
$\{\sin x, \cos x\} \subset C[-\pi, \pi]$ is linearly independent

Suppose $\alpha \sin x + \beta \cos x = 0$

put $x = 0$ we get $\beta = 0$

put $x = \pi/2$ we get $\alpha = 0$

<u>Exercise</u> $\{\sin x, \sin 2x, \sin 3x, \ldots \sin nx\} \subset C[-\pi, \pi]$ is Linearly independent

<u>Caution</u> $\{v_1, v_2 \ldots v_n\}$ is linearly dependent <u>does not imply</u> that each vector is in the span of the remaining vectors

<u>Example</u> $\{(1,0), (1,1), (2,2)\}$ is LD

and $(1,0) \notin span(\{(1,1), (2,2)\})$.

<u>Result</u> If $\{u_1, u_2 \ldots u_n\}$ is LD, then for any vector $v \in V$, then set $\{u_1, u_2 \ldots u_n, v\}$ is LD.

<u>Sol</u> Suppose $\alpha_1 u_1 + \ldots + \alpha_n u_n = 0$ with $\alpha_i \neq 0$ for atleast one i.

$\Rightarrow \alpha_1 u_1 + \ldots + \alpha_n u_n + 0v = 0$

$\Rightarrow \{u_1, u_2 \ldots u_n, v\}$ is LD.

(29k)

Exercise If $E = \{u_1, u_2, \ldots, u_n\}$ is linearly independent, then any subset of E is linearly independent.

Proof Let $F = \{u_{k_1}, u_{k_2}, \ldots, u_{k_r}\}$ be any subset of the linearly indep set $E = \{u_1, u_2, \ldots, u_n\}$.

If $\alpha_{k_1} u_{k_1} + \alpha_{k_2} u_{k_2} + \ldots + \alpha_{k_r} u_{k_r} = 0$ with $\alpha_i \in \mathbb{R}$ (or $\mathbb{C}$ the base field)

then $\sum_{i=1}^{n} \alpha_i u_i = 0$ where $\alpha_i = \begin{cases} \alpha_{k_i} & \text{when } i = k_i \\ 0 & \text{otherwise.} \end{cases}$

As E is a lin indep set, $\sum_1^n \alpha_i u_i = 0 \Rightarrow \alpha_i = 0$ for all i

$\Rightarrow \alpha_{k_j} = 0$ for all j

$\Rightarrow F$ is ~~lin~~ also lin. indep.

Exercise: Suppose $\{u_1, u_2 \ldots u_n\}$ is linearly independent and Y is a subspace of V such that

$span(\{u_1, u_2 \ldots u_n\}) \cap Y = \{0\}$.

Prove that every vector x in the span of $\{u_1, u_2 \ldots u_n\} \cup Y$ can be written uniquely as $x = d_1 u_1 + \ldots + d_n u_n + y$ with $d_1, d_2 \ldots d_n \in \mathbb{F}$ and $y \in Y$

Proof: Let W = span of $\{u_1, u_2, \ldots, u_n\} \cup Y$.

Let $x \in W$. Then x is a finite linear combination of vectors in $\{u_1, u_2, \ldots u_n\} \cup Y$ of the form $x = \sum_{i=1}^{n} \alpha_i u_i + \sum_{j=1}^{m} \beta_j y_j$, $y_j \in Y$, $[\alpha_i, \beta_j \in \mathbb{R}(\mathbb{C})]$

As Y is a subspace, $\sum_{j=1}^{m} \beta_j y_j = y \in Y$.

Suppose $x = \sum_{i=1}^{n} \gamma_i u_i + y'$ with $y' \in Y$, $\gamma_i \in \mathbb{R}(\mathbb{C})$

Then $0 = \sum_{i=1}^{n} (\alpha_i - \gamma_i) u_i + (y - y') \Rightarrow y - y' \in span\{u_1, u_2 \ldots u_n\}$

$\Rightarrow y - y' = 0$ (by our hyp $sp\{u_1 \ldots u_n\} \cap Y = 0$)

$\therefore$ ~~$x = \sum \alpha_i$~~ $\Rightarrow \sum_{i=1}^{n} (\alpha_i - \gamma_i) u_i = 0$

But $\{u_1, \ldots u_n\}$ is lin indep. $\therefore \alpha_i = \gamma_i \; \forall i$

Thus any $x \in W$ ~~is~~ can be uniquely expressed as $x = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_n u_n + y$ with $\alpha_i \in F$, $y \in Y$

//

(29m) LID3

Note 1. The set $\{e_1, \dots, e_n\}$ is LI in $\mathbb{R}^n$ and $\text{span}(\{e_1, e_2, \dots, e_n\}) = \mathbb{R}^n$.

2 - The set $\{1, x, \dots, x^n\}$ is LI in P_n and $P_n = \text{span}(\{1, x, \dots, x^n\})$

3. Exercise

If $\{p_1(x), \dots, p_r(x)\} \subset P$ be the set of polynomials such that $\deg p_1 < \deg p_2 \dots < \deg p_r$, then prove that $\{p_1(x) \dots p_r(x)\}$ is linearly independent.

Pf Assume on the contrary that they are LD. Then $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_r p_r = 0$ with $\alpha_i \in F$ ($\alpha_1 \dots \alpha_r$ not all zero). Let $j = \max\{i \mid \alpha_i \neq 0\}$. Then $\alpha_1 p_1 + \cdots + \alpha_j p_j = 0$

$$\Rightarrow \alpha_j p_j = -\{\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_{j-1} p_{j-1}\} \quad (*)$$

But $\deg \alpha_j \cdot p_j$ is $\deg p_j$ and $\deg(-(\alpha_1 p_1 + \cdots + \alpha_{j-1} p_{j-1})) \leq \deg p_{j-1}$.

$(*) \Rightarrow \deg p_j \leq \deg p_{j-1}$. ~~This contradicts our assumption~~ ~~which is not possible~~ ~~$\therefore$ each $\alpha_i = 0$ & hence $\{p_1, p_2, \dots p_r\}$ is lin indep~~ which contradicts what is given

$\therefore$ our assumption is wrong

$\therefore \{p_1(x), \dots p_r(x)\}$ is LI.

(29n) (1) dimensions.

By def^n., an infinite dim VS can not have a finite basis.

Is it possible for a finite dim space to have an infinite basis, or an infinite LI subset?

Answer is **NO** and we have the following result.

Th Let V be a finite dim VS with a basis consisting of n elements. Then every subset of V with more than n elements is LD.

Proof:- Let $\{u_1, \dots u_n\}$ be a basis of V.

and $\{x_1, x_2 \dots x_{n+1}\} \subset V$.

We show that $\{x_1, x_2, \dots x_{n+1}\}$ is LD.

If $\{x_1, x_2, \dots x_n\}$ is LD, then surely $\{x_1, x_2 \dots x_n, x_{n+1}\}$ is LD and we are done.

If $\{x_1, x_2 \dots x_n\}$ is LI, then we show $\{x_1, x_2 \dots x_{n+1}\}$ is LD.

$\because \{u_1, u_2, \dots u_n\}$ is a basis of V, $\exists$ scalars $\alpha_1, \alpha_2 \dots \alpha_n$ such that $x_1 = \alpha_1 u_1 + \dots + \alpha_n u_n$

$\because x_1 \neq 0$ one of $\alpha_1, \alpha_2 \dots \alpha_n$ is nonzero.

Without loss of generality, assume that $\alpha_1 \neq 0$.

Then ; $\alpha_1 u_1 = x_1 - \alpha_2 u_2 - \dots - \alpha_n u_n$

or $u_1 = \frac{1}{\alpha_1} x_1 - \frac{\alpha_2}{\alpha_1} u_2 - \dots - \frac{\alpha_n}{\alpha_1} u_n$

$\Rightarrow u_1 \in \text{span}\{x_1, u_2, \dots u_n\}$

Let $\alpha_1^{(2)}, \ldots \alpha_n^{(2)}$ be scalars $\ni$

$$x_2 = \alpha_1^{(2)} x_1 + \alpha_2^{(2)} u_2 + \cdots + \alpha_n^{(2)} u_n$$

$\because \{x_1, x_2\}$ is L I, at least one of $\alpha_2^{(2)}, \ldots \alpha_n^{(2)}$ is non zero. Without loss of generality assume that $\alpha_2^{(2)} \neq 0$.

Then $u_2 \in \operatorname{span}\{x_1, x_2, u_3, \ldots u_n)$

so that

$$V = \operatorname{span}\{x_1, u_2, \ldots u_n\} = \operatorname{span}\{x_1, x_2, u_3, \ldots u_n\}$$

Let $1 \leq k \leq n-1$ be such that

$$V = \operatorname{span}\{x_1, x_2, \ldots x_k, u_{k+1}, \ldots u_n\}.$$

Suppose $k < n-1$, $\because \{x_1, x_2 \ldots x_{k+1}\}$ is L I at least one of the scalars $\alpha_{k+1}^{(k+1)} \ldots \alpha_n^{(k+1)}$ is nonzero. Without loss of generality, assume that $\alpha_{k+1}^{(k+1)} \neq 0$. Then we have

$$u_{k+1} \in \operatorname{span}\{x_1, \ldots x_{k+1}, u_{k+2}, \ldots u_n\}$$ so that

$$V = \operatorname{span}\{x_1 \ldots x_k, u_{k+1}, \ldots u_n\}$$
$$= \operatorname{span}\{x_1 \ldots x_{k+1}, u_{k+2}, \ldots u_n\}$$

This proceedure leads to $V = \operatorname{span}\{x_1 \ldots x_{n-1}, u_n\}$ so that $\exists$ scalars $\alpha_1^{(n)} \ldots \alpha_n^{(n)}$ such that

(29p)

$$x_n = \alpha_1^{(n)} x_1 + \cdots + \alpha_{n-1}^{(n)} x_{n-1} + \alpha_n^{(n)} u_n$$

$\because \{x_1 \dots x_n\}$ is LI, it follows that $\alpha_n^{(n)} \neq 0$

$\therefore u_n \in \text{span}\{x_1 \dots x_n\}$

Consequently

$$V = \text{span}\{x_1, x_2, \dots x_{n-1}, u_n\}$$

$$= \text{span}\{x_1, x_2, \dots x_{n-1}, x_n\}$$

Thus $x_{n+1} \in \text{span}\{x_1 \dots x_n\}$

show that $\{x_1, x_2 \dots x_{n+1}\}$ is LD.

Let V be a vectorspace and W_1 & W_2 be subspaces of V. Then

$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$. (29a)

1. If $V = \mathbb{R}^3$ (1) W_1, W_2 are of dim 1, Then the are lines thro' origin

(2) If $W_1 \neq W_2$, then $W_1 + W_2$ will be the plane containing

a) both the lines & $W_1 \cap W_2 = \{\bar{0}\}$ $\therefore \dim(W_1+W_2) = \dim W_1 + \dim W_2$

---

b) If $W_1 = W_2$, then $W_1 + W_2 = W_1$, $W_1 \cap W_2 = W_1$

$\therefore 1 = \dim(W_1+W_2) = \dim W_1 (=1) + \dim W_2 (=1) - \dim(W_1 \cap W_2)(=1)$

$= 2 - 1 = 1$.

Illly one can analyse when W_1 is a line.

(2) W_2 is a place.

There are two cases $W_1 \subseteq W_2$ or $W_1 \cap W_2 = \{\bar{0}\}$:

a) In the first case $W_1 + W_2 = W_2$ so that

$2 = \dim(W_1+W_2) = (\dim W_1 = 1) + (\dim W_2 = 2) - (\dim W_1 \cap W_2 = 1)$

b) In the second case $W_1 + W_2 = \mathbb{R}^3$. so that

$3 = \dim(W_1+W_2) = (\dim W_1 = 1) + (\dim W_2 = 2) - (\dim W_1 \cap W_2 = 0)$

(3) Discuss if both W_1 & W_2 are planes thro origin.

a) Two case W_1 & W_2 intersect in a line or coincide

Is it possible that $W_1 \cap W_2 = \{\bar{0}\}$.

Proof:- $W_1 \cap W_2 \subseteq W_i$ for $i = 1, 2$

Let $u_1, u_2 \ldots u_k$ be a basis of $W_1 \cap W_2$

Extend this to a basis $\{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m\}$ of W_1

& a basis $\{u_1, u_2 \ldots u_k, w_1, w_2 \ldots w_n\}$ of W_2

Then $\dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (m+k) + (n+k) - k = m+n+k$.

Claim $\{u_1, u_2 \ldots u_k, v_1, v_2 \ldots v_m, w_1, w_2 \ldots w_n\}$ is a basis of

$W_1 + W_2$.

Let $x \in W_1 + W_2$. Then $x = w_1 + w_2$, $w_1 \in W_1$, $w_2 \in W_2$.

$w_1 = \sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j$ $\therefore x = \sum_{i=1}^{k} (\alpha_i + \gamma_i) u_i + \sum_{j=1}^{m} \beta_j v_j + \sum_{j=1}^{n} \delta_j w_j$

$w_2 = \sum_{i=1}^{k} \gamma_i u_i + \sum_{j=1}^{n} \delta_j w_j$

$\Rightarrow \{u_1, u_2, \ldots u_k, v_1, v_2 \ldots v_m, w_1, w_2 \ldots w_n\}$ spans $W_1 + W_2$. We now show that it is linearly indep.

~~Assum~~ Consider $\sum \alpha_i u_i + \sum \beta_j v_j + \sum \gamma_r w_r = 0$.

We need to show that $\alpha_i, \beta_j, \gamma_r$ are all zero.

Now $\sum \alpha_i u_i + \sum \beta_j v_j = - \sum \gamma_r w_r$

The exp on the rhs is an element of W_2 & on the LHS is an element of W_1

$\therefore - \sum \gamma_r w_r \in W_1 \cap W_2$.

$\therefore$ we can write
$$- \sum_{r=1}^{n} \gamma_r w_r = \sum_{j=1}^{k} \alpha_j u_j$$

so that $\sum_{j=1}^{k} \alpha_j u_j + \sum_{r=1}^{n} \gamma_r w_r = 0$

But $\{u_1, \ldots u_k, w_1 \ldots w_n\}$ is a basis of W_2 & $\therefore$ lin. ind

$\Rightarrow \alpha_j = 0, \gamma_r = 0$ for all j & r

In particular $\sum_{i=1}^{k} \alpha_i u_i + \sum_{j=1}^{m} \beta_j v_j = - \sum_{r=1}^{n} \gamma_r w_r = 0$

$\because \{u_1 \ldots u_k, v_1 \ldots v_m\}$ is a basis of W_1, it is linearly inde $\therefore \alpha_i = 0$ & $\beta_j = 0$

$\therefore \alpha_i, \beta_j, \gamma_r$ are all zero.

$\therefore$ result.

CHAPTER 3
LINEAR TRANSFORMATIONS

3.1 Definitions and Examples

Definition 3.1 Let V and W be vector spaces. A linear transformation T from V into W is a function that assigns to each vector $\mathbf{v} \in V$, a unique vector $T(\mathbf{v}) \in W$ and that satisfies, for **u** and **v** in V and each scalar α,

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}),$$

$$T(\alpha\, \mathbf{v}) = \alpha\, T(\mathbf{v}).$$

Notation $T: V \rightarrow W$ (also called linear operators) is a linear transformation from V to W. We shall denote $T(\mathbf{u})$ by $T\mathbf{u}$, $\mathbf{u} \in V$.

Example 3.2 $T: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $T(x_1, x_2) = (x_1, -x_2)$. T takes a vector in $\mathbf{R}^2$ and reflects it about the x-axis. T is a linear Transformation (Reflection about the x-axis).

Example 3.3 $T: \mathbf{R}^2 \rightarrow \mathbf{R}^3$ defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{pmatrix}. \text{ Let } \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbf{R}^2. \text{ Then } T(\mathbf{x} + \mathbf{y}) =$$

$$T\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] = T\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 \\ x_1 + y_1 - x_2 - y_2 \\ 3x_2 + 3y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 3x_2 \end{pmatrix} + \begin{pmatrix} y_1 + y_2 \\ y_1 - y_2 \\ 3y_2 \end{pmatrix} = T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Similarly, $T\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} = \alpha T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Example 3.4 (The zero transformation) Let V and W be vector spaces. $T: V \rightarrow W$ is defined by $T\mathbf{v} = \mathbf{0}$ for every $\mathbf{v} \in V$. $T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0} = T\mathbf{v}_1 + T\,\mathbf{v}_2$; $T(\alpha\, \mathbf{v}) = \mathbf{0} = \alpha\, \mathbf{0} = \alpha\, T\mathbf{v}$.

Example 3.5 (The identity transformation) $I_v: V \rightarrow V$, V is a vector space, defined by $I_v(\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in V$. I_v is a linear transformation called the identity transformation or the identity operator on V.

Example 3.6 (Rotation Transformation) Let $\mathbf{x} \in \mathbf{R}^2$ be rotated through an angle α in the anticlockwise direction. The new vector is $\mathbf{y}$. Let r be the length of $\mathbf{x}$, r does not change by rotation. Then $x_1 = r\cos\theta,\ y_1 = r\cos(\theta+\alpha);\ x_2 = r\sin\theta,\ y_2 = r\sin(\theta+\alpha)$

$$y_1 = r[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = x_1\cos\alpha - x_2\sin\alpha;$$
$$y_2 = r[\sin\theta\cos\alpha + \sin\alpha\cos\theta] = x_1\sin\alpha + x_2\cos\alpha$$

Thus $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let $A_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$. Then $\mathbf{y} = A_\alpha(\mathbf{x})$. T: $\mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(\mathbf{x}) = A_\alpha(\mathbf{x})$ is a linear transformation and A_α is called a rotation transformation.

Example 3.7 T: $\mathbf{R}^m \to \mathbf{R}^n$, $m \geq n$ defined by $T(x_1, x_2,\ldots, x_m) = (x_1, x_2,\ldots, x_n)$, that is, drop the last $(m - n)$ coordinates of vector $\mathbf{R}^m$. T is called the natural projection of $\mathbf{R}^m$ onto $\mathbf{R}^n$.

Example 3.8 T: $\mathbf{R}^3 \to \mathbf{R}^3$, defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ is the projection operator taking a vector in $\mathbf{R}^3$ and projecting it into xy plane. Similarly $T(x_1, x_2, x_3) = (x_1, 0, x_3)$ projects a vector in $\mathbf{R}^3$ into the xz plane.

Example 3.9 T: $\mathbf{R}^m \to \mathbf{R}^n$, $m \leq n$ defined by $T(x_1, x_2,\ldots, x_m) = (x_1, x_2,\ldots, x_m, 0, 0, \ldots, 0)$. T is 1-1 linear map called the natural inclusion of $\mathbf{R}^m$ into $\mathbf{R}^n$.

Example 3.10 (A Transpose operator) Define T: $M_{m \times n} \to M_{n \times m}$ by $T(A) = A'$. Since, $(A+B)' = T(A+B) = A' + B' = T(A) + T(B)$ and $(\alpha A)' = T(\alpha A) = \alpha A' = \alpha\, T(A)$, T is a linear transformation called transpose operator.

Example 3.11 (An Integral operator) J: $C[0,1] \to \mathbf{R}$ defined by $Jf = \int_0^1 f\,dx$. Since $\int_0^1 (f+g)\,dx = \int_0^1 f\,dx + \int_0^1 g\,dx$, $\int_0^1 (\alpha f)\,dx = \alpha\int_0^1 f\,dx$. J is a linear transformation called integral operator.

Example 3.12 (Differential operator) Let D: $\mathbf{C}^1[0,1] \to C[0,1]$ defined by $D\,f = f'$. Since, $(f+g)' = D(f+g) = f' + g' = Df + Dg$, $(\alpha f)' = D(\alpha f) = \alpha f' = \alpha Df$. D is a linear transformation called differential operator.

Matrix multiplication as ~ LT.

Let A be an m x n matrix. Define T: $\mathbf{R}^n \to \mathbf{R}^m$ by $T\mathbf{x} = A\mathbf{x}$. Since $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, $A(\alpha\mathbf{x}) = \alpha A\mathbf{x}$, $\forall\, \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$. T is a linear transformation. Thus, every m x n matrix gives rise to a linear transformation from $\mathbf{R}^n \to \mathbf{R}^m$. We shall see that a certain converse is true. i.e., <u>Every linear transformation between finite dimensional vector spaces can be represented by a matrix</u>.

Note Not every transformation that looks like linear is a linear.

Example 3.13 Define T: $\mathbf{R} \to \mathbf{R}$ by $Tx = 2x + 3$. Then $\{(x, Tx): x \in \mathbf{R}\}$ is a straight line in the xy plane. But T is not linear. $T(x + y) = 2(x + y) + 3 \neq Tx + Ty = 2x + 2y + 6$.

Example 3.14 T: $C[0,1] \to \mathbf{R}$ defined by $Tf = f(0) + 1$. T is not linear.
$T(f + g) = (f + g)(0) + 1 = f(0) + g(0) + 1$, $Tf + Tg = f(0) + 1 + g(0) + 1$. Therefore $T(f + g) \neq Tf + Tg$.

The only linear fns from IR to IR are fns of the form $f(x) = mx$ for some real number m. Thus, among all st. line fns, the only ones that are linear are the ones that pass thro' the origin.

3.2 Properties of Linear Transformation

Theorem 3.15 Let T: $V \to W$ be a Linear Transformation. Then, for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_n$ in V and all scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$:

(i) $T(\mathbf{0}) = \mathbf{0} \in W$

(ii) $T(\mathbf{u} - \mathbf{v}) = T\mathbf{u} - T\mathbf{v}$

(iii) $T(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_n \mathbf{v}_n) = \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \ldots + \alpha_n T\mathbf{v}_n$.

Proof Will be discussed during the lecture.

An important fact about linear transformation is that they are completely determined by what they do to basis vectors.

Theorem 3.16 Let V be a finite dimensional vector space with basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be n vectors in W. Suppose that T_1 and T_2 are two linear transformations from V to W such that $T_1 \mathbf{v}_i = T_2 \mathbf{v}_i = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$. Then for any $\mathbf{v} \in V$, $T_1 \mathbf{v} = T_2 \mathbf{v}$, i.e., $T_1 = T_2$.

Proof Will be discussed during the lecture.

<u>**Remark**</u> Theorem 3.16 tells us that if T: $V \to W$ and V is finite dimensional, then we need to know only what T does to the basis vectors in V. This determines T completely.

Example 3.17 Let T: $\mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(1, 0, 0) = (2, 3)$, $T(0, 1, 0) = (-1, 4)$, $T(0, 0, 1) = (5, -3)$. Then $T(3, -4, 5)$ is obtained as follows: Now,

$$(3, -4, 5) = 3\,(1, 0, 0) + (-4)\,(0, 1, 0) + 5\,(0, 0, 1),$$

Therefore $T(3, -4, 5) = 3T(1, 0, 0) + (-4)T(0, 1, 0) + 5T(0, 0, 1)$

$$= 3(2, 3) - 4(-1, 4) + 5(5, -3) = (35, -22).$$

<u>*Another question arises:*</u> If $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are n vectors in W, does there exist a linear transformation T such that $T\mathbf{v}_i = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$?

The answer is yes and we have the following theorem.

Theorem 3.18 Let V be a finite dimensional vector space with basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Let W be a vector space containing the n vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$. Then, there exists a unique linear transformation $T: V \to W$ such that $T\mathbf{v}_i = \mathbf{w}_i$ for $i = 1, 2, \ldots, n$.

Proof Will be discussed during the lecture.

<u>**Remark**</u> In Theorems 3.16 and 3.18

1) The vectors $\mathbf{w}_1, \ldots, \mathbf{w}_n$ need not be independent.
2) In fact, need not be even distinct.
3) Theorems are true for any finite–dimensional vector space and just not $\mathbf{R}^n$ alone.
4) Note W need not be finite dimensional..

Example 3.19 Find a linear transformation from $\mathbf{R}^2$ into a subspace of $\mathbf{R}^3$, i.e., from $\mathbf{R}^2$ to the plane $W = \{(x_1, x_2, x_3) \mid 2x_1 - x_2 + 3x_3 = 0\}$. Dim(W) = 2 and its basis is $\{(1, 2, 0), (0, 3, 1)\}$. Take $\hat{\mathbf{e}}_1 = (1,0)$, $\hat{\mathbf{e}}_2 = (0,1)$ as standard basis in $\mathbf{R}^2$. Define $T: \mathbf{R}^2 \to W$ by $T\hat{\mathbf{e}}_1 = (1,2,0)$, $T\hat{\mathbf{e}}_2 = (0,3,1)$.

Then T is completely determined and,

$$T(x_1, x_2) = T[x_1(1, 0) - x_2(0, 1)] = x_1T(1, 0) + x_2T(0, 1)$$
$$= x_1(1, 2, 0) + x_2(0, 3, 1)$$
$$= (x_1, 2x_1 + 3x_2, + x_2)$$

for $(x_1, x_2) \in \mathbf{R}^2$.

∴ Find $T\binom{5}{-1}$. $T\binom{5}{-1} = 5T\binom{1}{0} - 1T\binom{0}{1} = \begin{pmatrix}5\\11\\-1\end{pmatrix}$,

CHAPTER 4

THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

4.1 Linear Transformation and Matrices

Suppose A is a matrix of order m x n with real entries. Consider a transformation $T: \mathbf{R}^n \rightarrow \mathbf{R}^m$ given by $T(\mathbf{x}) = A\mathbf{x}$, $\mathbf{x} \in \mathbf{R}^n$. Here, we have regarded the n-tuple $\mathbf{x} \in \mathbf{R}^n$ as a column matrix of order n x 1. This is a standard practice and we shall follow it. It follows from the properties of matrices that:

(i) $T(\mathbf{x} + \mathbf{y}) = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

(ii) $T(\alpha\, \mathbf{x}) = A(\alpha \mathbf{x}) = \alpha\; A\mathbf{x} = \alpha\; T(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$.

In other words, T is a linear transformation. This is of course immediately verified. The remarkable fact is that the converse is also true, that is, every linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^m$ (or more generally between any two finite dimensional vector spaces) can be described by such a matrix multiplication.

To see this, let V be a real vector space of dimension n. (for convenience, we discuss the case of real vector space. All the results discussed here are also valid for complex vector spaces with obvious modifications).

Let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a basis of V. Then for each $\mathbf{x} \in V$, there exists a unique set of scalars $\alpha_1, \ldots, \alpha_n \in \mathbf{R}^m$ such that $\mathbf{x} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 \ldots + \alpha_n\ \mathbf{a}_n = \sum_{j=1}^{n} \alpha_j \mathbf{a}_j$. If we know $\alpha_1, \alpha_2 \ldots \alpha_n$, then we know $\mathbf{x}$ and vice versa. In other words, all the information about $\mathbf{x}$ is contained in these n real numbers. We express this fact by saying that the matrix of x with respect to the basis of A is the column matrix $\begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}$

Notation $[\mathbf{x}]_A$ = Matrix of **x** w.r.t. A. Thus $[\mathbf{x}]_A = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}$. Note that this matrix of **x** w.r.t. A depends not only on the vector **x** and the basis A, but also on the order in which the elements in the basis are written.

Example 4.1 Let V = R^3. Consider the standard basis E = { $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$} where $\mathbf{e}_1$= (1,0,0), $\mathbf{e}_2$ = (0,1,0) and $\mathbf{e}_3$ = (0,0,1). Let **x** = (1, −1, 2) = 1$\mathbf{e}_1$ − 1$\mathbf{e}_2$ + 2$\mathbf{e}_3$. Hence, $[\mathbf{x}]_E = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$. If we decide to write E as E = {$\mathbf{e}_2$, $\mathbf{e}_1$, $\mathbf{e}_3$}, then $[\mathbf{x}]_E$ will be $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

Example 4.2 Let V and **x** be as above. Consider another basis F = {$\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$} where $\mathbf{f}_1$ = (1,1,0) , $\mathbf{f}_2$ = (1, − 1,0) and $\mathbf{f}_3$ = (0,0,1). Then **x** = 0$\mathbf{f}_1$ + 1$\mathbf{f}_2$ + 2$\mathbf{f}_3$. Thus $[\mathbf{x}]_F = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

Since the matrix of **x** with respect to a basis depends on the order in which the elements of the basis are written, we need to fix some order in the elements to avoid ambiguity. Such a basis is called an <u>ordered basis</u> and henceforth we shall always deal with ordered basis.

Let A = {a_1, a_2, ... a_n} be an ordered basis for V.

Now let W be another real vector space with an <u>ordered basis</u> B = {$\mathbf{b}_1$, …,$\mathbf{b}_m$} and let T: V →W be a linear transformation. Recall that we know T completely if we know the image under T of each element in a basis of V. In other words, if we know T ($\mathbf{a}_j$) from j=1…, n, we can find T (**x**) for any arbitrary **x** ϵ V. These n vectors {T ($\mathbf{a}_1$), … ,T ($\mathbf{a}_n$)} completely determine T. Next each of these n vectors is completely determined by its matrix w.r.t. the basis B, $[T(\mathbf{a}_j)]_B$. Since for each j, $[T(\mathbf{a}_j)]_B$ is m x 1 matrix, these m x n scalars determine the operator T completely. It is natural to put these scalars in the form of a matrix of order m x n, called the <u>matrix of T</u> w.r.t. the <u>ordered basis</u> A and B, and denoted by $[T]_A^B$. Thus, $[T]_A^B$ = $[[T(\mathbf{a}_1)]_B \ldots , [T(\mathbf{a}_n)]_B]$. Hence if $[T]_A^B = [\alpha_{ij}]_{m\times n}$, then the j[th] column of $[T]_A^B$ is $[T(\mathbf{a}_j)]_B$, that is, $\begin{bmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{bmatrix} = [T(a_j)]_B$, $j = 1 \ldots n$. This means $T(\mathbf{a}_j) = \alpha_{1j}\,\mathbf{b}_1 + \ldots + \alpha_{mj}\,\mathbf{b}_m = \sum_{j=1}^{m} \alpha_{ij}\mathbf{b}_j$, $i = 1,\ldots n$. This is the basic relationship connecting $[\alpha_{ij}]$, the matrix of T with respect to the ordered bases A, B with the linear transformation.

Example 4.3 Let $V = \mathbf{R}^3$ with the standard ordered basis $A = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $W = \mathbf{R}^2$ with basis $B = \{\mathbf{e}_1, \mathbf{e}_2\}$. Let $T: V \rightarrow W$ be defined by $T(\mathbf{x}) = (-x_1, x_2 + x_3)$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$. Then $T(\mathbf{e}_1) = (-1, 0)$. Thus $[T(\mathbf{e}_1)]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. This becomes the first column of $[T]_A^B$. Similarly second and third columns can be computed. Thus we have $[T]_A^B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

Example 4.4 Let $T = \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be the linear transformation that rotates each vector anticlockwise by 45° and also doubles its magnitude. We consider the ordered bases $A = B = \{\mathbf{e}_1, \mathbf{e}_2\}$. Thus $T(\mathbf{e}_1) = (\sqrt{2}, \sqrt{2})$ and $T(\mathbf{e}_2) = (-\sqrt{2}, \sqrt{2})$. Hence, $[T]_A^B = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Example 4.5 For n = 0, 1, 2,... let $\mathbf{P}_n$ denote the vector space of all polynomials (with real coefficients) of degree $\le$ n. Let $p_j(t) = t^j$, $j = 0, 1, \ldots$. Then $\{p_0 \ldots p_n\}$ forms a basis of $\mathbf{P}_n$.

Example 4.6 Consider the linear transformation $D = \mathbf{P}_3 \rightarrow \mathbf{P}_2$, given by $D(p) = p'$, the derivative of p, for $p \in \mathbf{P}_3$, and ordered bases $A = \{p_0, p_1, p_2, p_3\}$ and $B = \{p_0, p_1, p_2\}$ of $\mathbf{P}_3$ and $\mathbf{P}_2$ respectively. To find $[D]_A^B$, note that $D(p_0) = 0$, $D(p_1) = 1 = p_0$, $D(p_2) = 2p_1$, and $D(p_3) = 3p_2$.

Thus $[D]_A^B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

We note that the matrix of a linear transformation depends on the choice of the ordered bases A and B. To illustrate this, we consider a different basis $C = \{q_0, q_1, q_2\}$ of $\mathbf{P}_2$ as follows: $q_0(t) = 1$, $q_1(t) = 1 + t$, $q_2(t) = 1 - t^2$. Now $D(p_0) = 0$, $D(p_1) = 1 = q_0$, $D(p_2) = 2q_1 - 2q_0$ and $D(p_3) = 3q_0 - 3q_3$. Then $[D]_A^C = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$.

What is the relationship between matrices of **x**, T and T(**x**)? The next theorem answers this question. It says that with every fixed choice of ordered bases, matrix of T(**x**) equals the product of matrix of T and matrix of **x**. This also establishes the claim made in the beginning of this chapter that every linear transformation between finite dimensional vector spaces is essentially same as multiplying by a fixed matrix.

Theorem 4.7 Let V, W be finite dimensional vector spaces. Let A = $\{\mathbf{a}_1, \dots \mathbf{a}_n\}$ be an ordered basis of V and B = $\{\mathbf{b}_1 \dots ,\mathbf{b}_m\}$ an ordered basis of W. Let $T : V \to W$ be a linear transformation. Then $\forall\, \mathbf{x} \in V$, $[T(\mathbf{x})]_B = [T]_A^B\, [\mathbf{x}]_A$.

Proof Let $[\mathbf{x}]_A = \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}$. This means $\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathbf{a}_j$. Hence $T(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j T(\mathbf{a}_j)$.

Suppose $[T]_A^B = [\alpha_{ij}]_{mxn}$. Then $T(\mathbf{a}_j) = \sum_{i=1}^{m} \alpha_{ij}\, \mathbf{b}_i, \quad j = 1,\dots n$. Hence

$$T(\mathbf{x}) = \sum_{j=1}^{n} \alpha_j T(\mathbf{a}_j) = \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} \alpha_{ij} \mathbf{b}_i$$

$$= \sum_{i=1}^{m} \mathbf{b}_i \sum_{j=1}^{n} \alpha_{ij}\alpha_j = \sum_{i=1}^{m} \beta_i \mathbf{b}_i \text{ where } \beta_i = \sum_{j=1}^{n} \alpha_{ij}\alpha_j = [T(\mathbf{x})]_B$$

Thus $\begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_m \end{bmatrix} = \left[\alpha_{ij}\right]_{mxn} \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}_{nx1}$.

Example 4.8 Consider Example 4.5 above. Let $p(t) = 2 - 3t + 5t^2 - t^3$.

Thus $[p]_A = \begin{bmatrix} 2 \\ -3 \\ 5 \\ -1 \end{bmatrix}$. $D(p)(t) = p'(t) = -3 + 10t - 3t^2$. Hence

$$[D(p)]_B = \begin{bmatrix} -3 \\ 10 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 5 \\ -1 \end{bmatrix} = [D]_A^B\, [p]_A$$

Next we discuss some relationships between the operations on linear transformations and the operations on matrices.

Theorem 4.9 Let V, W be finite dimensional real vector spaces, A = {$\mathbf{a}_1, \dots, \mathbf{a}_n$} be an ordered basis of V, B = {$\mathbf{b}_1, \dots, \mathbf{b}_n$} be an ordered basis of W. Let L (V, W) denote the set of all linear transformations from V to W and $\mathbf{R}^{m \times n}$ denote the set of all matrices of order m x n with real entries. Then the map $T \to [T]_A^B$ is a <u>one-one</u> and <u>onto linear transformation</u> from L (V, W) to $\mathbf{R}^{m \times n}$, that is,

(i) $[T+S]_A^B = [T]_A^B + [S]_A^B$, T, S ϵ L (V, W) (ii) $[\alpha T]_A^B = \alpha\, [T]_A^B$

Proof Straight forward

Corollary 4.10 Dim (L (V, W)) = mn

We next decide what happens to the corresponding matrix of the composition of two linear transformations

Theorem 4.11 Let V, W, Z be finite dimensional vector spaces with A = {$\mathbf{a}_1, \dots, \mathbf{a}_n$} ordered basis of V , B = {$\mathbf{b}_1, \dots, \mathbf{b}_m$} ordered basis of W and C = {$\mathbf{c}_1, \dots, \mathbf{c}_p$} ordered basis of Z. Let T: V → W and S: W → Z be linear transformations. Then ST: V → Z defined by (ST) (**x**) = S(T(**x**)) , **x** ϵ V is also a linear transformation and $[ST]_A^C = [S]_B^C\ [T]_A^B$

Proof That ST is a linear transformation is readily checked. Next for each j = 1, …n.

$[ST]_A^C [\mathbf{a}_j]_A = [\,(ST)(\mathbf{a}_j)]_C = [\,S(T(\mathbf{a}_j)]_C = [S]_B^C\, [T(\mathbf{a}_j)]_B = [S]_B^C\ [T]_A^B [\mathbf{a}_j]_A$. Note that

i.e. $a_j = 0\cdot a_1 + 0\cdot a_2 \dots + 1\cdot a_j + 0\cdot a_{j+1} \dots + 0\cdot a_n$, $[a_j]_A = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$.

$$[\mathbf{a}_j]_A = \begin{bmatrix}0\\ \cdot\\ 1\\ 0\\ \cdot\end{bmatrix}$$ with 1 in the j[th] row. Hence j[th] column of $[ST]_A^C$ and $[S]_B^C\ [T]_A^B$ coincide for each j.

Hence $[ST]_A^C = [S]_B^C\ [T]_A^B$

Corollary 4.12 Matrix multiplication is associative.

Proof Follows immediately from the above Theorem and the previous Theorem.

Quite frequently, we deal with linear transformation T from V to V itself. These are popularly called <u>linear operators</u>. In this case, it is customary to use only one basis to represent its matrix, $[T]_A^A$. We shall use the notation $[T]_A$ for $[T]_A^A$. Thus $[\,T(\mathbf{x})]_A = [T]_A\, [\mathbf{x}]_A$, ∀ **x**∈V. If I: V → V is the identify operator given by I(**x**) = **x** , **x** ∈ V, then $[\,I\,]_A = [\,\delta_{ij}]_{n \times n}$, the identify matrix . Similarly, matrix of the zero transformation is the zero matrix of appropriate order.

Next, if S, T $\in$ L (V) (= L (V,V)), then S T $\in$ L(V), and $[ST]_A = [S]_A [T]_A$. Thus in this case, the map $T \to [T]_A$ from L(V) onto $\mathbf{R}^{n \times n}$ preserves vector space operations and also products. One application of this is the following.

Corollary 4.13 Let V be a finite dimensional vector space with basis A and T $\in$ L(V). Then T is invertible if and only if $[T]_A$ is invertible, and in this case $\left[T^{-1}\right]_A = [T]_A^{-1}$.

Proof Suppose T is invertible. Then $\exists$ S $\in$ L(V) such that ST = I = TS. Then $T^{-1} = S$ and $[ST]_A = [S]_A [T]_A = [I]_A$ = Identify matrix. Thus $[T]_A [S]_A$ = Identity matrix. Thus $[T]_A$ is invertible and $[T]_A^{-1} = [S]_A = [T^{-1}]_A$.

Next we study what happens to the representing matrix $[T]_A$ when the ordered basis A is changed. In other words, what is the relationship between $[T]_A$ and $[T]_B$. If B is some other ordered basis of V. The following theorem answers this question:

Theorem 4.14 Let V be a finite dimensional vector space and A = $\{\mathbf{a}_1, \dots ,\mathbf{a}_n\}$, B = $\{\mathbf{b}_1, \dots ,\mathbf{b}_n\}$ be ordered bases of V. Let $M = [I]_A^B$. Then

Express a_j w.r.t B find $[a_j]_B$ for $j=1,\dots,n$.

(i) $[\mathbf{x}]_B = M [\mathbf{x}]_A, \forall\, \mathbf{x} \in V$ and (ii) $[T]_B = M[T]_A M^{-1}, \forall\, T \in L(V)$

arrange them as column vectors. This is $M = [[a_j]]_B$.

Proof Note that $M = [I]_A^B$ means that for each j, j^{th} column of M is $[I\, \mathbf{a}_j]_B$, that is, entries in the j^{th} column are obtained by expressing $\mathbf{a}_j$ as a linear combination of elements in B. From the previous corollary, we see that M is invertible. Now let $\mathbf{x} \in$ V. Then $[\mathbf{x}]_B = [I(\mathbf{x})]_B = [I]_A^B\, [\mathbf{x}]_A = M[\mathbf{x}]_A$.

Next we apply this to $[T(\mathbf{x})]$. Then $[T(\mathbf{x})]_B = M\, [T(\mathbf{x})]_A$. Hence, $[T]_B\, [\mathbf{x}]_B = M[T]_A\, [\mathbf{x}]_A$, that is, $[T]_B\, M[\mathbf{x}]_A = M[T]_A\, [\mathbf{x}]_A$. Since this holds $\forall\, \mathbf{x} \in$ V, we can take $\mathbf{x} = \mathbf{a}_j$ for each j =1,... ,n . Then the left side becomes j^{th} column of $[T]_B$ M and the right side becomes the j^{th} column of $M[T]_A$. Hence $[T]_B M = M[T]_A$, that is, $[T]_B = M\, [T]_A\, M^{-1}$.

Example 4.15 Let V = $\mathbf{R}^2$, A = $\{\mathbf{e}_1, \mathbf{e}_2\}$, B = $\{\mathbf{f}_1, \mathbf{f}_2\}$ where $\mathbf{f}_1$ = (1, 1) and $\mathbf{f}_2$ = (1, –1). Then

$$M = [I]_A^B = [[\mathbf{e}_1]_B\,,\ [\mathbf{e}_2]_B\,] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$ Consider $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \in \mathbf{R}^2$. Then $[\mathbf{x}]_A = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$e_1 = \gamma_1 f_1 + \gamma_2 f_2$

$(1,0) = \gamma_1 (1,1) + \gamma_2 (1,-1)$

$1 = \gamma_1 + \gamma_2 \Rightarrow \gamma_1 = \frac{1}{2}$, $\gamma_2 = \frac{1}{2}$ $\Rightarrow$ first column is $(\frac{1}{2}, \frac{1}{2})$

$$[x]_B = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix} = M\,[x]_A$$. T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ be defined by T ($\mathbf{x}$): = $(x_1 + x_2, 0)$, $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$. Then

$$[T]_A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad [T]_B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = M\,[T]_A\,M^{-1}.$$

This theorem motivates the following definition.

Definition 4.16 (Similarity) Let P and Q be square matrices of order n x n. We say that P is similar to Q if $\exists$ an invertible square matrix M such that $Q = MPM^{-1}$.

It is easy to see that this defines an equivalence relation on the set of all n x n matrices. Also the above theorem says that matrices $[T]_A$ and $[T]_B$, representing the same operator T with respect to different ordered bases A and B are similar. This raises a natural question. Given a linear operator $T \in L(V)$, can we find a basis A of V such that $[T]_A$ is very simple? What is meant by very simple? Other than the scalar matrices, the simplest examples of matrices are diagonal matrices.

Definition 4.17 (Diagonalizability) Let V, be a finite dimensional vector space. A linear operator $T \in L(V)$ is said to be diagonalizable if $\exists$ a basis A of V such that $[T]_A$ is a diagonal matrix. A square matrix P is said to be diagonalizable if it is similar to a diagonal matrix.

Thus the above question can be stated as: When is a linear operator T diagonalizable? Since an answer to this needs many other concepts like eigenvalues, eigenvectors etc., we shall discuss it later.

4.2 Kernel and Range of a Linear Transformation

Definition 4.18 Let V and W be a vector spaces. Let T: V $\rightarrow$ W be a linear transformation. Then

(i) Kernel of T = Ker T = $\{\mathbf{v} \in V \mid T\mathbf{v} = \mathbf{0}\}$. This is also known as null space of T.

(ii) The range of T = $\{\mathbf{w} \in W \mid \mathbf{w} = T\mathbf{v}$ for some $\mathbf{v} \in V\}$.

Note

1. Ker T is nonempty, since $T(\mathbf{0}) = \mathbf{0}$. Therefore, $\mathbf{0} \in$ Ker T for any linear transformation.
2. Range T is the set of 'images' of vectors in V under the transformation T.

Theorem 4.19 If T: V $\rightarrow$ W is a linear transformation, then (i) Ker T is a subspace of V (ii) Range T is a subspace of W.

Proof Will be discussed during the lecture.

Example 4.20 (Kernel and Range of Zero Transformation) Let $T\mathbf{v} = \mathbf{0}\ \forall\ \mathbf{v} \in V$, then Ker T = V, Range T = $\{\mathbf{0}\}$.

Example 4.21 (Kernel and Range of Identity Transformation) Let $T\mathbf{v} = \mathbf{v}\ \forall\ \mathbf{v} \in V$, then Ker T = $\{\mathbf{0}\}$, Range T = V.

Example 4.22 (Kernel and Range of Projection Operator) Let T: $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ is a projection from $\mathbf{R}^3$ to x_1x_2 plane. If $T(x_1, x_2, x_3) = (x_1, x_2, x_3) = (0, 0, 0) = \mathbf{0} \Rightarrow x_1 = 0,\ x_2 = 0$. Therefore, Ker T = $\{(x_1, x_2, x_3) \mid x_1 = 0, x_2 = 0\}$ = x_3 axis and Range T = $\{(x_1, x_2, x_3) \mid x_3 = 0\}$ = x_1x_2 plane. Dim(Ker T) = 1 and Dim(Range T) = 2.

More Examples will be discussed during the lecture.

Definition 4.23 If T is a linear transformation from V $\rightarrow$ W, then we define

Nullity of T = $\nu(T)$ = Dim(Ker T), Rank of T = $\rho(T)$ = Dim(Range T).

4.3 One – One and Onto Linear Transformation

Definition 4.24 (One – One Transformation) Let T: V$\rightarrow$ W be a Linear Transformation. Then T is one-one if $T\mathbf{v}_1 = T\mathbf{v}_2 \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$. i.e., T is 1–1 if every vector **w** in the Range of T is the image of exactly one vector in V.

Theorem 4.25 Let T: V$\rightarrow$ W be a Linear Transformation. Then T is one-one iff Ker T = $\{\mathbf{0}\}$.

Proof Will be discussed during the lecture.

Definition 4.26 (Onto Transformation) Let T: V$\rightarrow$ W be a Linear Transformation. Then T is said to be onto if every $\mathbf{w} \in W$, there is at least one $\mathbf{v} \in V$ such that $T\mathbf{v} = \mathbf{w}$. i.e., T is onto iff Range of T = W.

Example 4.27 T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ defined by $T(x_1, x_2) = (x_1 - x_2, 2x_1 + x_2)$. Then T(1, 0) = (1, 2) and T(0, 1) = (–1, 1) . The matrix of the linear transformation is $A_T = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. Since $|A_T| \neq 0$, A_T and hence T is invertible. Thus T is 1–1 and onto.

Example 4.28 T: $R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1 - x_2, 2x_1 - 2x_2)$. The matrix of the linear transformation is $A_T = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$. $|A_T| = 0$, $\rho(A_T) = 1$, $\nu(A_T) = 1 \Rightarrow \nu(T) = 1$. For example $T(1, 1) = \mathbf{0} = T(0, 0)$. Therefore, T is not 1–1. Range T $\neq \mathbf{R}^2$. Therefore, T is not onto. It is easy to see that Ker T = span{(1, 1)} and Range T = span {(1, 2)}.

Theorem 4.29 If T: V→ W is a linear transformation from an n dimensional vector space V to a vector space W, then Rank of T + Nullity of T = n.

Proof Will be discussed during the lecture.

Theorem 4.30 Let T: V→ W be a linear transformation. Suppose that Dim(V) = Dim(W) = n. Then (i) If T is 1–1, then T is onto. (ii) If T is onto, then T is 1–1.

Proof Will be discussed during the lecture.

Theorem 4.31 Let T: V→ W be a linear transformation. Suppose that Dim(V) = n and Dim(W) = m. Then (i) If n > m, then T is not 1–1. (ii) If m > n, then T is not onto.

Proof Will be discussed during the lecture.

4 .4 Isomorphic Vector Spaces

Definition 4.32 (Isomorphism) Let T: V→ W be a linear transformation. Then, T is an isomorphism if T is 1–1 and onto.

Definition 4.33 The vector spaces V and W are said to be isomorphic, if there exists an isomorphism T from V to W and we write V ≅ W if V is isomorphic to W

Example 4.34 (An isomorphism between $\mathbf{R}^3$ and $\mathbf{P}_2$) Let T: $\mathbf{R}^3 \to \mathbf{P}_2$ be defined by $T(a, b, c) = a + b\,x + c\,x^2$. <u>**Verify**</u> T is linear. Suppose $T(a, b, c) = 0 = 0 + 0\,x + 0\,x^2$. Then $a = b = c = 0$. i.e., Ker T = {**0**} ⇒ T is 1–1. If $p(x) = a_0 + a_1 x + a_2 x^2$, then $p(x) = T(a_0, a_1, a_2) \Rightarrow$ Range T = $\mathbf{P}_2$. Therefore, T is onto. Therefore, $\mathbf{R}^3 \cong \mathbf{P}_2$.

<u>**Remark**</u> $\text{Dim}(\mathbf{R}^3) = \text{Dim}(\mathbf{P}_2) = 3$. Once we know that T is 1–1, we know that T is onto. It was unnecessary to show that T is onto.

The following theorem illustrates the similarity of two isomorphic vector spaces.

Theorem 4.35 Let $T: V \rightarrow W$ be an isomorphism.

(i) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span V, then $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n$ span W.

(ii) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent in V, then $T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n$ are linearly independent in W.

(iii) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis in V, then $\{ T\mathbf{v}_1, T\mathbf{v}_2, \ldots, T\mathbf{v}_n \}$ is a basis in W.

(iv) If V is finite dimensional, then W is finite dimensional and Dim(V) = Dim(W).

Proof Will be discussed during the lecture.

Theorem 4.36 Let V and W be two <u>real</u> finite dimensional vector spaces with Dim(V) = Dim(W). Then, $V \cong W$.

Proof Will be discussed during the lecture.

Corollary 4.37 Every real vector space of dimension n is isomorphic to $\mathbf{R}^n$.

Proof of Theorem 3-15

Let $T: V \to W$ be a linear transformation. Then for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ in V and all scalars $\alpha_1, \alpha_2, \ldots \alpha_n$:

(i) $T(\mathbf{0}) = \mathbf{0}$ (Note: $\mathbf{0}$ on the LHS is the zero vector $\in V$, $\mathbf{0}$ on the RHS is the zero vector $\in W$).

(ii) $T(\mathbf{u}-\mathbf{v}) = T\mathbf{u} - T\mathbf{v}$

(iii) $T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n) = \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \cdots \alpha_n T\mathbf{v}_n$

Proof:- (i) $T(\mathbf{0}) = T(\mathbf{0}+\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0})$

Thus $\mathbf{0} = T(\mathbf{0}) - T(\mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) - T(\mathbf{0}) = T(\mathbf{0})$

(ii) $T(\mathbf{u}-\mathbf{v}) = T[\mathbf{u} + (-1)\mathbf{v}] = T\mathbf{u} + T[(-1)\mathbf{v}]$

$= T\mathbf{u} + (-1)T\mathbf{v} = T\mathbf{u} - T\mathbf{v}$

(iii) We prove this part by induction. For $n = 2$, we get

$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = T(\alpha_1\mathbf{v}_1) + T(\alpha_2\mathbf{v}_2) = \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2$

Thus, the equation holds for $n=2$. We assume that it holds for $n=k$ and prove it for $n=k+1$:

$T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1})$

$= T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) + T(\alpha_{k+1}\mathbf{v}_{k+1})$

and using the induction hypothesis for $n=k$, this is equal to $(\alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \cdots + \alpha_k T\mathbf{v}_k) + \alpha_{k+1} T\mathbf{v}_{k+1}$, which is what we wanted to show.

Remark Note that part (ii) of the above theorem is a special case of part (iii).

An important fact about linear transformation is that they are completely determined by what they do to basis vectors.

Proof of Theorem 3.16

Let V be a finite-dimensional vector space with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n$ be n vectors in W. Suppose T_1 and T_2 are two linear transformations from V to W such that $T_1 \mathbf{v}_i = T_2 \mathbf{v}_i = \mathbf{w}_i$, for $i = 1, 2 \ldots n$. Then, for any vector $\mathbf{v} \in V$, $T_1 \mathbf{v} = T_2 \mathbf{v}$. That is $T_1 = T_2$.

Proof: Since B is a basis for V, there exists a unique set of scalars $\alpha_1, \alpha_2 \ldots \alpha_n$ such that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots \alpha_n \mathbf{v}_n$. Then from part (iii) of Theorem 3.15,

$$T_1 \mathbf{v} = T_1(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n) = \alpha_1 T_1 \mathbf{v}_1 + \alpha_2 T_1 \mathbf{v}_2 + \cdots + \alpha_n T_1 \mathbf{v}_n$$
$$= \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \cdots + \alpha_n \mathbf{w}_n$$

Similarly,

$$T_2 \mathbf{v} = T_2(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n) = \alpha_1 T_2 \mathbf{v}_1 + \alpha_2 T_2 \mathbf{v}_2 + \cdots + \alpha_2 T_2 \mathbf{v}_n$$
$$= \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \cdots + \alpha_n \mathbf{w}_n$$

Thus $T_1 \mathbf{v} = T_2 \mathbf{v}$.

Theorem 3.16 tells us that if $T : V \to W$ is finite-dimensional, then we need to know only what T does to basis vectors in V. This determines T completely. To see this, let $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$ be a basis in V and let $\mathbf{v}$ be another vector in V. Then,

$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_n \mathbf{v}_n$, so

$T\mathbf{v} = \alpha_1 T\mathbf{v}_1 + \alpha_2 T\mathbf{v}_2 + \cdots \alpha_n T\mathbf{v}_n$.

Then, we can compute $T\mathbf{v}$ for any vector $\mathbf{v} \in V$ if we know $T\mathbf{v}_1, T\mathbf{v}_2, \ldots T\mathbf{v}_n$.

Another question arises: if $w_1, w_2, \ldots w_n$ are n vectors in W, does there exist a linear transformation T such that $Tv_i = w_i$ for $i = 1, 2 \ldots n$?
The answer is Yes and is shown in Theorem 3·17.

<u>Theorem 3·17</u> Let V be a finite dimensional vector space with basis $B = \{v_1, v_2, \ldots v_n\}$. Let W be a vector space containing the n vectors $w_1, w_2 \ldots w_n$. Then there exists a <u>unique</u> linear transformation $T: V \to W$ such that $Tv_i = w_i$, for $i = 1, 2 \ldots n$.

<u>Proof:</u> Define a function T as follows:

(i) $Tv_i = w_i$

(ii) If $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, then

$$Tv = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n \longrightarrow (*)$$

Because B is a basis for V, T is defined for every $v \in V$; and since W is a vector space, $Tv \in W$. Thus, it only remains to show that T is linear. This follows from $(*)$. For if

$$u = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$
$$v = \beta_1 v_1 + \beta_2 v_2 + \ldots + \beta_n v_n \text{, then}$$

$$T(u+v) = T\left[(\alpha_1+\beta_1) v_1 + (\alpha_2+\beta_2) v_2 + \ldots (\alpha_n+\beta_n) v_n\right]$$
$$= (\alpha_1+\beta_1) w_1 + (\alpha_2+\beta_2) w_2 + \ldots + (\alpha_n+\beta_n) w_n$$
$$= (\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n) + (\beta_1 w_1 + \beta_2 w_2 + \ldots + \beta_n w_n)$$
$$= Tu + Tv.$$

Similarly $T(\alpha v) = \alpha Tv$, so T is linear.
The uniqueness follows from Theorem 3·16.

<u>Remark</u> In Theorems 3-16 and 3-17, the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n$ need not be distinct or even independent.
Moreover, the theorems are true if V is any finite-dimensional vector space, not just $\mathbb{R}^n$.
Note also that W does not have to be finite-dimensional.

<u>Example</u> Find a linear transformation from $\mathbb{R}^2$ into the plane $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 2x - y + 3z = 0 \right\}$

$\because$ $2x - y + 3z = 0$, $y = 2x + 3z$,
any $(x, y, z) \in W$ is such that
$(x, y, z) = (x, 2x + 3z, z) = x(1, 2, 0) + z(0, 3, 1)$.
W is a two-dimensional subspace of $\mathbb{R}^3$
W has basis vectors $\mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$
Using the standard basis in $\mathbb{R}^2$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
we define the linear transformation T by
$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$.
Then, T is completely determined.
For example $T\begin{pmatrix} 5 \\ -7 \end{pmatrix} = T\left[5\begin{pmatrix} 1 \\ 0 \end{pmatrix} - 7\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$$= 5\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - 7\begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -11 \\ -7 \end{pmatrix}.$$

We have seen that if A is an $m \times n$ matrix, then the transformation $T\mathbf{x} = A\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear. We now show that every linear transformation between finite-dimensional vector spaces can be represented by a matrix.

Theorem 3.18 Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then, there exists a unique $m \times n$ matrix A_T such that $T\mathbf{x} = A_T\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$.

Proof: Let $\mathbf{w}_1 = T\mathbf{e}_1$, $\mathbf{w}_2 = T\mathbf{e}_2, \ldots \mathbf{w}_n = T\mathbf{e}_n$. Let A_T be the matrix whose columns are $\mathbf{w}_1, \mathbf{w}_2, \ldots \mathbf{w}_n$. Then $A_T\mathbf{e}_i = \mathbf{w}_i$. Now take any vector $\mathbf{x}$ in $\mathbb{R}^n$.

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

then

$$T\mathbf{x} = x_1T\mathbf{e}_1 + x_2T\mathbf{e}_2 + \cdots + x_nT\mathbf{e}_n$$
$$= x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n$$

But

$$A_T\mathbf{x} = x_1A_T\mathbf{e}_1 + x_2A_T\mathbf{e}_2 + \cdots + x_nA_T\mathbf{e}_n$$
$$= x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots + x_n\mathbf{w}_n$$

so that $T\mathbf{x} = A_T\mathbf{x}$. Uniqueness is guaranteed by Theorem 3.16.

Definition 3.19 The matrix A_T is called the transformation matrix corresponding to T.

Example 3.20 Find the transformation matrix A_T corresponding to the projection of a vector in $\mathbb{R}^3$ onto the xy-plane.

Here $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. In particular, $T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Thus,

$$A_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

CHAPTER 5
INNER PRODUCT SPACES

Just as a vector space generalizes the concept of addition of two vectors and the product of a scalar and a vector, the concept of an inner product is a generalization of the usual concept of the dot product of two vectors. This extension enables us to generalize other notions that depend on the concept of dot product, such as length (magnitude) of a vector, distance between two vectors, orthogonality, angle between the two vectors. All these in turn are highly useful in developing the practical notions of approximation, convergence etc. It is convenient to give the definition of an inner product in the context of a complex vector space. For a complex number $z, \bar{z}$ will denote its complex conjugate.

5.1 Definitions and Examples

Definition 5.1 Let V be a vector space over C. An inner product on V is a map $< , >: V \times V \to C$ satisfying the following axioms. We denote the image of $(\mathbf{a}, \mathbf{b}) \in V \times V$ under $< , >$ by $< \mathbf{a}, \mathbf{b} >$.

(i) $< \mathbf{a}, \mathbf{a} > \geq 0 \ \forall \ \mathbf{a} \in V$ and

$< \mathbf{a}, \mathbf{a} > = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$

(ii) $< \mathbf{a} + \mathbf{b}, \mathbf{c} > = < \mathbf{a}, \mathbf{c} > + < \mathbf{b}, \mathbf{c} > \forall \ \mathbf{a,b,c} \in V$

(iii) $< \alpha \mathbf{a}, \mathbf{b} > = \alpha < \mathbf{a}, \mathbf{b} > \ \forall \ \mathbf{a}, \mathbf{b} \in V, \alpha \in C$

(iv) $< \mathbf{b}, \mathbf{a} > = \overline{< \mathbf{a}, \mathbf{b} >} \ \forall \ \mathbf{a}, \mathbf{b} \in V$

Remark

1. If V is a vector space over **R**, the above definition is modified in an obvious manner. $< , >$ is a map from V x V to **R** and the condition (iv) becomes

   $< \mathbf{b}, \mathbf{a} > = < \mathbf{a}, \mathbf{b} > \ \forall \ \mathbf{a}, \mathbf{b} \in V$.
2. Condition (i) is sometimes expressed by saying that the inner product is positive definite. Similarly (ii) and (iii) are expressed by saying that the inner product is linear in the first variable; and (iv) by saying that it is conjugate symmetric.

Is an inner product also linear in the second variable? In general no. This will be clear by looking at some examples.

Definition 5.2 An <u>Inner Product space</u> is a pair (V, < , >) where V is a vector space over **C** (or **R**) and < , > is an inner product on V.

Examples 5.3 $V = \mathbf{C}^n$. For $\mathbf{z} = < z_1 , \dots , z_n >$, $\mathbf{w} = < w_1 , \dots , w_n >$ define $<\mathbf{z},\mathbf{w}> = \sum_{j=1}^{n} z_j \overline{w_j}$. It is easy to check that this map satisfies all the conditions in the Definition 5.1. This will be the case usually. Also note that $<\mathbf{z},\lambda\mathbf{w}> = \sum_{j=1}^{n} z_j \overline{\lambda w_j} = \overline{\lambda} <\mathbf{z},\mathbf{w}> \neq \lambda <\mathbf{z},\mathbf{w}>$, when $\lambda \neq \overline{\lambda}$ and $<\mathbf{z},\mathbf{w}> \neq 0$. This shows that in general, the inner product is not linear in the second variable.

Examples 5.4 $V = \mathbf{R}^n$ For $\mathbf{x} = (x_1 , \dots x_n)$, $\mathbf{y} = (y_1 , \dots y_n)$ define $<\mathbf{x},\mathbf{y}> = \sum_{j=1}^{n} x_j y_j$. This is a real inner product space.

The next example is a small modification of Example 5.3.

Examples 5.5 Let $t_1 , \dots t_n$ be any positive real numbers. Let $V = \mathbf{C}^n$. For **z**, **w** $\in \mathbf{C}^n$, define $<\mathbf{z},\mathbf{w}> = \sum_{j=1}^{n} t_j z_j \overline{w_j}$. The numbers t_j's are called weights. We can consider a similar modification of Example 5.4 also. Also note that Example 5.3 is a special case of this when $t_j = 1$, for $j = 1, \dots n$

✓**Examples 5.6** Let $V = \mathbf{C}^{n \times n}$, the vector space of all matrices of order n x n with complex entries. Recall that for $A=[\alpha_{ij}] \in V$, Trace of A denoted by tr (A) is the sum of all diagonal elements of A. Thus, tr (A) = $\sum_{j=1}^{n} \alpha_{jj}$. Also A* denotes the conjugate transpose of A; thus if $A^* = [\beta_{ij}]$, then $\beta_{ij} = \overline{\alpha_{ji}}$ for all i, j = 1, … n. Now for A , B $\in$ V , define <A, B> = tr (AB*). It is easy to show that this defines an inner product on V if we note the following: If $A = [\alpha_{ij}]$ and $B=[\beta_{ij}]$, then $<A , B> = tr (AB^*) = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij}\overline{\beta_{ij}}$. Note that if we consider matrices with real entries, then we get an example of a real ~~inner product~~ space.

✓**Examples 5.7** Let V = **C** [0,1], the vector space of all complex valued continuous functions on [0,1]. For f, g $\in$ V, define $< f , g > = \int_0^1 f(t)\overline{g(t)}\,dt$. Again it is easy to see that this is an inner product. A small modification of this can be obtained by introducing a weight function w (t), such that, w (t) > 0 for all t (for example $w(t) = e^t$) and defining a new inner product $< f, g > = \int_0^1 w(t)f(t)\overline{g(t)}\,dt$. Note, if we consider real valued fn, then we get an example of a real inner product space.

In example 5.6, if we consider matrices with real entries and make obvious modifications in the definition of the inner product, we get an example of a real inner product space. Similar comment holds if in Example 5.7, we consider real valued functions instead of complex valued functions. Next recall that $Q \in \mathbb{C}^{n \times n}$ is called Hermitian if $Q^* = Q$ and it is called positive definite if $\mathbf{x}^* Q\mathbf{x} > 0 \ \ \forall\ \mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq 0$. It is easy to see that for such a positive definite matrix, Q induces an inner product on $\mathbb{C}^n$ in a natural way as follows: For $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$, define $<\mathbf{z}, \mathbf{w}> = \mathbf{w}^* Q\mathbf{z}$, where we regard the element $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ as column matrices and the 1 x 1 matrix $\mathbf{w}^*Q\mathbf{z}$ is identified with a complex scalar.

It is also true that every inner product on $\mathbb{C}^n$ (or for that matter on every finite dimensional inner product space) arises in this fashion. To see this, recall that any inner product is linear in the first variable. This means that, if we fix an element $\mathbf{y}$ in V, and define a map f_y: $V \to \mathbb{C}$, by $f_y(\mathbf{x}) = <\mathbf{x}, \mathbf{y}>$, $\mathbf{x} \in V$, then f_y is a linear map; in fact, a linear functional. This, in particular implies that $f_y(\mathbf{0}) = <\mathbf{0}, \mathbf{y}> = 0, \ \forall\ \mathbf{y} \in V$. Also $f_y(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha f_y(\mathbf{a}) + \beta f_y(\mathbf{b})$, hence $<\alpha\mathbf{a} + \beta\mathbf{b}, \mathbf{y}> = \alpha <\mathbf{a}, \mathbf{y}> + \beta <\mathbf{b}, \mathbf{y}>$, for all $\mathbf{a}, \mathbf{b}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{C}$.

Is the inner product also linear in the second variable?

To answer this, consider for $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$ and $\alpha \in \mathbb{C}$.

$$\begin{aligned} <\mathbf{a}, \mathbf{b}+\mathbf{c}> &= \overline{<\mathbf{b}+\mathbf{c}, \mathbf{a}>} = \overline{<\mathbf{b}, \mathbf{a}> + <\mathbf{c}, \mathbf{a}>} \\ &= \overline{<\mathbf{b}, \mathbf{a}>} + \overline{<\mathbf{c}, \mathbf{a}>} = <\mathbf{a}, \mathbf{b}> + <\mathbf{a}, \mathbf{c}> \end{aligned}$$

Also, $<\mathbf{a}, \alpha\mathbf{b}> = \overline{<\alpha\mathbf{b}, \mathbf{a}>} = \overline{\alpha <\mathbf{b}, \mathbf{a}>} = \overline{\alpha} \overline{<\mathbf{b}, \mathbf{a}>}$. This shows that when V is a complex inner product space, inner product is not linear in the second variable. It is a map that is called conjugate linear. To summarize, inner product on a complex inner product space is linear in the first variable and conjugate linear in the second variable.

Note that if V is a real inner product space, then the inner product is linear in both the variables.

Now let $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_m \in V$. Let $\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathbf{a}_j, \mathbf{y} = \sum_{k=1}^{m} \beta_k \mathbf{b}_k$. Then

$$\begin{aligned} <\mathbf{x}, \mathbf{y}> &= \left\langle \sum_{j=1}^{n} \alpha_j \mathbf{a}_j, \sum_{k=1}^{m} \beta_k \mathbf{b}_k \right\rangle = \sum_{j=1}^{n} \alpha_j \left\langle \mathbf{a}_j, \sum_{k=1}^{m} \beta_k \mathbf{b}_k \right\rangle \\ &= \sum_{j=1}^{n} \alpha_j \sum_{k=1}^{m} \overline{\beta}_k \left\langle \mathbf{a}_j, \mathbf{b}_k \right\rangle = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_j \overline{\beta}_k \left\langle \mathbf{a}_j, \mathbf{b}_k \right\rangle \end{aligned}$$

Now suppose A = {$\mathbf{a}_1$, ... $\mathbf{a}_n$} is an ordered basis of V. Then $\mathbf{x}, \mathbf{y} \in$ V can be uniquely written as $\mathbf{x} = \sum_{j=1}^{n} \alpha_j \mathbf{a}_j, \mathbf{y} = \sum_{k=1}^{m} \beta_k \mathbf{b}_k$. Then as above, $<\mathbf{x},\mathbf{y}> = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_j \overline{\beta}_k \langle \mathbf{a}_j, \mathbf{b}_k \rangle$.

In other words, if we know the inner product of pair of ~~basie~~ basis elements, then we can find the inner product of every two elements in V.

Also this inner product can be expressed in a convenient matrix form. Let $G_A = [< \mathbf{a}_j , \mathbf{a}_k >]_{n \times n}$. This is called the <u>matrix of the inner product < , > with respect to the ordered basis</u> A. Recall that **x** and **y** also have representations as column matrices with respect to A, given by,

$$[\mathbf{x}]_A = \begin{bmatrix} \alpha_1 \\ . \\ . \\ . \\ \alpha_n \end{bmatrix}, \quad [\mathbf{y}]_A = \begin{bmatrix} \beta_1 \\ . \\ . \\ . \\ \beta_n \end{bmatrix}.$$

Then $<\mathbf{x},\mathbf{y}> = \sum_{j=1}^{n} \sum_{k=1}^{m} \alpha_j \overline{\beta}_k \langle \mathbf{a}_j, \mathbf{b}_k \rangle = [\mathbf{y}]_A^* G_A [\mathbf{x}]_A$, where we identify matrix of the order 1 x 1 with the corresponding scalar. Identifying **x** and **y** with their matrices, the above formula can be written as <**x**, **y**> = **y*** G**x**, where we write G = G_A. From this it is easy to prove that the matrix G of the inner product satisfies the following properties:

(i) G* = G, (ii) G is positive definite

We now discuss some geometric concepts depending on the concept of an inner product. First is the concept of <u>norm</u> which is an analogue of the concept of a magnitude of the usual two or three dimensional physical vectors.

Definition 5.8 Let V be an inner product space and let $\mathbf{a} \in$ V. Then the <u>norm of **a**</u> denoted by $||\mathbf{a}||$ is the non-negative square root of <**a**, **a**>. Thus $||\mathbf{a}|| = < \mathbf{a}, \mathbf{a}>^{1/2}$ or $||\mathbf{a}||^2 = <\mathbf{a} , \mathbf{a}>$. Since $< \mathbf{a}, \mathbf{a}> \geq 0 \ \forall \ \mathbf{a} \in$ V, this is well defined.

Examples 5.9 Let V = $\mathbf{C}^2$ as in Example 5.3. Consider **a** = (1 + i , 2 – i).

Then $||\mathbf{a}||^2 = < \mathbf{a}, \mathbf{a}> = (1 + i)\ \overline{(1+i)} + (2 - i)\ \overline{(2-i)} = 2 + 5$. Thus $||\mathbf{a}||^2 = 7$.

Examples 5.10 Let V = **C**[0,1] as in Example 5.7 above. Let f(t) = 2 + 5it , t $\in$ [0,1].

Then $||\mathbf{f}||^2 = < \mathbf{f}, \mathbf{f} > = \int_0^1 f(t)\ \overline{f(t)}\ dt = \int_0^1 |f(t)|^2\ dt = \int_0^1 (4+25t^2)\,dt = 4 + \frac{25}{3} = \frac{37}{3}$. Thus $||\mathbf{f}|| = \sqrt{\frac{37}{3}}$.

Next we see a few properties of norm.

Theorem 5.11 Let V be an inner product space. Then

(i) $||\mathbf{a}|| \geq 0 \ \forall \ \mathbf{a} \in$ V and $||\mathbf{a}|| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$

(ii) $\|\lambda \mathbf{a}\| = |\lambda| \; \|\mathbf{a}\| \; \forall \; \mathbf{a} \in V$ and $\lambda \in \mathbf{C}$

(iii) Cauchy – Schwarz inequality: $|\langle \mathbf{a}, \mathbf{b} \rangle| \le \|\mathbf{a}\| \; \|\mathbf{b}\| \; \forall \; \mathbf{a}, \mathbf{b} \in V$

(iv) $\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\| \; \forall \; \mathbf{a}, \mathbf{b} \in V$

Proof (i) Follows from the Definition of Inner product

(ii) Let $\mathbf{a} \in V$, $\lambda \in \mathbf{C}$. Then

$$\|\lambda \mathbf{a}\|^2 = \langle \lambda \mathbf{a}, \lambda \mathbf{a} \rangle = \lambda \langle \mathbf{a}, \lambda \mathbf{a} \rangle = \lambda \; \bar{\lambda} \langle \mathbf{a}, \mathbf{a} \rangle = |\lambda|^2 \|\mathbf{a}\|^2$$

(iii) Let $\mathbf{a}, \mathbf{b} \in V$ and $t \in \mathbf{R}$. Then

$$0 \le \langle t\mathbf{a} + \mathbf{b}, t\mathbf{a} + \mathbf{b} \rangle = t^2 \langle \mathbf{a}, \mathbf{a} \rangle + t(\langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{a} \rangle) + \langle \mathbf{b}, \mathbf{b} \rangle = t^2 \|\mathbf{a}\|^2 + 2t \operatorname{Re} \langle \mathbf{a}, \mathbf{b} \rangle + \|\mathbf{b}\|^2.$$

The right hand side of the above is a quadratic expression in t and it is nonnegative for all real t. Hence the discriminant is ≤ 0. That is, $4(\operatorname{Re} \langle \mathbf{a}, \mathbf{b} \rangle)^2 - 4 \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \le 0$. This gives $|\operatorname{Re} \langle \mathbf{a}, \mathbf{b} \rangle| \le \|\mathbf{a}\| \; \|\mathbf{b}\|$ and completes the proof if V is a real vector space, as in this case, since $\langle \mathbf{a}, \mathbf{b} \rangle$ is real, $\operatorname{Re} \langle \mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{b} \rangle$.

If V is a complex inner product space, we need some more work.

Let $z = \langle \mathbf{a}, \mathbf{b} \rangle$, and $\lambda = 1$, if $z = 0$

$$= \frac{|z|}{z}, \text{ if } z \ne 0.$$

Then $|\lambda| = 1$ and $\lambda z = |z|$. Now $|\langle \mathbf{a}, \mathbf{b} \rangle| = |z| = \lambda z = \operatorname{Re}(\lambda z)$ (Since $\lambda z = |z|$ is real)

Therefore, $|\langle \mathbf{a}, \mathbf{b} \rangle| = \operatorname{Re}(\lambda \langle \mathbf{a}, \mathbf{b} \rangle) = (\operatorname{Re} \langle \lambda \mathbf{a}, \mathbf{b} \rangle) \le \|\lambda \mathbf{a}\| \; \|\mathbf{b}\| = |\lambda| \; \|\mathbf{a}\| \; \|\mathbf{b}\| = \|\mathbf{a}\| \; \|\mathbf{b}\|$.

(iv) Let $\mathbf{a}, \mathbf{b} \in V$. Then $\|\mathbf{a} + \mathbf{b}\|^2 = \langle \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} \rangle = \langle \mathbf{a}, \mathbf{a} \rangle + \langle \mathbf{b}, \mathbf{a} \rangle + \langle \mathbf{a}, \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{b} \rangle$

$$= \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2 \operatorname{Re} \langle \mathbf{a}, \mathbf{b} \rangle \le \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2 \|\mathbf{a}\| \; \|\mathbf{b}\| \quad \text{by (iii) above}$$

$$= (\|\mathbf{a}\| + \|\mathbf{b}\|)^2.$$

Hence, $\|\mathbf{a} + \mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$.

If we apply the Cauchy-Schwarz inequality to inner products in the above examples, we get the following inequalities:

(i) $\left| \sum_{j=1}^{n} z_j \overline{w_j} \right| \le \left(\sum_{j=1}^{n} |z_j|^2 \right)^{1/2} \left(\sum_{j=1}^{n} |w_j|^2 \right)^{1/2} \; \forall \; \mathbf{z} = (z_1, \ldots, z_n), \; \mathbf{w} = (w_1, \ldots, w_n) \in \mathbf{C}^n$

(ii) $|\operatorname{tr}(AB^*)| \le (\operatorname{tr}(AA^*))^{1/2} \; (\operatorname{tr}(BB^*))^{1/2} \; \forall \; A, B \in \mathbf{C}^{n \times n}$

(iii) $\left| \int_0^1 f(t) \overline{g(t)} \, dt \right| \le \left(\int_0^1 |f(t)|^2 \, dt \right)^{1/2} \left(\int_0^1 |g(t)|^2 \, dt \right)^{1/2} \; \forall \; f, g \in \mathbf{C}[0, 1]$

$\langle A, B \rangle = \operatorname{tr}(AB^*)$

$|\langle A, B \rangle| \le \|A\| \, \|B\|$

$= [\operatorname{tr}(AA^*)]^{1/2} [\ldots]^{1/2}$

Definition 5.12 Let V be a vector space over **R** or **C**. A norm on V is a function $\|\ \|: V \to \mathbf{R}$ satisfying the following:

(i) $\|\mathbf{a}\| \geq 0 \ \forall\ \mathbf{a} \in V$ and $\|\mathbf{a}\| = 0 \Leftrightarrow \mathbf{a} = \mathbf{0}$

(ii) $\|\lambda\mathbf{a}\| = |\lambda|\ \|\mathbf{a}\| \ \forall\ \mathbf{a} \in V$ and $\lambda \in \mathbf{R}$ or $\mathbf{C}$

(iii) $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \ \forall\ \mathbf{a}, \mathbf{b} \in V$

A **normed linear space** is a pair $(V, \|\ \|)$, where V is a vector space and $\|\ \|$ is a norm on V.

The above Theorem shows that if a vector space has an inner product, then it also has a norm induced by that inner product by the relation $\|\mathbf{a}\| = <\mathbf{a}, \mathbf{a}>^{1/2}$. But there can be other norms on the vector spaces that do not arise in this fashion from any inner product. Here are a few examples:

For $\mathbf{z} \in C^n$, define $\|\mathbf{z}\|_1 = \sum_{j=1}^{n} |z_j|$, $\|\mathbf{z}\|_\infty = \max\{|z_j|, j = 1, 2 \ldots n\}$. Then $\|\ \|_1$ and $\|\ \|_\infty$ are norms on $\mathbf{C}^n$.

Similarly, for $f \in \mathbf{C}[0,1]$, define $\|f\|_1 = \int_0^1 |f(t)|dt$ and $\|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$. Then $\|\ \|_1$ and $\|\ \|_\infty$ are norms on $\mathbf{C}[0,1]$.

How do we know that these norms are not induced by any inner product?

Exercise

(i) Show that if $\|\ \|$ on V is induced by an inner product, then it satisfies the **parallelogram identity**: $\forall\ \mathbf{a}, \mathbf{b} \in V$, $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2$.

(ii) Show that $\|\ \|_1$ and $\|\ \|_2$ do not satisfy the parallelogram identity.

5.2 Orthogonal Bases and Gram-Schmidt Orthonormalization

Another geometric concept that depends on the notion of the inner products is that of an angle between two vectors.

Definition 5.13 Let V be an inner product space and $\mathbf{a}, \mathbf{b} \in V$. We say that a is orthogonal to b, denoted by $\mathbf{a} \perp \mathbf{b}$ if $<\mathbf{a}, \mathbf{b}> = 0$.

A subset $A \subseteq V$ is called orthogonal if any two distinct vectors in A are orthogonal to each other.

An orthogonal set is called orthonormal if every vector in it is of norm 1.

Thus, A is orthonormal if $\forall\ \mathbf{a}, \mathbf{b} \in A$, $<\mathbf{a}, \mathbf{b}> = 0$ if $\mathbf{a} \neq \mathbf{b}$.

$= 1$ if $\mathbf{a} = \mathbf{b}$

Example 5.14 Let $\mathbf{a} = (1, 1)$, $\mathbf{b} = (1, -1) \in \mathbf{R}^2$. Then $<\mathbf{a}, \mathbf{b}> = 0$. Thus $\{\mathbf{a}, \mathbf{b}\}$ is an orthogonal set. However, it is not orthonormal. On the other hand, the standard ordered basis of $\mathbf{R}^2$, namely, $\{\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0,1)\}$ is an orthonormal set. Similarly standard bases in each $\mathbf{R}^n$ and $\mathbf{C}^n$ are orthonormal sets.

Example 5.15 Let $V = \mathbf{C}([-\pi, \pi])$ with the inner product $< f, g> = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt$, for $f, g \in V$. Let $f_0(t) = 1$, $f_1(t) = t \ \forall\, t \in [-\pi, \pi]$. Then $< f_0, f_1 > = 0$. Thus $\{f_0, f_1\}$ is an orthogonal set. It is not orthonormal because $< f_0, f_0 > = 2\pi$

Now let $g_n(t) = \dfrac{e^{int}}{\sqrt{2\pi}}$ $t=[-\pi, \pi]$. Then $<g_n, g_m> = \int_{-\pi}^{\pi} \dfrac{e^{int}}{\sqrt{2\pi}} \dfrac{e^{-imt}}{\sqrt{2\pi}}\,dt = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)t}\,dt$

$= 0$ if $n \neq m$

$= 1$ if $n = m$

Thus $\{g_n = n = 0, \pm 1, \pm 2, \ldots\}$ is an infinite orthonormal set in V.

Theorem 5.16 (Pythogoras Theorem) Let V be an inner product space and $\mathbf{a}, \mathbf{b} \in V$. If $\mathbf{a} \perp \mathbf{b}$, then $||\mathbf{a} + \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$.

Proof $||\mathbf{a} + \mathbf{b}||^2 = <\mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b}> = <\mathbf{a}, \mathbf{a}> + <\mathbf{b}, \mathbf{a}> + <\mathbf{a}, \mathbf{b}> + <\mathbf{b}, \mathbf{b}> = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$, since $\mathbf{a} \perp \mathbf{b}$, $<\mathbf{a}, \mathbf{b}> = 0 = <\mathbf{b}, \mathbf{a}>$.

There are several advantages of working with orthonormal sets. For example, suppose $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is a linearly independent set and $\mathbf{x} \in$ span (A). Then $\mathbf{x}$ can be uniquely expressed as $\mathbf{x} = \alpha_1 \mathbf{a}_1 + \ldots + \alpha_n \mathbf{a}_n$.

Given a vector $\mathbf{x}$, if we want to compute the coefficient α_j, then we have to solve a system of n equations in n unknowns $\alpha_1, \ldots, \alpha_n$. However, if $\mathbf{a}_1, \ldots \mathbf{a}_n$ is an orthonormal set, this problem becomes very easy. Consider for example,

$<\mathbf{x}, \mathbf{a}_1> = <\alpha_1 \mathbf{a}_1 + \ldots + \alpha_n \mathbf{a}_n, \mathbf{a}_1>$

$= \alpha_1 <\mathbf{a}_1, \mathbf{a}_1> + \alpha_2 <\mathbf{a}_2, \mathbf{a}_1> + \ldots + \alpha_n <\mathbf{a}_n, \mathbf{a}_1>$

$= \alpha_1$.

Thus $\alpha_1 = <\mathbf{x}, \mathbf{a}_1>$. Similarly, $\alpha_2 = <\mathbf{x}, \mathbf{a}_2>, \ldots \alpha_n = <\mathbf{x}, \mathbf{a}_n>$. This also means that if $\mathbf{x} = 0$, then each $\alpha_j = 0$, for $j = 1, \ldots, n$. We have proved the following:

Theorem 5.17 Every orthonormal set is linearly independent.

The converse of the above theorem is obviously false. For example, in $\mathbf{R}^3$, the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, where $\mathbf{a}_1 = (1, 1, 0)$, $\mathbf{a}_2 = (0, 1, 1)$, $\mathbf{a}_3 = (1, 0, 1)$, is linearly independent but not orthonormal. On the other hand, given a linearly independent set A in an inner product space, we can find an orthonormal set U such that span (A) = span (U). We now describe the procedure for doing this known as **GRAM – SCHMIDT PROCESS**.

Theorem 5.18 Let A = $\{\mathbf{a}_1, \ldots\}$ be a linearly independent set of vectors in an inner product space V. Then we can obtain an orthonormal set U = $\{\mathbf{u}_1,\ldots\}$ such that for each j, span $(\{\mathbf{a}_1,\ldots\mathbf{a}_j\})$ = span $(\{\mathbf{u}_1,\ldots,\mathbf{u}_j\})$.

Proof By mathematical induction. Let j = 1. Since A is linearly independent $\mathbf{a}_1 \neq 0$. Let $u_1 = \dfrac{\mathbf{a}_1}{\|\mathbf{a}_1\|}$. Then clearly $\{\mathbf{u}_1\}$ is an orthonormal set and span $(\{\mathbf{a}_1\})$ = span $(\{\mathbf{u}_1\})$. Next, suppose that for j = m, we have constructed an orthonormal set $\{\mathbf{u}_1,\ldots,\mathbf{u}_m\}$ such that span$(\{\mathbf{a}_1,\ldots,\mathbf{a}_m\})$ = span $(\{\mathbf{u}_1,\ldots,\mathbf{u}_m\})$. Consider j = m+1. Let $\mathbf{b}_{m+1} = \mathbf{a}_{m+1} - \sum_{j=1}^{m}\langle \mathbf{a}_{m+1}, \mathbf{u}_j\rangle \mathbf{u}_j$. First note that $\mathbf{b}_{m+1} \neq 0$, or otherwise, $\mathbf{a}_{m+1} \in$ span $(\{\mathbf{u}_1,\ldots,\mathbf{u}_m\})$ = span $(\{\mathbf{a}_1,\ldots,\mathbf{a}_m\})$ which is a contradiction. Next for k=1, … , m

$$\begin{aligned}\langle \mathbf{b}_{m+1}, \mathbf{u}_k\rangle &= \langle \mathbf{a}_{m+1}, \mathbf{u}_k\rangle - \left\langle \sum_{j=1}^{m}\langle \mathbf{a}_{m+1}, \mathbf{u}_j\rangle \mathbf{u}_j, \mathbf{u}_k\right\rangle \\ &= \langle \mathbf{a}_{m+1}, \mathbf{u}_k\rangle - \sum_{j=1}^{m}\langle \mathbf{a}_{m+1}, \mathbf{u}_j\rangle\langle \mathbf{u}_j, \mathbf{u}_k\rangle \quad (\because \langle \mathbf{u}_j, \mathbf{u}_k\rangle = 0 \text{ for } j \neq k) \\ &= \langle \mathbf{a}_{m+1}, \mathbf{u}_k\rangle - \langle \mathbf{a}_{m+1}, \mathbf{u}_k\rangle = 0\end{aligned}$$

Thus $\mathbf{b}_{m+1} \perp \mathbf{u}_k$ for k = 1, … ,m. Let $\mathbf{u}_{m+1} = \dfrac{\mathbf{b}_{m+1}}{\|\mathbf{b}_{m+1}\|}$. Then $\{\mathbf{u}_1,\ldots,\mathbf{u}_m,\mathbf{u}_{m+1}\}$ is an orthonormal set.

It also follows that

$$\begin{aligned}\text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}\}) &\subseteq \text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{b}_{m+1}\}) \\ &\subseteq \text{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{a}_{m+1}\}) \\ &\subseteq \text{span}(\{\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{a}_{m+1}\})\end{aligned}$$

Similarly, we can show span $(\{\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{a}_{m+1}\}) \subseteq$ span $(\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{u}_{m+1}\})$.

Remarks 5.19 The above proof shows that this **GRAM-SCHMIDT PROCESS** can be also used to check whether a given set is linearly independent. If A=$\{\mathbf{a}_1 \ldots\}$ is linearly dependent, then for some m, $\mathbf{a}_{m+1} \in$ span $(\{\mathbf{a}_1, \ldots, \mathbf{a}_m\})$ = span $(\{\mathbf{u}_1, \ldots, \mathbf{u}_m\})$. Hence at this stage $\mathbf{b}_{m+1}$ will become 0.

Examples of applying Gram-Schmidt Process

Example 5.20 Let us apply this method to the set A = $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subseteq \mathbf{R}^3$ considered above.

$\mathbf{a}_1 = (1,1,0)$, $\mathbf{a}_2 = (0,1,1)$, $\mathbf{a}_3 = (1,0,1)$. $\|\mathbf{a}_1\| = \sqrt{2}$. Hence $\mathbf{u}_1 = \frac{\mathbf{a}_1}{\|\mathbf{a}_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

$$\mathbf{b}_2 = \mathbf{a}_2 - \langle \mathbf{a}_2, \mathbf{u}_1\rangle \mathbf{u}_1 = (0,1,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) = (0,1,1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) = -\left(\frac{1}{2}, \frac{1}{2}, 1\right)$$

$\|\mathbf{b}_2\|^2 = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2}$; $\|\mathbf{b}_2\| = \frac{\sqrt{3}}{\sqrt{2}}$. Hence $\mathbf{u}_2 = \frac{\mathbf{b}_2}{\|\mathbf{b}_2\|} = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$.

$$\mathbf{b}_3 = \mathbf{a}_3 - \langle \mathbf{a}_3, \mathbf{u}_1\rangle \mathbf{u}_1 - \langle \mathbf{a}_3, \mathbf{u}_2\rangle \mathbf{u}_2 = (1, 0, 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$= (1, 0, 1) - \left(\frac{1}{2}, \frac{1}{2}, 0\right) - \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}\right) = \left(\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$\mathbf{u}_3 = \frac{\mathbf{b}_3}{\|\mathbf{b}_3\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Thus U = $\left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{6}}(-1,1,2), \frac{1}{\sqrt{3}}(1, -1, 1)\right\}$.

Example 5.21 Let V = C([−1, 1]) with the inner product, $\langle f, g\rangle = \int_{-1}^{1} f(t)\overline{g(t)}\,dt$, f, g $\in$ V. Let A= $\{f_0, f_1, \ldots f_n, \ldots\}$, where $f_0(t) = 1$, $f_1(t) = t$, … $f_n(t) = t^n \ldots$ $t \in [-1, 1]$. Then A is a linearly independent set. Now we apply Gram-Schmidt process to A. This will be more involved and time consuming compared to last example because calculation of inner products involve computing integrals.

$f_0 = 1$, $\|f_0\|^2 = \int_{-1}^{1} 1\,dt = 2$, $\|f_0\| = \sqrt{2}$, $p_0 = \frac{f_0}{\|f_0\|} = \frac{f_0}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$g_1 = f_1 - \langle f_1, p_0\rangle p_0 = f_1$ as $\langle f_1, p_0\rangle = \int_{-1}^{1} f(t)\overline{p_0(t)}\,dt = \frac{1}{\sqrt{2}}\int_{-1}^{1} t\,dt = 0$, $\|f_1\|^2 = \int_{-1}^{1} t^2\,dt = \frac{2}{3}$, $\|f_1\| = \frac{\sqrt{2}}{\sqrt{3}}$

Thus $p_1 = \frac{f_1}{\|f_1\|} = \frac{3}{2} f_1$

$g_2 = f_2 - \langle f_2, p_0\rangle p_0 - \langle f_2, p_1\rangle p_1$; $\langle f_2, p_0\rangle = \int_{-1}^{1} t^2 \frac{1}{\sqrt{2}}\,dt = \frac{1}{\sqrt{2}}\frac{2}{3} = \frac{\sqrt{2}}{3}$; $\langle f_2, p_1\rangle = \int_{-1}^{1} t^2 \frac{\sqrt{3}}{\sqrt{2}} t\,dt = 0$

Thus $g_2 = f_2 - \frac{\sqrt{2}}{3}\frac{1}{\sqrt{2}} = f_2 - \frac{1}{3}$. Then $\|g_2\|^2 = \int_{-1}^{1}\left(t^2 - \frac{1}{3}\right)^2 dt = \frac{8}{45}$

$p_2 = \frac{g_2}{\| g_2 \|} = \frac{\sqrt{45}}{\sqrt{8}}\left(f_2 - \frac{1}{3}\right) = \frac{3}{2}\frac{\sqrt{5}}{\sqrt{2}}\left(f_2 - \frac{1}{3}\right)$. Thus $p_2(t) = \frac{\sqrt{5}}{\sqrt{2}}\frac{1}{2}(3t^2 - 1)$.

Proceeding in this way,for each n, we get a polynomial p_n, such that $U = \{p_0, p_1, \ldots p_n\}$ is an orthonormal set. Suitable scalar multiples of these polynomials are known as Legendre polynomials. These are well known polynomials with many interesting properties and several applications to differential equations.

Similarly many other sets of polynomials can be obtained by applying Gram-Schmidt process in a suitable inner product space. Some well known examples include Hermite polynomials, Laguerre polynomials.0

Theorem 5.22 Every finite dimensional inner product space has an orthonormal basis.

Proof Follows by applying Gram-Schmidt process to any basis of the space.

Theorem 5.23 Let V be an inner product space and $U = \{\mathbf{u}_1, \ldots \mathbf{u}_n\}$ be an orthonormal set. For $x \in V$, define $\mathbf{u} = \sum_{j=1}^{n} <\mathbf{x}, \mathbf{u}_j> \mathbf{u}_j$, Then

(i) $<\mathbf{x} - \mathbf{u}, \mathbf{y}> = \mathbf{0} \;\; \forall\, \mathbf{y} \in \text{span}(U)$.

(ii) (Bessel's inequality) $\|\mathbf{u}\|^2 = \sum_{j=1}^{n} |<\mathbf{x}, \mathbf{u}_j>|^2 \leq \|\mathbf{x}\|^2$

(iii) If U is a basis of V, then (a) $\mathbf{x} = \sum_{j=1}^{n} <\mathbf{x}, \mathbf{u}_j> \mathbf{u}_j$ and (b) $\|\mathbf{x}\|^2 = \sum_{j=1}^{n} |<\mathbf{x}, \mathbf{u}_j>|^2$

In this case (iii) is called the Fourier expansion of **x** with respect to the orthonormal basis U and (ii) is known as the PARSEVAL'S identity.

Proof Consider for k = 1 , ... , n

$$<\mathbf{u}, \mathbf{u}_k> = \left\langle \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j, \mathbf{u}_k \right\rangle = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_k \rangle = \langle \mathbf{x}, \mathbf{u}_k \rangle$$

(Note: $\langle \mathbf{u}_j, \mathbf{u}_k \rangle = 0$ if $j \neq k$
$= 1$ if $j = k$)

Thus $<\mathbf{x} - \mathbf{u}, \mathbf{u}_k> = 0 \;\; \forall\, \mathbf{u}_k \in U$. Hence $<\mathbf{x} - \mathbf{u}, \mathbf{y}> = \mathbf{0} \;\forall\, \mathbf{y} \in \text{span}(U)$. This proves (i).

$$\text{Next, } \|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u} \rangle = \left\langle \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_j \rangle \mathbf{u}_j, \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_k \rangle \mathbf{u}_k \right\rangle = \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_j \rangle \left\langle \mathbf{u}_j, \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_k \rangle \mathbf{u}_k \right\rangle$$

$$= \sum_{j=1}^{n} \langle \mathbf{x}, \mathbf{u}_j \rangle \sum_{j=1}^{n} \overline{\langle \mathbf{x}, \mathbf{u}_k \rangle} \langle \mathbf{u}_j, \mathbf{u}_k \rangle = \sum_{j=1}^{n} \left| \langle \mathbf{x}, \mathbf{u}_j \rangle \right|^2$$

Again the last step follows by noting that, since U is an orthonormal set, $\langle \mathbf{u}_j, \mathbf{u}_k \rangle = 0$ if $j \neq k$
$= 1$ if $j = k$

Now, $\|\mathbf{x}\|^2 = \|\mathbf{x} - \mathbf{u}\|^2 + \|\mathbf{u}\|^2 \geq \|\mathbf{u}\|^2$. This proves (ii).

Further if U is a basis of V, then V = span (U). Thus $\mathbf{x} \in$ span (U), hence $\mathbf{x} - \mathbf{u} \in$ span (U). Now by (i), $<\mathbf{x} - \mathbf{u} , \mathbf{x} - \mathbf{u}> = \mathbf{0}$. Hence, $\mathbf{u} = \mathbf{x}$. Now (iii) (a) and (iii) **(b)** follow immediately, from (i) and (ii).

5.3 Some Applications of the Gram-Schmidt Process

QR-Factorization of a matrix

Recall the following basic fact about the Gram-Schmidt Process.

Proposition 5.24 Let A = $\{\mathbf{a}_1 \ldots\}$ be a linearly independent set of vectors in an inner product space V. Then by applying the Gram-Schmidt process to A, we can obtain an orthonormal set U= $\{\mathbf{u}_1, \ldots\}$ such that for each j, span $(\{\mathbf{a}_1 \ldots ,\mathbf{a}_j\})$ = span $(\{\mathbf{u}_1 \ldots , \mathbf{u}_j\})$.

Now let A be a matrix of order m x n with real entries. Let $\mathbf{a}_1 \ldots , \mathbf{a}_n$ denote columns of A. Each $\mathbf{a}_j$ can be regarded as a vector in $\mathbf{R}^m$. Note that if $\{\mathbf{a}_1, \ldots ,\mathbf{a}_n\}$ is an orthonormal set, then $A^T A = I_n$, where I_n denotes the identity matrix of order n. This is because ij^{th} entry of $A^T A$ is the inner product of i^{th} and j^{th} columns of A (If A has complex entries, then a similar argument will give $A^* A = I_n$. For the sake of convenience, we shall only discuss the matrices with real entries).

In general, if the columns $\mathbf{a}_1 \ldots ,\mathbf{a}_n$ of A form a linearly independent set, then we can apply the Gram-Schmidt process 5.24 to it and obtain an orthonormal set, say, $\mathbf{q}_1, \ldots ,\mathbf{q}_n$ in **R** . Let Q be the matrix formed with these columns. Then, as we have observed above $Q^T Q = I_n$. Further, since for each j, span $(\{\mathbf{a}_1,\ldots,\mathbf{a}_j\})$ = span $(\{\mathbf{q}_1,\ldots ,\mathbf{q}_j\})$, each $\mathbf{a}_j$ can be expressed as a linear combination of $\mathbf{q}_1 \ldots ,\mathbf{q}_j$. Thus we can find real numbers r_{ij} such that $\mathbf{a}_j = r_{1j}\, \mathbf{q}_1 + \ldots + r_{jj}\, \mathbf{q}_j$. Now let $r_{ij} = 0$ if $i > j$ and R be the matrix with entries r_{ij}. In other words R is upper triangular. Also no r_{jj} can be zero (why?) so that R is invertible. The above equation can be written in the matrix form as A=QR. This factorization of A is known as QR-Factorization and is quite important in numerical linear algebra. Note that we have proved the following:

Corollary 5.25 Let A be a matrix of order m x n with real entries. If A has linearly independent columns, then there exists a matrix Q of order m x n whose columns form an orthonormal set in R^n, and an upper triangular invertible matrix **R** of order n x n, such that A = QR.

Example 5.26 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then applying the Gram-Schmidt process 5.24 to the linearly independent set {(1,0,1), (1,1,1)} in $\mathbf{R}^3$, we get the orthonormal set of vectors $\{(1/\sqrt{2},\ 0,\ 1/\sqrt{2}),\ (0,1,0)\}$. Thus the matrix Q in 5.25 is given by $Q = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 1/\sqrt{2} & 0 \end{bmatrix}$. We have A = QR where R is an upper triangular invertible matrix of order 2 x 2. To find R, we can solve the equation A=QR obtaining $R=Q^T A$. Thus $R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1 \end{bmatrix}$.

Exercises: Find a QR-Factorization of the following matrices.

1. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ 2. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ 3. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5.4 Best approximations

We frequently encounter a situation where we want to approximate an element **x** in an inner product space V by some element in a subset W of V. For example, we may want to find a vector in the plane $x_1 + 2x_2 + 4x_3 = 0$ which is closest to the vector (1,1,1) or we may want to find a polynomial of degree ≤ 3 that approximates the function exp(x) as closely as possible. This is made precise in the following Definition.

Definition 5.27 Let V be an inner product space, $\mathbf{x} \in V$ and W be a nonempty subset of V. Then a vector $\mathbf{u} \in W$ is said to be **best approximation** to **x** from W if $\|\mathbf{x} - \mathbf{u}\| \le \|\mathbf{x} - \mathbf{y}\|$ for every $\mathbf{y} \in W$. Note that if such a **u** exists, then $\|\mathbf{x} - \mathbf{u}\|$ is the minimum distance from **x** to an element in W and it is called **error of approximation**. Since the norms in many familiar inner product spaces are defined in terms of sums (or integrals) of squares, these best approximations are also called **least square approximation**. Another commonly used term is **projection of x on** W. In general, such a best approximation may or may not exist, it may not be unique. But the situation is very satisfactory if W is a finite dimensional subspace. In this case, there exists a unique best approximation and moreover, we can find it by using the Gram-Schmidt process 5.24.

Theorem 5.28 Let V be an inner product space, $\mathbf{x} \in V$ and W be a subspace of V. Then

1. A vector $\mathbf{u} \in W$ is a best approximation to $\mathbf{x}$ from W if and only if $< \mathbf{x} - \mathbf{u}, \mathbf{y} > = 0$ for all $y \in W$. (that is, $\mathbf{x} - \mathbf{u}$ is orthogonal to every element in W).
2. If a best approximation to $\mathbf{x}$ from W exists, then it is unique.
3. If W is finite-dimensional and $\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}$ is an orthonormal basis of W, then $\mathbf{u} = \sum_{j=1}^{n} <\mathbf{x}, \mathbf{u}_j> \mathbf{u}_j$ is the unique best approximation to $\mathbf{x}$ from W.

Proof 1. Suppose $<\mathbf{x} - \mathbf{u}, \mathbf{y}> = 0 \ \forall \ \mathbf{y} \in W$. Then for every $\mathbf{w} \in W$, $\mathbf{u} - \mathbf{w} \in W$. Hence $<\mathbf{x} - \mathbf{u}, \mathbf{u} - \mathbf{w}> = 0$. By Pythagoras theorem,

$||\mathbf{x} - \mathbf{w}||^2 = ||\mathbf{x} - \mathbf{u} + \mathbf{u} - \mathbf{w}||^2 = ||\mathbf{x} - \mathbf{u}||^2 + ||\mathbf{u} - \mathbf{w}||^2 \geq ||\mathbf{x} - \mathbf{u}||^2$.

Thus $\mathbf{u}$ is a best approximation to $\mathbf{x}$ from W.

Conversely, suppose $\mathbf{u}$ is a best approximation to $\mathbf{x}$ from W. Let $y \in W$. Suppose $<\mathbf{x} - \mathbf{u}, \mathbf{y}> \neq 0$. Replacing y by $-y$ if V is real or by any one of $-y$, iy, $-iy$ if V is complex inner product space, we may assume that

$$\text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}> \ < \ 0 \qquad (5.1)$$

Now $\forall \ t \in \mathbf{R}$, $\mathbf{u} - t\,\mathbf{y} \in W$. Hence

$$||\mathbf{x} - \mathbf{u}||^2 \leq ||\mathbf{x} - \mathbf{u} + t\,\mathbf{y}||^2 = <\mathbf{x} - \mathbf{u} + t\,\mathbf{y}, \mathbf{x} - \mathbf{u} + t\,\mathbf{y}> = ||\mathbf{x} - \mathbf{u}||^2 + 2\,t\,\text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}> + t^2\,||\mathbf{y}||^2.$$

Thus $t^2\,||\mathbf{y}||^2 + 2\,t\,\text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}> \geq 0 \ \forall \ t \in \mathbf{R}$. Hence $\forall \ t > 0$, $\frac{t}{2}\,||\mathbf{y}||^2 \geq -\ \text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}>$. Since this holds $\forall \ t \geq 0$, we must have $-\ \text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}> \leq 0$, that is, $\text{Re} <\mathbf{x} - \mathbf{u}, \mathbf{y}> \ \geq \ 0$. This contradicts (5.1). Hence $< \mathbf{x} - \mathbf{u}, \mathbf{y} > = 0$, $\forall \ \mathbf{y} \in W$. This proves 1.

2. Suppose $\mathbf{u}, \mathbf{v} \in W$ and both $\mathbf{u}, \mathbf{v}$ are best approximations to $\mathbf{x}$ from W. Then $\mathbf{u} - \mathbf{v} \in W$. Hence $<\mathbf{x} - \mathbf{u}, \mathbf{u} - \mathbf{v}> = 0 = <\mathbf{x} - \mathbf{v}, \mathbf{u} - \mathbf{v}>$. Now, $||\mathbf{u} - \mathbf{v}||^2 = <\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}> = <\mathbf{x} - \mathbf{v} - (\mathbf{x} - \mathbf{u}), \mathbf{u} - \mathbf{v}>$ $= <\mathbf{x} - \mathbf{v}, \mathbf{u} - \mathbf{v}> - <\mathbf{x} - \mathbf{u}, \mathbf{u} - \mathbf{v}> = 0$. Hence $\mathbf{u} = \mathbf{v}$. This proves 2.

3. We have already proved (Theorem 5.23 (i)) that if

$$\mathbf{u} = \sum_{j=1}^{n} <\mathbf{x}, \mathbf{u}_j> \mathbf{u}_j, \text{ then } <\mathbf{x} - \mathbf{u}, \mathbf{y}> = 0 \ \forall \ \mathbf{y} \in \text{span}(\{\mathbf{u}_1,\ldots,\mathbf{u}_n\}) = W.$$

Hence by 1, $\mathbf{u}$ is a best approximation to $\mathbf{x}$ from W and it is unique by 2.

Note that the above theorem gives a method of finding the best approximation to $\mathbf{x}$ from W, when W is finite dimensional, namely first find an orthonormal basis of W and then use the above formula. It is also worthwhile to note that in applying the Gram-Schmidt process to a linearly independent set $\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$, at each stage j+1, we actually subtract from $\mathbf{a}_{j+1}$, its projection on

W_j = span ({$\mathbf{a}_1,\ldots,\mathbf{a}_j$}) = span ({$\mathbf{u}_1,\ldots,\mathbf{u}_j$}) and hence the resulting vector is orthogonal to each vector in W_j and in particular to each of $\mathbf{u}_1,\ldots,\mathbf{u}_j$. Then we divide this vector by its norm to get the next unit vector $\mathbf{u}_{j+1}$.

Exercises Find the indicated best approximations.

1. The best approximation of (1,2,1) from span ({3,1,2), (1,0,1)} in $\mathbf{R}^3$.

2. The best approximation of (1,2,1) from the plane $x_1 + x_2 + x_3 = 0$ in $\mathbf{R}^3$.

3. The best approximation of (1, 0, −1,1) from span ({(1,0,1,1), (0,0,1,1)}) in $\mathbf{R}^4$.

4. The best approximation of $f(x) = e^x$ from $\mathbf{P}_3$ in the inner product space $\mathbf{C}[-1, 1]$ with the usual inner product.

5.5 Best approximate solutions

Consider a system of equations $A\mathbf{x} = \mathbf{b}$, where A is a matrix of order m x n and $\mathbf{b} \in \mathbf{R}^m$. If this system does not have a solution, then we look for approximate solutions, which can be considered best in a certain sense. The following definition makes this notion precise.

Definition 5.29 Let V, W be inner product spaces, $T:V \to W$ be a linear transformation and $\mathbf{b} \in W$. A vector $\mathbf{u} \in V$ is said to be a **best approximate solution** of the equation $T\,\mathbf{x} = \mathbf{b}$ if $\|T\mathbf{u} - \mathbf{b}\| \leq \|T\mathbf{x} - \mathbf{b}\|$ for all $\mathbf{x} \in V$. A best approximate solution is also called a **least square solution.** Note that if $\mathbf{u}$ is a best approximation solution to $T\mathbf{x} = \mathbf{b}$, then $T\mathbf{u}$ is the best approximation to $\mathbf{b}$ from the Range of T.

For any $\mathbf{x} \in V$, the number $\|T\mathbf{x} - \mathbf{b}\|$ is called **error of approximation.** Thus a best approximate solution $\mathbf{u}$ minimizes this error. In general, such a best approximate solution need not exist and also it may not be unique. We shall only consider the case when $V = \mathbf{R}^n$, $W = \mathbf{R}^m$ and the linear transformation $T : V \to W$ is represented by a matrix A of order m x n with respect to the standard bases in $\mathbf{R}^n$ and $\mathbf{R}^m$. Henceforth we shall not distinguish between T and A. Thus $\mathbf{u} \in \mathbf{R}^n$ is a best approximate (least square) solution of $A\mathbf{x} = \mathbf{b}$ means that $\|A\mathbf{u} - \mathbf{b}\| \leq \|A\mathbf{x} - \mathbf{b}\|$ for all $\mathbf{x} \in \mathbf{R}^n$. Also note that the range of A is same as the column space of A and a vector $\mathbf{y} \in \mathbf{R}^m$ is orthogonal to the range of A if and only if $A^T\mathbf{y} = \mathbf{0}$.

Theorem 5.30 Let A be a matrix of order m x n with real entries and let $\mathbf{b} \in \mathbf{R}^m$. Then $\mathbf{u} \in \mathbf{R}^n$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ if and only if $A^T (A\mathbf{u} - \mathbf{b}) = \mathbf{0}$.

Proof Let $\mathbf{u} \in \mathbf{R}^n$. Then $\mathbf{u} \in \mathbf{R}^n$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ if and only if $A\mathbf{u}$ is a best approximation to $\mathbf{b}$ from the range of A if and only if $A\mathbf{u} - \mathbf{b}$ is orthogonal to the range of A (Theorem 5.28) if and only if $A^T (A\mathbf{u} - \mathbf{b}) = 0$.

The above theorem gives a method to find a least square solution of $A\mathbf{x} = \mathbf{b}$, namely that to find such a solution, we must solve the equation $A^T A\mathbf{u} = A^T \mathbf{b}$.

Again, in general, this new system may or may not have solution. The following corollary gives conditions for the existence as well uniqueness of such a least square solution.

Corollary 5.31 Let A, **b** as in Theorem 5.30. If the columns of A form a linearly independent set, then there exists a unique least square solution of $A\mathbf{x} = \mathbf{b}$.

Proof By Corollary 5.25, there exists a matrix Q of order m x n whose columns form an orthonormal set in $\mathbf{R}^n$, and an upper triangular invertible matrix R of order n x n, such that A = QR. Now the equation $A^T A\mathbf{u} = A^T \mathbf{b}$ becomes $R^T Q^T Q R\mathbf{u} = R^T Q^T \mathbf{b}$. Since $Q^T Q = I_n$, this becomes $R^T R\mathbf{u} = R^T Q^T \mathbf{b}$, that is, $R\mathbf{u} = Q^T \mathbf{b}$, because R is invertible. Again since R is invertible, this last equation has a unique solution.

This corollary also gives a method of finding the unique least square solution, namely, we first find the QR – Factorization of A and then solve the system of equation $R\mathbf{u} = Q^T\mathbf{b}$. Since R is upper triangular, this can be solved very easily by back substitution.

Example 5.32 According to Hooke's law, the distance that a spring stretches to is proportional to the force applied. The following table gives the data obtained by attaching four different weights to the spring and measuring the resultant lengths.

Weight in kg.	2.0	4.0	5.0	6.0
Length in cm.	6.5	8.5	11.0	12.5

Let w denote the weight and l the corresponding length. By Hooke's law, we expect l and w to satisfy the relationship of the type $l = r_0 + r_1 w$. Thus there are only two unknowns r_0 and r_1 and theoretically, only two measurements should suffice to determine their values. In practice, however, there are some errors in measurements. Hence more measurements are made than actually necessary. Each such measurement leads to one equation and we obtain what is known as an **over determined system**. It is unlikely that such a system will have an exact solution. Hence we look for a least square (or best approximate) solution. Thus we are looking for a least square solution of $A\mathbf{r} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{bmatrix}$, $\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 6.6 \\ 8.5 \\ 11.0 \\ 12.5 \end{bmatrix}$. Since A has linearly independent columns, by Corollary 5.31, there exists a unique least square solution $\mathbf{u}$. This can be obtained either by solving the system $A^T A\mathbf{u} = A^T \mathbf{b}$ (see Theorem 5.30) or as mentioned after Corollary 5.31, by first finding the QR-Factorization of A and then solving $R\mathbf{u} = Q^T\mathbf{b}$. As remarked earlier, the second method is usually easier and faster. By any of this method, we can obtain $\begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 3.1 \\ 1.5 \end{bmatrix}$. Thus the function that best fits the data points is $l = 3.1 + 1.5\,w$. This is called the **lease square fit** of the data points. This is linear least square fit. Similarly one can consider quadratic, cubic and other higher order least square fits.

Exercises Find the least square solution for each of the following system $A\mathbf{x} = \mathbf{b}$.

1. $A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \\ 2 & -1 \end{bmatrix}$ $\qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ $\qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 0 & 5 \\ 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $\qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -2 \\ 0 \end{bmatrix}$

5.6 Contraction Mapping Principle and Applications

Contraction mapping principle

One of the popular methods of finding approximate solution to an equation of the type $f(\mathbf{x}) = \mathbf{0}$ is to convert it to the equation $g(\mathbf{x}) = \mathbf{x}$, start with some initial guess $\mathbf{x}_0$ and find subsequent approximations by the scheme $x_{n+1} = g(x_n)$ for n = 1,2, … . This is called an iterative method and each x_n is called iterate.

Definition 5.33 Let X be a nonempty set and T: X → X be a map. A point p ∈ X is called a **fixed point** of T if T(p) = p.

In general, a map may or may not have any fixed point. Even if it has a fixed point, it may or may not be unique. For example, T : **R** → **R** given by T(x) = x + 2 has no fixed point, S: **R**→ **R** given by $S(x) = x^2$ has two fixed points, namely 0 and 1, rotation of a plane has a unique fixed point, whereas projection on X – axis has infinitely many fixed points.

Definition 5.34 Let (V, || · ||) be a normed linear space and B be a nonempty subset of V. A map f : B → B is called a contraction on B if there exists a constant α such that 0 < α < 1 and $\|f(\mathbf{x}) - f(\mathbf{y})\| \le \alpha \|\mathbf{x} - \mathbf{y}\|$ for all **x** , **y** ∈ B.

Proposition 5.35 Let (V, || · ||) be a normed linear space and B be a nonempty subset of V. Let f : B → B be a contraction on B. Suppose there exists a constant α such that 0 < α < 1 and $\|f(\mathbf{x}) - f(\mathbf{y})\| \le \alpha \|\mathbf{x} - \mathbf{y}\|$ for all **x** , **y** ∈ B. Further, let $\mathbf{x}_0 \in B$ and let $\mathbf{x}_{n+1} = f(\mathbf{x}_n)$ for n=1 , … . Then

1. If f has a fixed point, it is unique.
2. $\|\mathbf{x}_{n+1} - \mathbf{x}_n\| \le \alpha^n \|\mathbf{x}_1 - \mathbf{x}_0\|$ for all n = 1, ….
3. For m > n ≥ 1, $\|\mathbf{x}_m - \mathbf{x}_n\| \le \dfrac{\alpha^n}{1-\alpha} \|\mathbf{x}_1 - \mathbf{x}_0\|$.

Proof: Suppose **x**, **y** are two fixed points of f. If **x** and **y** are distinct, then $0 < \|\mathbf{x} - \mathbf{y}\| = \|f(\mathbf{x}) - f(\mathbf{y})\| \le \alpha \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x} - \mathbf{y}\|$. Contradiction. This proves uniqueness.

Next, $\|\mathbf{x}_{n+1} - \mathbf{x}_n\| = \|f(\mathbf{x}_n) - f(\mathbf{x}_{n-1})\| \le \alpha \|\mathbf{x}_n - \mathbf{x}_{n-1}\| \le \alpha^2 \|\mathbf{x}_{n-1} - \mathbf{x}_{n-2}\| \le \cdots \le \alpha^n \|\mathbf{x}_1 - \mathbf{x}_0\|$

Now suppose m > n. Then $\|\mathbf{x}_m - \mathbf{x}_n\| \le \|\mathbf{x}_m - \mathbf{x}_{m-1}\| + \|\mathbf{x}_{m-1} - \mathbf{x}_{m-2}\| + \cdots + \|\mathbf{x}_{n+1} - \mathbf{x}_n\|$

$$\le (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n) \|\mathbf{x}_1 - \mathbf{x}_0\| \le \frac{\alpha^n}{1-\alpha} \|\mathbf{x}_1 - \mathbf{x}_0\|.$$

Note that the above proposition does not say anything about the existence of a fixed point. This is obtained by recourse to another famous theorem known as Banach's Contraction Mapping principle. We shall only discuss a very toned down version of this principle.

Definition 5.36 Let (V, || · ||) be a normed linear space and let, as usual, **N** denote the set of all natural numbers. A *sequence* in V is a function from **N** into V. We shall denote the elements of such a sequence by $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots$ and the sequence itself by $\{\mathbf{x}_n\}$. Such a sequence is said to converge to a vector **x** ∈ V if the sequence $\{\|\mathbf{x}_n - \mathbf{x}\|\}$ of real numbers converges to **0**. A sequence $\{\mathbf{x}_n\}$ is said to be bounded if there exists a positive real number M such that $\|\mathbf{x}_n\| \le M$ for all n.

Let $n_1, n_2, \ldots n_j, \ldots$ be natural numbers such that $n_1 < n_2 < \ldots < n_j < n_{j+1} < \ldots$. Then the sequence $\{x_{n1}, x_{n2}, \ldots\}$ is called a subsequence of $\{x_n\}$. A subset B of V is called *closed* if $x_n \in B$ for all n and $\{x_n\}$ converges to **x**, then **x** also belongs to B. For example, V itself is a closed set. Also every closed ball, that is, a set of the type $\{x \in: \|x - a\| \le r\}$ for some $a \in V$ and $r > 0$ is a closed set. On the other hand, the open interval (0,1) is not a closed set in **R**.

Theorem 5.37 (Bolzano-Weierstrass Theorem): Let $(\|\cdot\|)$ be a norm on R^n. Then every bounded sequence in $(R^n, \|\cdot\|)$ has a convergent subsequence.

Since a proof of this theorem involves some concepts from real analysis, we shall not prove it here. A proof can be found in any book on real analysis, for example Rudin.

Theorem 5.38 Let $(\|\cdot\|)$ be a norm on R^n, and B a closed subset of R^n. Let $f : B \to B$ be a contraction. Then f has a unique fixed point in B.

Proof We need to prove only the existence of a fixed point. The uniqueness follows from Proposition 5.35. Let $x_0 \in B$ and for n = 1,2, ... , define $x_{n+1} = f(x_n)$. Since f is a contraction, there exists a constant α such that $0 < \alpha < 1$ and $\|f(x) - f(y)\| \le \alpha \|x - y\|$ for all $x, y \in B$. Then by Proposition 5.35, for each $n \ge 1$, $\|x_n\| \le \|x_n - x_1\| + \|x_1\| \le \frac{\alpha}{1-\alpha} \|x_1 - x_0\| + \|x_1\|$. Thus $\{x_n\}$ is a bounded sequence in R^n and hence by Theorem 5.37 has a convergent subsequence, say $\{x_{nj}\}$ converging to some $x \in R^n$. Since B is closed, $x \in B$. We shall prove that $f(x) = x$. To prove this, we shall show that $\|f(x) - x\| < \varepsilon$ for every $\varepsilon > 0$. So let $\varepsilon > 0$. First since $\alpha^n \to 0$ and $\|x_{n_j} - x\| \to 0$, we can choose n, sufficiently large so that $\|x_{n_j} - x\| < \varepsilon/3$. Then $\|f(x) - f(x_{n_j})\| \le \|x - x_{n_j}\| < \varepsilon/3$. Also $\|f(x_{n_j}) - x_{n_j}\| = \|x_{n_j+1} - x_{n_j}\| < \alpha^{n_j}\|x_1 - x_0\| < \varepsilon/3$ by Proposition 5.35. Hence $\|f(x) - x\| \le \|f(x) - f(x_{n_j})\| + \|f(x_{n_j}) - x_{n_j}\| + \|x_{n_j} - x\| < \varepsilon$.

With all notations as above, the vector $x - x_n$ is called the error. The following Corollary gives an estimate for its norm.

Corollary 5.39 With all the notations as in Theorem 5.38, we have $\|x - x_n\| \le \frac{\alpha^n}{1-\alpha} \|x_1 - x_0\|$.

Proof Let $\varepsilon > 0$. There exists $m > n$ such that $\|x - x_m\| < \varepsilon$. Then $\|x - x_n\| \le \|x - x_m| + \|x_m - x_n\| \le \varepsilon + \frac{\alpha^n}{1-\alpha} \|x_1 - x_0\|$. Since this holds for every $\varepsilon > 0$, the conclusion follows.

Exercises:

1. Let $X = \{x \in \mathbf{R}: x \geq 1\}$ and let $f(x) = x / 2 + 1/ x$. Show that f is a contraction on X.

2. Show that the error bounds given in Corollary 5.39 form a monotonically decreasing sequence.

3. Prove that if g is a continuously differentiable function defined on a closed interval J and there exists α such that $|g'(x)| \leq \alpha < 1$ for all $x \in J$, then for any $x_0 \in J$ the sequence defined by $x_{n+1} = g(x_n)$ converges to a fixed point of g. (Hint: Use Mean Value Theorem).

4. In order to solve the equation $x^3 + x - 1 = 0$, it can be converted to the form $x = g(x)$ in one of the following ways and then fixed point iteration method can be applied. Which of these formulation(s) will lead to a convergent iterative scheme?

(a) $x = 1 / (1+x^2)$ (b) $x = 1 - x^3$ (c) $x = x^{1/2} (1+x^2)^{1/2}$

5. Let f be a twice continuously differentiable function defined on an interval [a, b], let $x^* \in [a, b]$ be such that $f(x^*) = 0$, $f'(x^*) \neq 0$. Show that the function g defined by $g(x) = x - f(x)/f'(x)$ defines a contraction in an interval containing x* (Hint: Observe that g is continuous and $g'(x^*) = 0$). The iterative method to solve $f(x) = 0$ using this g is called Newton Raphson Method. Give a geometric interpretation of this method.

6. Let $C = [c_{ij}]$ be a matrix of order n x n satisfying $\sum_{j=1}^{n} |c_{ij}| < 1$ for $i = 1, \ldots, n$

Show that for every $\mathbf{d} \in \mathbf{R}^n$ the system of equation $\mathbf{x} = \mathbf{Cx} + \mathbf{d}$ has a unique solution and this solution can be obtained as a limit of an iterative scheme $\mathbf{x}_{m+1} = \mathbf{Cx}_m + \mathbf{d}$ starting with any initial vector $x_0 \in \mathbf{R}$ (Hint: First show that $\|C\|_\infty < 1$. Then show that $f : \mathbf{R}^n \to \mathbf{R}^n$ defined by $f(\mathbf{x}) = \mathbf{Cx} + \mathbf{d}$ is a contraction on $\mathbf{R}^n$ with respect to $\|\cdot\|_\infty$ norm).

7. Show that if a matrix $A = [a_{ij}]$ of order n x n satisfies $|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$ for $i = 1, \ldots, n$ then for every $\mathbf{b} \in \mathbf{R}^n$, Jacobi iteration method for solving the system $\mathbf{Ax} = \mathbf{b}$ converges. In particular, such a matrix is invertible (Such a matrix is called row diagonally dominant).
(Hint: Observe that in Jacobi's method the system $\mathbf{Ax} = \mathbf{b}$ is converted to the equivalent system $\mathbf{x} = \mathbf{Cx} + \mathbf{d}$, where the entries c_{ij} of C are given by $c_{ij} = 0$ if $i = j$ and $c_{ij} = - a_{ij} / a_{ii}$ if $i \neq j$. Then use the previous exercise).

CHAPTER 6

SOLVING SYSTEMS OF LINEAR EQUATIONS

Our objective in this section is to discuss numerical aspects of solving a system of linear equations of the form

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 \ldots + a_{2n}x_n = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 \ldots + a_{nn}x_n = b_n \end{array}\right\} \begin{array}{l} \text{system of } n \text{ equations in } n \text{ unknowns} \\ x_1, x_2, \ldots, x_n \\ a_{ij},\ b_i \in R \end{array} \qquad (6.1)$$

Matrices are useful devices for representing system of equations. Equation (6.1) is written as $A\mathbf{x} = \mathbf{b}$;

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- Construct general purpose algorithms for solving this problem: $A\mathbf{x} = \mathbf{b}$
- Analyse the errors associated with the computed solution and study methods for controlling and reducing them.
- give an introduction to the iterative algorithms.

Elementary operations

(i) interchanging two equations in the system: $E_i \leftrightarrow E_j$

(ii) multiplying an equation by a nonzero number: $\lambda E_i \rightarrow E_i$

(iii) adding to an equation a multiple of some other equations $E_i + \lambda E_j \rightarrow E_i$

Theorem 6.1 If one system of equations is obtained from another by a finite sequence of elementary operations, then the two systems are equivalent.

If A^{-1} exists, then $A\mathbf{x} = \mathbf{b}$ has the solution $\mathbf{x} = A^{-1}\mathbf{b}$. If A^{-1} is already available then this is a good method for computing x. If not, then A^{-1} should not be computed solely for the purpose of obtaining x.

More efficient procedures are discussed in what follows.

Direct methods: Decomposition, Gauss Elimination method.

Iterative Methods: Gauss – Jacobi, Gauss-Seidel method.

LU decomposition Suppose A can be factored into A = LU, L is a lower triangular matrix, U is a upper triangular matrix then $A\mathbf{x} = LU\mathbf{x} = \mathbf{b}$. Set $U\mathbf{x} = \mathbf{z}$ (solve for $\mathbf{x}$ by backward substitution). Then $L\mathbf{z} = \mathbf{b}$ (solve for $\mathbf{z}$ by forward substitution).

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix}; \quad U = \begin{pmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

If A = LU, then A has LU decomposition. If L is <u>unit</u> lower triangular $l_{ii} = 1$, $1 \le i \le n$, then we have Dolittle factorization. If U is <u>unit</u> upper triangular ; $u_{ii} = 1$, $1 \le i \le n$,then we have Crouts factorization. If $U = L^T$, then $A = L\,L^T$ gives Cholesky's factorization .

A Sufficient condition for a square matrix A to have LU decomposition, is given below.

Theorem 6.2 If all n leading principal minors of the n x n matrix A are non singular, then A has an LU decomposition.

Proof Will be discussed during the lecture.

Theorem 6.3 If A is a real, symmetric, positive definite matrix, then it has a unique factorization, $A = LL^T$, in which L is a lower triangular matrix with a positive diagonal.

Proof Will be discussed during the lecture.

Example 6.4 $A = \begin{pmatrix} 60 & 30 & 20 \\ 30 & 20 & 15 \\ 20 & 15 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} \begin{pmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = LU\,(\text{Dolittle})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{pmatrix} \begin{pmatrix} 60 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = LD\hat{U} = \hat{L}\hat{U}$$

$$
= \begin{pmatrix} 60 & 0 & 0 \\ 30 & 5 & 0 \\ 20 & 5 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \hat{L}\sqrt{D}\sqrt{D}\,\hat{U} = \tilde{L}\tilde{L}^{T} \text{ (Crout)}
$$

$$
= \begin{pmatrix} \sqrt{60} & 0 & 0 \\ \frac{1}{2}\sqrt{60} & \sqrt{5} & 0 \\ \frac{1}{3}\sqrt{60} & \sqrt{5} & \frac{1}{\sqrt{3}} \end{pmatrix}\begin{pmatrix} \sqrt{60} & \frac{1}{2}\sqrt{60} & \frac{1}{3}\sqrt{60} \\ 0 & \sqrt{5} & \sqrt{5} \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ (Cholesky)}
$$

<u>Pivoting and constructing algorithm</u>

In what follows, an abstract version of Gauss Elimination Method (GEM) in the guise of LU decomposition is presented. The application of GEM to **Ax** = **b** reduces the matrix A to an upper triangular matrix and the system is then solved by back substitution

Example 6.5
$$
\left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 12 \\ 12 & -8 & 6 & 10 & 34 \\ 3 & -13 & 9 & 3 & 27 \\ -6 & 4 & 1 & -18 & -38 \end{array}\right) \simeq \left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 12 \\ 0 & -4 & 2 & 2 & 10 \\ 0 & -12 & 8 & 1 & 21 \\ 0 & 2 & 3 & -14 & -26 \end{array}\right).
$$

($R_2 - 2R_1 \rightarrow R_2$; $R_3 - \frac{1}{2} R_1 \rightarrow R_3$; $R_4 + R_1 \rightarrow R_4$); $(2, \frac{1}{2}, -1)$ are the multipliers and 6 is the pivotal element.

$$
\simeq \left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 12 \\ 0 & -4 & 2 & 2 & 10 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 4 & -13 & -21 \end{array}\right)_{\substack{R_3 - 3R_2 \rightarrow R3; \\ R_4 + \frac{1}{2}R_2 \rightarrow R4}} \simeq \left(\begin{array}{cccc|c} 6 & -2 & 2 & 4 & 12 \\ 0 & -4 & 2 & 2 & 10 \\ 0 & 0 & 2 & -5 & -9 \\ 0 & 0 & 0 & -3 & -3 \end{array}\right)_{R_4 - 2R_3 \rightarrow R4}
$$

(3, (− 1/2) are multipliers, − 4 is the pivotal element) (2 is the multiplier, 2 is the pivotal element)

The backward subsituton yields $x_1 = 1$, $x_2 = -3$, $x_3 = -2$, $x_4 = 1$.

Multipliers used are exhibited in a unit lower triangular matrix $L = \begin{pmatrix} 2 & 1 & 0 & 0 \\ \frac{1}{2} & 3 & 1 & 0 \\ -1 & -\frac{1}{2} & 2 & 1 \end{pmatrix}$ and we note that the coefficient matrix of final system is upper triangular given by

$$U = \begin{pmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \text{ Check: } A = L\,U$$

Note The entire elimination process breaks down if any of the pivot elements is zero.

Theorem 6.6 If all the pivot elements $a_{kk}^{(k)}$ are non zero in the process just described, then A = LU.

Proof Will be discussed during the lecture.

Pivoting The GEM, described, is not satisfactory since it fails on systems that are in fact easy to solve.

Example 6.7 $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ Apply GEM. The method fails because there is no way of adding a multiplier of the first equation to the second in order to get a-0-coefficient for x_1 in the second equation. The same difficulty is encountered in the following case also.

$\begin{pmatrix} \varepsilon & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. GEM yields $\begin{pmatrix} \varepsilon & 1 \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\frac{1}{\varepsilon} \end{pmatrix}$. $x_2 = \dfrac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}} \approx 1$;

$x_1 = (1 - x_2)\,\frac{1}{\varepsilon} = 0$. In the computer, if ϵ is small enough, $2 - \frac{1}{\varepsilon}$ is computed to be the same as $-\frac{1}{\varepsilon}$ and denominator $1 - \frac{1}{\varepsilon}$ is computed as $-\frac{1}{\varepsilon}$. Therefore, x_2 is computed as 1 and x_1 is computed as 0. But exact solution $x_1 = x_2 = 1$. Therefore, computed solution is exact for x_2 but is extremely inaccurate for x_1.

We will further show that it is not actually the smallness of the co-efficient a_{11} that is causing the trouble. Rather, it is the smallness of a_{11} relative to other elements in its row.

Example 6.8 Consider the equivalent system $\begin{pmatrix} 1 & \frac{1}{\varepsilon} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \\ 2 \end{pmatrix}$.GEM yields

$\begin{pmatrix} 1 & \frac{1}{\varepsilon} \\ 0 & 1-\frac{1}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\varepsilon} \\ 2-\frac{1}{\varepsilon} \end{pmatrix}$. Solution is $x_2 = \dfrac{2-\frac{1}{\varepsilon}}{1-\frac{1}{\varepsilon}} \approx$ [redacted] $x_1 = \frac{1}{\varepsilon} - \frac{1}{\varepsilon} x_2 \approx 0$. Again, for small ε, x_2 is computed as [redacted] and x_1 as 0, which is wrong. The difficulties disappear if the order of the equation is changed. Therefore, interchange of equations leads to

$$\begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ and GEM gives $$\begin{pmatrix} 1 & 1 \\ 0 & 1-\varepsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1-2\varepsilon \end{pmatrix}$$ and the solution is

$$x_2 = \frac{1-2\varepsilon}{1-\varepsilon} \simeq 1;\ x_1 = 2 - x_1 = 2 - 1 = 1.$$

Conclusion A good algorithm must incorporate the interchanging of equations in a system when circumstances require it.

Partial pivoting If only row interchanging is used to bring the element of large magnitude of the pivotal column to the pivotal position at each step of diagonalisation, then the process is partial pivoting. In this process, the matrix may have larger element in non-pivotal column, but the largest element in the pivotal column only is brought to pivotal (diagonal) position in the process by making use of row transformations.

Example 6.9

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 3x_1 + 3x_2 + 4x_3 &= 20 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned}\right\} \qquad \text{Apply Partial pivoting}$$

Largest element in the first column is in the second equation ($3x_1$) which is not the pivotal position, perform row transformation $R_1 \leftrightarrow R_2$. System is $\begin{aligned} 3x_1 + 3x_2 + 4x_3 &= 20 \\ x_1 + x_2 + x_3 &= 6 \\ 2x_1 + x_2 + 3x_3 &= 13 \end{aligned}$. Augumented matrix is

$$\left(\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \end{array}\right)_{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \simeq \left(\begin{array}{ccc|c} 3 & 3 & 4 & 20 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \\ 2 & -1 & -\frac{1}{3} & -\frac{1}{3} \end{array}\right) \simeq \left(\begin{array}{ccc|c} 3 & 3 & 20 & 20 \\ 0 & -1 & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \end{array}\right)$$

(There is a '0' at the pivotal position in the second row second column. Therefore, apply R2 ↔ R3.) (Upper Triangular)

Now using back substitution:

$$-\frac{1}{3}x_3 = -\frac{2}{3} \Rightarrow x_3 = 2; \quad -x_2 + \frac{1}{3}x_3 = -\frac{1}{3} \Rightarrow x_2 = 1; \quad 3x_1 + 3x_2 + 4x_3 = 20 \Rightarrow x_1 = 3.$$

Complete pivoting In this process, the largest element (in magnitude) of the whole coefficient matrix A is first brought to (1, 1) positions and then leaving first row, first column, the largest among the remaining elements is brought to (2, 2) position and so on by performing both row and column transformation. Since there is also a column transformation, there will be a change of position of the individual elements of the unknown vector **x**. Therefore, in the end, the elements of **x** have to be rearranged by applying inverse column transformations in the reverse order to all the column transformations performed.

Example 6.10

$$\left.\begin{aligned} 3x_1+3x_2+4x_3&=20\\ x_1+x_2+x_3&=6\\ 2x_1+x_2+3x_3&=13 \end{aligned}\right\}$$

largest element is at first row third column. Do column transformation $C_1 \leftrightarrow C_3$

$$\simeq \left(\begin{array}{ccc|c} 4 & 3 & 3 & 20\\ 1 & 1 & 1 & 6\\ 3 & 1 & 2 & 13 \end{array}\right).$$

<u>**Note**</u> the order of the individual elements of the unknown vector **x**. It is now (x_3, x_2, x_1).

do $R_2 \to R_2 - \frac{1}{4}R_1$; $R_3 \to R_3 - \frac{3}{4}R_1$. Then the system is equivalent to

$$\simeq \left(\begin{array}{ccc|c} 4 & 3 & 3 & 20\\ 0 & \frac{1}{4} & \frac{1}{4} & -2\\ 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{3}{5} \end{array}\right)$$

Now, the element with the largest magnitudes is in the third row (leaving the first row aside)

do $R_2 \leftrightarrow R_1$ $$\simeq \left(\begin{array}{ccc|c} 4 & 3 & 3 & 20\\ 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{3}{5}\\ 0 & \frac{1}{4} & \frac{1}{4} & -2 \end{array}\right)_{R_3 \to R_3 - (-\frac{1}{5}R_2)} \simeq \left(\begin{array}{ccc|c} 4 & 3 & 3 & 20\\ 0 & -\frac{5}{4} & -\frac{1}{4} & \frac{3}{5}\\ 0 & 0 & \frac{1}{5} & \frac{2}{5} \end{array}\right) \Rightarrow x_1=3, x_2=1, x_3=2$$

<u>**Tridiagonal system**</u> The matrix $A = (a_{ij})$ is tridiagonal if $a_{ij} = 0$ for $|i-j|>1$.

Solution by GEM (without pivoting) Consider the system $A\mathbf{x} = \mathbf{b}$ given by

$$\begin{aligned} b_1x_1 - c_1x_2 &= d_1\\ -a_2x_1 + b_2x_2 + c_2x_3 &= d_2\\ \ddots \qquad \vdots \qquad &\quad \vdots\\ -a_ix_{i-1} + b_ix_i + c_ix_{i+1} &= d_i\\ \ddots \qquad \vdots \qquad &\quad \vdots\\ -a_nx_{n-1} + b_nx_n &= d_n \end{aligned}$$

where a_i, b_i, c_i and d_i are known.

In GEM, the first equation is used to eliminate x_1 from the second equation, the new second equation is used to eliminate x_2 from the third equation and so on so that the unknowns $x_n, x_{n-1}, \ldots x_1$ are found in turn by back substitution.

Assume that the following stage of elimination has been reached.

$$\alpha_{i-1} x_{i-1} - c_{i-1} x_i = S_{i-1}$$
$$-a_i x_{i-1} + b_i x_i - c_i x_{i+1} = d_i$$

where $\alpha_1 = b_1$, $S_1 = d_1$. Elimination of x_{i-1} leads to

$$\left(b_i - \frac{a_i c_{i-1}}{\alpha_{i-1}}\right) x_i - c_i x_{i+1} = d_i + \frac{a_i S_{i-1}}{\alpha_{i-1}} \quad \text{i.e. } \alpha_i x_i - c_i x_{i+1} = S_i \qquad (6.2)$$

where $\alpha_i = \left(b_i - \dfrac{a_i c_{i-1}}{\alpha_{i-1}}\right)$, $S_i = d_i + \dfrac{a_i S_{i-1}}{\alpha_{i-1}}$, $i = 1, 2, 3 \ldots$

The last pair of simultaneous equations are

$$\alpha_{n-1} x_{n-1} - c_{n-1} x_n = S_{n-1}$$
$$-a_n x_{n-1} + b_n x_n = d_n$$

Elimination of x_{n-1} gives

$$\left(b_n - \frac{a_n c_{n-1}}{\alpha_{n-1}}\right) x_n = d_n + \frac{a_n S_{n-1}}{\alpha_{n-1}} \quad \text{i.e } \alpha_n x_n = S_n \qquad (6.3)$$

Equations (6.2) and (6.3) show that the solution can be calculated from

$$x_n = \frac{S_n}{\alpha_n}, \quad x_i = \frac{1}{\alpha_i}(S_i + c_i x_{i+1}), \quad i = n-1, n-2, \ldots, 1$$

where $\alpha_1 = b_1$, $\alpha_i = b_i - \dfrac{a_i c_{i-1}}{\alpha_{i-1}}$, $S_1 = d_1$, $S_i = d_i + \dfrac{a_i S_{i-1}}{\alpha_{i-1}}$, $i = 1, 2, 3 \ldots n$.

Comment on the stability of GEM

Theorem 6.11 The non-pivoting GEM for solving the set of linear equations **Ax** = **b**, with a tridiagonal matrix A, is always stable (i.e. with no growth of round-off errors) if

(i) $a_i > 0$, $b_i > 0$ and $c_i > 0$

(ii) $b_i > a_{i+1} + c_{i-1}$, $i = 1, 2, \ldots n$, defining $c_0 = a_{n+1} = 0$

(iii) $b_i > a_i + c_i$, $i = 1, 2, \ldots n$, defining $a_1 = c_n = 0$

It is to be noted that conditions (i) and (ii) which ensure that the forward elimination is stable, state that the diagonal element must exceed the sum of the moduli of the other elements in the same column of the matrix A of coefficients.

Conditions (i) and (iii), which ensure that the back substitution is stable, state that the diagonal element must exceed the sum of the moduli of the other elements in the same row.

Proof Will be discussed during the lecture.

Diagonally Dominant Matrices

Sometimes a system of equations has the property that GEM without pivoting can be safely used. One class of matrices for which this is true is the class of diagonally dominant matrices. This property is expressed by the inequality

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad 1 \le i \le n \tag{6.4}$$

If the coefficient matrix has this property, then in the first step of GEM, we can use row 1, as the pivot row. Therefore, the pivot element a_{11} is not zero by (6.4). After step 1 has been completed, we would like to know that row 2 can be used as the next pivot row. This is answered by the next Theorem.

Theorem 6.12 GEM without pivoting preserves the diagonal dominance of a matrix.

Proof Will be discussed during the lecture.

Corollary 6.13 Every diagonally dominant matrix is non singular and has an LU – factorization.

We know that if all the pivot elements $a_{kk}^{(k)}$ are non zero in the GEM, then A = LU. This result together with previous theorem implies that a diagonally dominant matrix A has a LU – decomposition in which L is unit lower triangular. The matrix U, by the preceeding theorem, is diagonally dominant. Hence, its diagonal elements are nonzero. Thus, L and U are non singular.

Norms and Analysis of Errors

Recall the definition of vector norms introduced earlier.

Vector norms On a vector space V, a norm is a function $\|\cdot\|$ from V to the set of non negative reals that obeys the three postulates:

$\|\mathbf{x}\| > 0$ if $\mathbf{x} \neq 0$, $\mathbf{x} \in V$

$\|\lambda\mathbf{x}\| = |\lambda| \, \|\mathbf{x}\|$, if $\lambda \in \mathbf{R}$, $\mathbf{x} \in V$

$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\| + \|\mathbf{y}\|$, if $\mathbf{x}, \mathbf{y} \in V$

Think of $\|\mathbf{x}\|$ as the length or magnitude of the vector $\mathbf{x}$.

The norm on R^n is the Euclidean norm defined by $\|\mathbf{x}\|_2 = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}$, where $x = (x_1, x_2, \ldots x_n)$.

Other norms are $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$, ($l_\infty$ - norm); $\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$ (l_1 – norm)

Matrix norms

We define matrix norm in such a way that it is intimately related to a vector norm. If a vector norm $\| \cdot \|$ has been specified, the matrix norm subordinate to it is defined by

$$\| A \| = \sup \{ \| A\mathbf{u} \| : \mathbf{u} \in \mathbf{R}^n, \| \mathbf{u} \| = 1 \} \tag{6.5}$$

This is also called the matrix norm associated with the given vector norm. A is an n x n matrix.

Theorem 6.14 If $\| \cdot \|$ is any norm on $\mathbf{R}^n$, then (6.5) defines a norm on the linear space of all n x n matrices. That is, $\| A \|$ satisfies $\| A \| > 0$

$$\| \lambda A \| = |\lambda| \, \| A \|$$

$$\| A + B \| \le \| A \| + \| B \|$$

In addition, $\|A\mathbf{x}\| \le \|A\| \, \|\mathbf{x}\|, \mathbf{x} \in \mathbf{R}^n$.

Remark Matrix norm subordinate to a vector norm also satisfies

$\| I \| = 1$ and $\|AB\| \le \|A\| \, \|B\|$.

In particular $\| A^2 \| \le \| A \|^2$ and by induction $\| A^n \| \le \| A \|^n$ for all n.

For the vector norm $\| \mathbf{x} \|_\infty = \max_{1 \le i \le n} |\mathbf{x}_i|$, we now compute its subordinate matrix norm.

$$\|A\|_\infty = \sup_{\|u\|_\infty = 1} \|A\mathbf{u}\|_\infty = \sup_{\|u\|_\infty=1} \left\{ \max_{1 \le i \le n} |(Au)_i| \right\} = \max_{1 \le i \le n} \left\{ \sup_{\|u\|_\infty = 1} |(Au)_i| \right\}$$

$$= \max_{1 \le i \le n} \left\{ \sup_{\|u\|_\infty = 1} \left| \sum_{j=1}^{n} a_{ij} u_j \right| \right\} = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Thus, if the vector norm $\| \cdot \|_\infty$ is defined by $\| \mathbf{x} \|_\infty = \max_{1 \le i \le n} |x_i|$, then its subordinate matrix norm is

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

Note We have used the fact that two maximization processes can be interchanged.

Also $\sup \left| \sum_{j=1}^{n} a_{ij} u_j \right|$ for fixed i and $\| \mathbf{u} \|_\infty = 1$ is obtained by putting

$u_j = +1$ if $a_{ij} \ge 0$

and $u_j = -1$, if $a_{ij} < 0$.

The condition number and perturbations

Given $A\mathbf{x} = \mathbf{b}$, we see how the solution $\mathbf{x}$ changes as the right hand side vector $\mathbf{b}$ changes (Assume $|A| \neq 0$). Look at two specific systems $\left.\begin{array}{l} A\mathbf{x}_1 = \mathbf{b}_1 \\ A\mathbf{x}_2 = \mathbf{b}_2 \end{array}\right\} \Rightarrow \mathbf{x}_1 - \mathbf{x}_2 = A^{-1}(\mathbf{b}_1 - \mathbf{b}_2)$. Therefore, relative error in $\mathbf{x}_2$ as an approximation to $\mathbf{x}_1$ is given by $\dfrac{\|\mathbf{x}_1 - \mathbf{x}_2\|}{\|\mathbf{x}_1\|} \leq \|A^{-1}\| \dfrac{\|\mathbf{b}_1 - \mathbf{b}_2\|}{\|\mathbf{x}_1\|}$.

Our interest is to bound the change in solution by something that does not depend on the solution. Thus, we want to get rid of the $\mathbf{x}_1$ in the denominator. To do this, note $\|A\| \; \|\mathbf{x}_1\| \geq \|\mathbf{b}_1\|$, so that $\dfrac{1}{\|\mathbf{x}_1\|} \leq \dfrac{\|A\|}{\|\mathbf{b}_1\|}$ and therefore

$$\boxed{\frac{\|\mathbf{x}_1 - \mathbf{x}_2\|}{\|\mathbf{x}_1\|} \leq \|A\| \|A^{-1}\| \frac{\|\mathbf{b}_1 - \mathbf{b}_2\|}{\|\mathbf{b}_1\|}}$$

The multiplying coefficient $\|A\| \|A^{-1}\|$ is interesting. It depends entirely on the matrix in the problem not on the right hand side vector, yet it shows up as an amplifier to the relative change in the right hand side vector. We call it the condition number.

Definition 6.15 For a given matrix $A \in \mathbf{R}^{n \times n}$ and a given matrix norm $\| \cdot \|$, the condition number with respect to the given norm is defined by $K(A) = \|A\| \|A^{-1}\|$. If A is singular, we take $K(A) = \infty$.

The justification for taking $K(A) = \infty$, if A is given as singular is presented below.

Theorem 6.16 Let $A \in \mathbf{R}^{n \times n}$ be given and non singular, Then , for any singular matrix $B \in \mathbf{R}^{n \times n}$, we have $\dfrac{1}{K(A)} \leq \dfrac{\|A - B\|}{\|A\|}$

Remarks The theorem tells us that if A is close to a singular matrix, then the reciprocal of the condition number will be near zero. That is K (A) itself will be 'large'.

Thus, the condition number measures how close the matrix is to being singular; if K(A) is large, then we know that A is close to being singular,

Solving systems that are nearly singular can produce large errors.

Example 6.17

Let $\varepsilon > 0$, $A = \begin{pmatrix} 1 & 1+\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}$, $\|A\|_\infty = 2+\varepsilon$

$$A^{-1} = \varepsilon^{-2}\begin{pmatrix} 1 & -1-\varepsilon \\ -1+\varepsilon & 1 \end{pmatrix}; \quad \|A^{-1}\|_\infty = \varepsilon^{-2}(2+\varepsilon)$$

$K(A) = \left(\frac{2+\varepsilon}{\varepsilon}\right)^2 > \frac{4}{\varepsilon^2}$. If $\varepsilon \leq 0.01$, then $K(A) \geq 40{,}000$. In such a case, a small relative perturbation in **b** may induce a relative perturbation 40, 000 times greater in the solution of the system $A\mathbf{x} = \mathbf{b}$.

Theorem 6.18 (Effects of perturbation in b) Let $A \in R^{n \times n}$ (non singular) and $\mathbf{b} \in R^n$ be given and define $\mathbf{x} \in R^n$ as the solution of the linear system $A\mathbf{x} = \mathbf{b}$. Let $\delta\,\mathbf{b} \in R^n$ be a small perturbation of **b** and define $\mathbf{x} + \delta\,\mathbf{x} \in R^n$ as the solution of the system $A(\mathbf{x} + \delta\,\mathbf{x}) = \mathbf{b} + \delta\,\mathbf{b}$. Then

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq K(A)\frac{\|\delta\,\mathbf{b}\|}{\|\mathbf{b}\|}.$$

Proof Will be discussed during the lecture.

A similar result involves the residual of a solution. Let $\mathbf{x}_c$ be a 'computed' solution to $A\mathbf{x} = \mathbf{b}$. Then, the residual is the vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$. That is , r is the amount by which $\mathbf{x}_c$ fails to solve the system. If $\mathbf{r} = \mathbf{0}$, then $\mathbf{x}_c$ is exact. Therefore, one might then think that if r is small, then $\mathbf{x}_c$ is close to the exact solution; this does not happen always.

Example 6.19 Consider the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 3.02 & -1.05 & 2.53 \\ 4.33 & 0.56 & -1.78 \\ -0.83 & -0.54 & 1.47 \end{pmatrix}; \quad \mathbf{b} = \begin{pmatrix} -1.61 \\ 7.23 \\ -3.38 \end{pmatrix}$$

Solve by GEM, using pivoting and carrying three digits rounded, we get the system as

$$\left(\begin{array}{ccc|c} 4.33 & 0.56 & -1.78 & 7.23 \\ 0 & -1.44 & 3.77 & -6.65 \\ 0 & 0 & -0.00362 & 0.00962 \end{array}\right)$$

and the solution is $\mathbf{x}_c = (0.880, -2.34, -2.66)$. Compute $A\mathbf{x}_c = (-1.6047, 7.2348, -3.3716)$, which is close to '**b**'. The exact solution is $(1, 2, -1)$.

The following theorem gives the effects of residual on accuracy.

Theorem 6.20 Let $A \in R^{n \times n}$ (non singular) and $\mathbf{b} \in R^n$ be given and define $\mathbf{x}_c \in \mathbf{R}^n$ as a computed solution and $\mathbf{x}$ is the exact solution of the linear system $A\,\mathbf{x} = \mathbf{b}$. Let $\mathbf{r} \in \mathbf{R}^n$ be the residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$. Then $\frac{\|\mathbf{x}-\mathbf{x}_c\|}{\|\mathbf{x}\|} \le K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$.

Proof Will be discussed during the lecture.

Remark The above result shows that we can not trust the solution to a problem involving an ill–conditioned matrix unless we take special care in the solution.

Relation between residual error $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$ and the error $\mathbf{e} = \mathbf{x} - \mathbf{x}_c$, where $A\mathbf{x} = \mathbf{b}$

Consider $A\mathbf{e} = A\,(\mathbf{x} - \mathbf{x}_c) = A\mathbf{x} - A\mathbf{x}_c = \mathbf{b} - A\mathbf{x}_c = \mathbf{r}$. Therefore, $A\mathbf{e} = \mathbf{r}$. It can be shown that

$$\frac{1}{K(A)}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \le \frac{\|\mathbf{e}\|}{\|\mathbf{x}\|} \le K(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \qquad (6.6)$$

(6.6) shows that the relative error in the computed solution vector $\mathbf{x}_c$ can be as great as the relative residual multiplied by the condition number. It can also be as small as the relative residual divided by K(A). Therefore, when the condition number is large, the residual gives little information about the accuracy of $\mathbf{x}_c$. Conversely, when the condition number is near unity, the relative residual is a good measure of the relative error of $\mathbf{x}_c$.

Iterative improvement to correct $\mathbf{x}_c$

Define $\mathbf{e} = \mathbf{x} - \mathbf{x}_c^{(0)}$, $\mathbf{r} = \mathbf{b} - A\mathbf{x}_c$. Then $A\mathbf{e} = \mathbf{r}$, solve for $\mathbf{e}$. Apply this as a correction to $\mathbf{x}_c^{(0)}$. Thus, the corrected $\mathbf{x}_c^{(1)}$ is $\mathbf{x}_c^{(1)} = \mathbf{x}_c^{(0)} + \mathbf{e}$. If $\frac{\|\mathbf{e}\|}{\|\mathbf{x}_c^{(1)}\|}$ is small, then it means that $\mathbf{x}_c^{(1)}$ is close to $\mathbf{x}$. This is so if the computation of $\mathbf{r}$ is as precise as possible.

Example 6.21 $A = \begin{pmatrix} 4.23 & -1.06 & 2.11 \\ -2.53 & 6.77 & 0.98 \\ 1.85 & -2.11 & -2.32 \end{pmatrix}$; $\mathbf{b} = \begin{pmatrix} 5.28 \\ 5.22 \\ -2.58 \end{pmatrix}$.

Exact solution:

$\mathbf{x} = (1, 1, 1)$, $\mathbf{x}_c^{(0)} = (0.991,\ 0.997,\ 1.000)$

$A\,\mathbf{x}_c = (5.24511, 5.22246, -2.59032)$; $\quad \mathbf{r} = (0.0349, -0.00246, 0.0103)$

Solve $A\mathbf{e} = \mathbf{r}$; $\mathbf{e} = (0.00822, 0.00300, -0.00000757)$

Therefore, $\mathbf{x}_c^{(1)} = \mathbf{x}_c^{(0)} + \mathbf{e} = (0.999, 1.000, 1.000)$

In general, repeat iterations until corrections are negligible.

In what follows, we will be discussing the iterative methods for the solution of a system of linear algebraic equations. In order to determine whether an iterative method converges or diverges, we need to find out if the error increases or decreases as iterations are continued. If we are solving for one variable, then the error is a simple number, whose absolute value will tell us whether the error grows or decays. However, when N variables are solved simultaneously, then the error is a vector quantity and we need to come up with a way of determining the magnitude of this vector quantity.

The need for a matrix norm arises when one tries to answer the following:

1. Under what conditions can we solve the system $A\mathbf{x} = \mathbf{b}$?
2. How stable is the solution?

i.e. if the matrix A or right hand side vector **b** is perturbed, then how different is the new solution from the previously obtained solution to the unperturbed system?

The concept of a norm for a matrix allows us to define a quantity known as the condition number of the matrix.

Systematic iterative methods for solving large linear systems of algebraic equations

Consider a system of N equations in N unknowns

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1N} x_N = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2N} x_N = b_2$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$a_{N1} x_1 + a_{N2} x_2 + \dots + a_{NN} x_N = b_N, \qquad a_{ii} \neq 0 \text{ , } i = 1 \text{ (1) } N.$$

Rewrite this as

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12} x_2 \cdots - a_{1N} x_N]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21} x_1 \cdots - a_{2N} x_N]$$

$$\vdots$$

$$x_N = \frac{1}{a_{NN}}[b_N - a_{N1} x_N \cdots - a_{NN-1} \ x_{N-1}]$$

Gauss – Jacobi Method: (GJM)

$$x_i^{(n+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(n)} - \sum_{j=i+1}^{N} a_{ij}\ x_j^{(n)}\right],\ i = 1\,(1)\,N.$$

Gauss – Seidel Method: (GSM)

$$x_i^{(n+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(n+1)} - \sum_{j=i+1}^{N} a_{ij}\ x_j^{(n)}\right],\ i = 1\,(1)\,N.$$

Successive Over Relaxation – method: (SOR)

$$x_i^{(n+1)} = x_i^{(n)} + \frac{w}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}\, x_j^{(n+1)} - \sum_{j=i}^{N} a_{ij}\ x_j^{(n)}\right], i = 1\,(1)\,N,\ 1 < w < 2.\ w = 1 \text{ gives GSM.}$$

Given

$$A\mathbf{x} = \mathbf{b} \tag{6.7}$$

express A = D – L – U, D – diagonal matrix, L – strictly lower triangular, U – strictly upper triangular, for a 4 x 4 system with

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \qquad D = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix},$$

$$-L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix}, \qquad -U = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore. (6.7) becomes $(D - L - U)\,\mathbf{x} = \mathbf{b}$

$$D\,\mathbf{x} = (L + U)\,\mathbf{x} + \mathbf{b}$$

<u>GJM</u>: $D\,\mathbf{x}^{(n+1)} = (L + U)\,\mathbf{x}^{(n)} + \mathbf{b}$ (or) $\mathbf{x}^{(n+1)} = D^{-1}(L + U)\,\mathbf{x}^{(n)} + D^{-1}\,\mathbf{b}$

$D^{-1}(L + U)$ is a point Jacobi iteration matrix.

<u>GSM</u>: $D\,\mathbf{x}^{(n+1)} = L\,\mathbf{x}^{(n+1)} + U\,\mathbf{x}^{(n)} + \mathbf{b}$ (or)

$(D - L)\,\mathbf{x}^{(n-1)} = U\,\mathbf{x}^{(n)} + \mathbf{b}$ (or) $\mathbf{x}^{(n+1)} = (D - L)^{-1}\,U\,\mathbf{x}^{(n)} + (D - L)^{-1}\,\mathbf{b}$

$(D - L)^{-1}\,U$ is the point Gauss–Seidel iteration matrix.

The correction vector $\mathbf{d}^{(n)} = \mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}$ of the SOR method is defined by w times the displacement vector given by GSM. Therefore,

$D\,(\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)}) = D\,\mathbf{d}_1^{(n)} = L\,\mathbf{x}^{(n+1)} + U\,\mathbf{x}^{(n)} + \mathbf{b} - D\,\mathbf{x}^{(n)}$

Therefore the SOR iteration, defined by $\mathbf{d}^{(n)} = w\,\mathbf{d}_1^{(n)}$ can be written as

$$\mathbf{x}^{(n+1)} - \mathbf{x}^{(n)} = w\,D^{-1}\,[L\,\mathbf{x}^{(n+1)} + U\,\mathbf{x}^{(n)} + \mathbf{b} - D\,\mathbf{x}^{(n)}]$$

$$\therefore (I - w\,D^{-1}L)\,\mathbf{x}^{(x+1)} = \{(1-w)\,I + w\,D^{-1}U\}\,\mathbf{x}^{(n)} + w\,D^{-1}\,\mathbf{b}$$

$$\therefore \mathbf{x}^{(x+1)} = (I - w\,D^{-1}L)^{-1}\,\{(1-w)\,I + w\,D^{-1}U\}\,\mathbf{x}^{(n)} + (I - w\,D^{-1}L)^{-1}\,w\,D^{-1}\,\mathbf{b}.$$

Therefore, point SOR iteration matrix is $(I - w\,D^{-1}L)^{-1}\,\{(1-w)\,I + w\,D^{-1}U\}$.

Remark 'point' refers to the fact that algebraic equations approximate a differential equation at a number of mesh points and the iterative procedure expresses the next iterative value at only one mesh point in terms of known iterative values at other mesh points.

A necessary and sufficient condition for the convergence of iterative methods

Each of the three iterative methods (GJM, GSM, SOR) can be written as

$$\mathbf{x}^{(n+1)} = Q\,\mathbf{x}^{(n)} + C \qquad (6.8)$$

where Q is the iteration matrix and C is a column vector of known values. (6.8) has been derived after rearranging the system in the form

$$\mathbf{x} = Q\,\mathbf{x} + C \qquad (6.9)$$

The error $\mathbf{e}^{(n)}$ in the n^{th} approximation to the exact solution is defined by $\mathbf{e}^{(n)} = \mathbf{x} - \mathbf{x}^{(n)}$. Then

$$\mathbf{x}^{(n+1)} - \mathbf{x} = Q\,\mathbf{x}^{(n)} - Q\,\mathbf{x} \Rightarrow \mathbf{e}^{(n+1)} = Q\,\mathbf{e}^{(n)}$$

$$\therefore \mathbf{e}^{(n)} = Q\,\mathbf{e}^{(n-1)} = Q^2\,\mathbf{e}^{(n-2)} = \ldots = Q^n\,\mathbf{e}^{(0)}$$

The sequence of iterative values $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots \mathbf{x}^{(n)}$ will converge to $\mathbf{x}$ as $n \to \infty$ if $\lim_{n\to\infty} \mathbf{e}^{(n)} = 0$.

Since $\mathbf{x}^{(0)}$ and therefore $\mathbf{e}^{(0)}$ is arbitrary, the iteration will converge iff $\lim_{n\to\infty} Q^{n} = 0$

Assume that the matrix Q of order N has N linearly independent eigenvectors $\mathbf{v}_s$ with the corresponding eigenvalue λ_s, $s = 1,2, \ldots N$. Then, these eigenvectors can be used as a basis for N-dimensional vector space and the arbitrary error vector $\mathbf{e}^{(0)}$, with its N components, can be expressed uniquely as a linear combination of them, namely, $\mathbf{e}^{(0)} = \sum_{i=1}^{N} C_i\,\mathbf{v}_i, C_i$ – scalars.

Therefore, $\mathbf{e}^{(1)} = Q\,\mathbf{e}^{(0)} = \sum_{s=1}^{N} C_s Q\,v_s = \sum_{s=1}^{N} C_s\,\lambda_s\,v_s$ because $Q\,v_s = \lambda_s\,v_s$ by definition of an eigenvalue, λ_s is the eigenvalue corresponding to v_s. Similarly $\mathbf{e}^{(n)} = \sum_{s=1}^{N} C_s\,\lambda_s^n\,v_s = Q^n\,\mathbf{e}_0$.

Therefore, $\mathbf{e}^{(n)}$ will tend to the null vector as $n \to \infty$, for arbitrary $\mathbf{e}^{(0)}$, if $|\lambda_s| < 1$ for all s. That is, the iteration will converge for arbitrary $\mathbf{x}^{(0)}$ if the spectral radius $\rho(Q)$ of Q is less than one.

Remark Though this proof works only for diagonalizable M, it is possible to give a proof for an arbitrary matrix.

$\rho(Q) = \max_i |\lambda_i| < 1$, ρ is called the **spectral radius** of the matrix Q. Thus Jacobi iterates converge if $\rho(D^{-1}(L + U)) < 1$ and Gauss-Seidel iterates converges if $\rho((D - L)^{-1} U) < 1$.

Theorem 6.22 A sufficient condition for convergence is that $\|Q\| < 1$.

Proof Will be discussed during the lecture.

CHAPTER 7

EIGENVALUES, EIGENVECTORS, DIAGONALIZATION

7.1 Definitions and Examples

Definition 7.1 Let $T: V \to V$ be a linear transformation. In a great variety of applications, it is useful to find a vector $\mathbf{v} \in V$ such that $T\mathbf{v}$ and $\mathbf{v}$ are parallel. i.e. we seek a vector $\mathbf{v}$ and a scalar λ such that

$$T\mathbf{v} = \lambda\,\mathbf{v} \tag{7.1}$$

If $\mathbf{v} \neq 0$ and λ satisfy (7.1), then <u>λ is called</u> <u>an eigenvalue of T</u> and $\mathbf{v}$ is called <u>an eigenvector of T corresponding to the eigenvalue λ</u>.

If V is finite–dimensional, then T can be represented by a matrix A_T. Therefore, we discuss the eigenvalues and eigenvectors of n x n matrices.

Definition 7.2 Let A be n x n matrix with real or complex entries. The number λ (real or complex) is called an eigenvalue of A, if there is a non zero vector $\mathbf{v} \in \mathbb{C}^n$ such that $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq 0$ is called an eigenvector corresponding to λ.

Theorem 7.3 Let A be n x n matrix. Then λ is an eigenvalue of A iff $\rho(\lambda) = \det(A - \lambda I) = 0$.

Theorem 7.4 Let λ be an eigenvalue of n x n matrix A. Let $E_\lambda = \{\mathbf{v} \mid A\,\mathbf{v} = \lambda\,\mathbf{v}\}$. E_λ is a subspace of $\mathbb{C}^n$.

Proof Let $\mathbf{v} \in \mathbb{C}^n$ then $\mathbf{v} \in E_\lambda \Leftrightarrow A\,\mathbf{v} = \lambda\,\mathbf{v} \Leftrightarrow (A - \lambda I)\,\mathbf{v} = 0 \Leftrightarrow \mathbf{v} \in \operatorname{Ker}\{(A - \lambda I)\}$. Therefore, E_λ is the Kernel of the matrix $(A - \lambda I)$.

Note If A is real matrix, we know Ker(A) is a subspace of $\mathbf{R}^n$. Therefore, extending that result shows that E_λ is a subspace of $\mathbb{C}^n$.

Theorem 7.5 Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof Will be discussed during the lecture.

<u>**Remark**</u> It is significant that one can define eigenvalues and eigenvectors for a linear transformation $T: V \to V$ without any reference to a matrix representation and without even assuming that V is finite–dimensional.

Example 7.6 Not every linear transformation has eigenvectors. Rotation of the plane anticlockwise through a positive angle θ is a linear transformation. If $0 < \theta < 180^0$, then no vector is mapped onto one parallel to it – that is, no vector is an eigenvector.

If $\theta = 180^0$, then every non zero vector is an eigenvector and they all have the same associated eigenvalue $\lambda_1 = -1$.

Example 7.7 The linear transformation, T: $\mathbf{R}^2 \rightarrow \mathbf{R}^2$ that reflects vectors in the line $x + 2y = 0$ maps (2, –1) onto itself and (1, 2) onto (–1, –2). Therefore, $T\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

This shows that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are eigenvectors of T with corresponding eigenvalues 1 and –1 respectively.

<u>Remark</u> For a linear transformation T: $\mathbf{R}^n \rightarrow \mathbf{R}^n$, one can find the eigenvalues and eigenvectors of T by finding those of its standard matrix representation.

Example 7.8 T: $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ is defined by $T(x_1\ x_2, x_3) = (x_1, -8x_1 + 4x_2 - 6x_3, 8x_1 + x_2 + 9x_3)$.

The matrix of the linear transformation with respect to the standard bases is $A = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$, whose characteristic polynomial is $|A - \lambda I| = 0$ given by $(1 - \lambda)(\lambda - 6)(\lambda - 7) = 0$. The eigenvalues are 1, 6 and 7 and the corresponding eigenvectors are $\begin{pmatrix} -15/16 \\ -1/2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ respectively.

Example 7.9 Let D_∞ be the vector space of all functioning mapping **R** into **R** and having derivatives of all orders. Let T: $D_\infty \rightarrow D_\infty$ be defined by $T(f) = f'$. The eigenvalues and eigenvectors of T can be described as follows.

We need to find scalars λ and non zero functions f such that $T f = \lambda f$, that is $\lambda f = f'$.

<u>Consider two cases</u>: if $\lambda = 0$, and $\lambda \neq 0$.

If $\lambda = 0$, then we need to solve $f' = 0$. The only solutions of $f' = 0$ are the constant functions. Thus, in this case, non zero constant functions are eigenvectors corresponding to the eigenvalue 0.

If $\lambda \neq 0$, then we need to solve $f' = \lambda f$ which gives $f = k e^{\lambda x}$ as an eigenvector for every non zero scalar k. The eigenvectors are of the form $f = k e^{\lambda x}$.

Remark Computing the eigenvalues of a matrix is one of the toughest jobs in Linear Algebra. Many algorithms have been developed, but no single method is considered the best for all cases. We now discuss the algebraic eigenvalue problem of locating and computing eigenvalues in special cases.

The following theorems show how to locate, crudely, the eigenvalues of any matrix with practically no computation.

Theorem 7.10 (Gerschgorin) Modulus of every eigenvalue of a square matrix A is less than or equal to the largest sum of the moduli of the elements along any row or any column.

Proof Will be discussed during the lecture.

Theorem 7.11 (Brauer) Let P_k be the sum of the moduli of the elements along the k^{th} row excluding the diagonal element a_{kk}. Then, every eigen value of A lies inside or on the boundary of at least one of the circles $|\lambda - a_{kk}| = P_k$, $k = 1,2, \dots n$.

Proof We have $\lambda_i - a_{kk} = a_{k1}\left(\frac{x_{i,1}}{x_{i,k}}\right) + a_{k2}\left(\frac{x_{i,2}}{x_{i,k}}\right) + \dots + a_{kn}\left(\frac{x_{i,n}}{x_{i,k}}\right)$. Therefore, $|\lambda_i - a_{kk}| = \sum_{\substack{i=1 \\ i\neq k}}^{n} |a_{ki}| = P_k$.

Thus, all the eigenvalues of A lie inside or on the union of the above circles. Since A and A^T have the same eigenvalues, we find that all the eigenvalues lie in the union of the n circles

$$|\lambda_i - a_{kk}| \le \sum_{\substack{j=1 \\ j\neq k}}^{n} |a_{jk}|, k = 1, \dots, n$$

- The bounds obtained here are all independent. Hence, all the eigenvalues of A must lie in the intersection of these bounds.
- These circles are called the Gerschgorin circles and the bounds are the Gerschgorin bounds.
- If A is a real symmetric matrix, then we obtain an interval which contains all the eigenvalues of A.
- The eigen values of the matrix A are given by the diagonal elements when it has one of the following three forms: A = D or A = L or A = U. Therefore, methods of finding the eigenvalues of A will generally be based on reducing A to either D or L or U or LU.

In addition, iterative methods are developed to compute largest or smallest eigenvalue in magnitude.

Example 7.12 Estimate the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ using Gerschgorin bounds.

(i) the eigenvalues lie in the regions $|\lambda| \leq 5$, $|\lambda| \leq 6$.

(ii) the union of the circles $|\lambda - 1| \leq 3$, $|\lambda - 1| \leq 2$, $|\lambda + 1| \leq 4$ and

(iii) the union of the circles $|\lambda - 1| \leq 2$, $|\lambda - 1| \leq 5$, $|\lambda + 1| \leq 2$.

The first union gives $|\lambda - 1| \leq 3$, $|\lambda + 1| \leq 4$ and the second union gives $|\lambda - 1| \leq 5$. The intersection of the circles give the required region.

7.2 Power Method

The simplest methods for approximating eigenvalues are based on the observations that the eigenvectors represent the direction in which the matrix operates and the eigenvalues represent the gain along those directions. Thus, the quantity $A^N\mathbf{x}$ should eventually begin to line up in the direction of the eigenvector associated with the largest (in absolute value) eigenvalue. While this is not an efficient approach for finding all the eigenvalues and eigenvectors of a matrix, it is useful for finding some of them and much of the theory of more general methods is based on the essential ideas of the power methods.

Theorem 7.13 Let $A \in \mathbb{R}^{n \times n}$ be given, and assume that

1. A has n linearly independent eigenvectors, $\mathbf{x}_k$, $1 \leq k \leq n$
2. The eigenvalues λ_k satisfy $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$
3. The vector $\mathbf{z} \in \mathbb{R}^n$ is such that $\mathbf{z} = \sum_{k=1}^{n} \xi_k \mathbf{x}_k$ and $\xi_1 \neq 0$

Then $\lim_{N\to\infty} \dfrac{A^N\mathbf{z}}{\lambda_1^N} = C\mathbf{x}_1$ for some $C \neq 0$ and $\lim_{N\to\infty} \dfrac{(\mathbf{z},\ A^N\mathbf{z})}{(\mathbf{z},\ A^{N-1}\mathbf{z})} = \lambda_1$.

Proof Will be discussed during the lecture.

<u>**Remark**</u> The result of this theorem can be used to construct the most <u>dominant eigenvalue</u>, that is, the largest in absolute value and the corresponding eigenvector.

Example 7.14 Let $A = \begin{pmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{pmatrix}$. Let the initial vector be $\mathbf{x} = (-1, 1, 1)$

$x^{(1)} = Ax^{(0)} = 2(-1.00000, 0.333333, 0.333333)$

$x^{(2)} = Ax^{(1)} = -2(-1.00000, -0.111111, -0.111111)$

$x^{(3)} = Ax^{(2)} = 22(-1.00000, -0.407407, -0.407407)$

$x^{(4)} = Ax^{(3)} = 8.90909(-1.00000, -0.604938, -0.604938)$

$\vdots \qquad \vdots$

$x^{(28)} = Ax^{(27)} = 6.00007(-1.00000, -0.999977, -0.999977)$

The dominant eigenvalue is 6 and the corresponding eigenvector is (1, 1, 1).

Theorem 7.15 If λ is an eigenvalue of A and if A is non-singular,then λ^{-1} is an eigenvalue of A^{-1}.

Proof Let $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq 0$. Then $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Hence $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$ and λ^{-1} is an eigenvalue of A^{-1}.

Inverse Power Method

The preceding theorem suggests a means of computing the smallest eigenvalue of A. Suppose the eigenvalues of A can be arranged as follows: $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_{n-1}| > |\lambda_n| > 0$. This implies that A is nonsingular, since 0 is not an eigenvalue. The eigenvalues of A^{-1} are the numbers λ_i^{-1}, and they are arranged like this: $|\lambda_n^{-1}| > |\lambda_{n-1}^{-1}| \geq \ldots \geq |\lambda_1^{-1}| > 0$. Consequently, we can compute λ_n^{-1} by applying the power method to A^{-1}.

It is not a good idea to compute the inverse, A^{-1}, first and then use $A^{-1}\mathbf{x}^{(k)} = \mathbf{x}^{(k+1)}$. Rather, we obtain $\mathbf{x}^{(k-1)}$ by solving the equation $A\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)}$. This is done efficiently by using GEM. This method is the inverse power method.

Example 7.16 Let $A = \begin{pmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{pmatrix}$

Its LU factorization is $\begin{pmatrix} 6 & 5 & -5 \\ 2 & 6 & -2 \\ 2 & 5 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{10}{13} & 1 \end{pmatrix} \begin{pmatrix} 6 & 5 & -5 \\ 0 & \frac{13}{3} & -\frac{1}{3} \\ 0 & 0 & \frac{12}{13} \end{pmatrix}$.

We begin with the vector $\mathbf{x} = (3, 7, -13)^T$. $A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \Rightarrow U\mathbf{x}^{(k+1)} = L^{-1}\mathbf{x}^{(k)}$. In each step, we obtain $\mathbf{x}^{(k+1)}$ by solving $U\mathbf{x}^{(k+1)} = L^{-1}\mathbf{x}^{(k)}$. Then a ratio is computed and printed: namely. $r_k = \dfrac{\mathbf{x}_1^{(k-1)}}{\mathbf{x}_1^{(k)}}$. Before proceeding to the next step, $\mathbf{x}^{(k+1)}$ is normalized: that is divided by its l_∞ - norm. Then, one obtains

$\mathbf{x}^{(0)}$ = (3.000 , 7.000 , –13.000)

$\mathbf{x}^{(1)}$ = (–0.801653, –0.008264 , –1.00000), r_0 = –5.8889

$\mathbf{x}^{(2)}$ = (–0.950887, –0.17735, –1.00000), r_1= 1.19759

$\mathbf{x}^{(3)}$ = (–0.987589, –0.007125, –1.00000), r_2= –1.02750

$\vdots$

$\mathbf{x}^{(11)}$ = (–1.00000, 0.00000 , –1.00000), r_{10} = 1.00000

Remark It is the inverse power method that opens up the possibility of efficient computation of more eigenvalues, because by introducing shifts, we can converge to almost any eigenvalue we want. The method will be discussed during the lecture.

Note The power methods are very useful and efficient for finding single eigenvalues and eigenvectors; for finding many – or all – of the eigenvalues and eigenvectors of a matrix, power methods are too costly. The standard algorithm for computing all the eigenvalues and eigenvectors of a general matrix has been the QR iteration and this method will be discussed in detail during the lecture.

7.3 Diagonalization

Let V be a finite dimensional vector space and T: V → V be a linear transformation. Let A = $\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ be a basis of V and suppose $[\alpha_{ij}]=[\mathrm{T}]_{\mathrm{A}}$, the matrix of T with respect to the basis A. Recall that this means that $\mathrm{T}(\mathbf{a}_j)=\sum_{i=1}^{n}\alpha_{ij}\,\mathbf{a}_i$, $j=1,\ldots,n$.

Recall that the operator T is called diagonalizable if there is a basis B of V such that $[T]_{\mathrm{B}}$ is a diagonal matrix. In this section, we shall discuss the condition under which T is diagonalizable. We say that a square matrix M = $[\alpha_{ij}]$ of order n x n is diagonalizable if the corresponding operator T given by $\mathrm{T}(\mathbf{a}_j)=\sum_{i=1}^{n}\alpha_{ij}\,\mathbf{a}_i$, $j=1,\ldots,n$ is diagonalizable.

Suppose T is diagonalizable, this means there is a basis B such that the matrix N = $[\beta_{ij}]$ = $[\mathrm{T}]_{\mathrm{B}}$ is a diagonal matrix. We have already seen that in such a situation, these exists an invertible matrix P such that N = P^{-1} M P. M is said to be similar to N. Thus a matrix is diagonalizable iff it is similar to a diagonal matrix. Since N = $[\beta_{ij}]=[\mathrm{T}]_{\mathrm{B}}$, we have $\mathrm{T}(\mathbf{b}_j)=\sum_{i=1}^{n}\beta_{ij}\,\mathbf{b}_i$ .....$j=1,\ldots,n$. If N is a diagonal matrix, then $\beta_{ij} = 0$ for $i \neq j$. Then the above becomes $T(\mathbf{b}_i) = \beta_{ii}\,\mathbf{b}_i, i = 1.....n$. In other words, $T(\mathbf{b}_i)$ is a scalar multiple of $\mathbf{b}_j$. This naturally leads to the following Definition.

Definition 7.17 A nonzero vector $\mathbf{x} \in V$ is called an <u>eigenvector</u> of T if $\exists$ a scalar λ such that $T(\mathbf{x}) = \lambda\mathbf{x}$.

λ is called an <u>eigenvalue</u> of T <u>corresponding to the eigenvector **x**</u>. Also **x** is called an <u>eigenvector</u> of T <u>corresponding to the eigenvalue λ</u>.

In this terminology, saying that an operator T is diagonalizable is equivalent to saying that V has a basis consisting of eigenvectors of T. In particular, if Dim(V) = n, then T has n linearly independent eigenvectors. This fact deserves to be stated prominently.

$T : V \rightarrow V$ is diagonalizable $\Leftrightarrow$ V has a basis consisting of eigenvectors of T.

Next, we discuss how to determine whether a given operator is diagonalizable. For this purpose, it is convenient to assume that V is a complex vector space of dimension n and T is a linear operator in V. Suppose **x** is an eigenvector of T and λ is the corresponding eigenvalue. Then $T(\mathbf{x}) = \lambda\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$. Thus, $(T - \lambda I)(\mathbf{x}) = \mathbf{0}$, with $\mathbf{x} \neq \mathbf{0}$. Hence $\mathbf{0} \neq \mathbf{x} \in \mathcal{N}(T - \lambda I) = (\mathrm{Ker}(T - \lambda I))$, where $\mathcal{N}(T - \lambda I)$ denotes the Null space of $(T - \lambda I)$. Hence $T - \lambda I$ is not invertible. If A is any basis of V, then the matrix $[T - \lambda I]_A$ is not inversible. Hence $\det([T - \lambda I]_A) = 0$. $\det([T - \lambda I]_A)$ is a polynomial in λ of degree n. This is known as the <u>Characteristic polynomial of T</u>.

Note that, this is independent of the matrix representation of $T - \lambda I$, that is, independent of the choice of basis A. This can be seen as follows: If B is any other basis, then there exists an invertible matrix P such that $[T - \lambda I]_B = P^{-1}[T - \lambda I]_A\, P$. Hence $\det([T - \lambda I]_B) = \det(P^{-1})\det([T - \lambda I]_A)\det(P) = \det([T - \lambda I]_A)$, because $\det(P^{-1})\det(P) = \det(I) = 1$.

The equation $\det([T - \lambda I]_A) = 0$ is called the <u>characteristic equation</u> of T.

Eigenvalues of T are the roots of the characteristic equation, that is, the zeros of the characteristic polynomial. Thus, in principle, all the eigenvalues of T can be found by determining all the roots of the characteristic equation. In practice, this is an impossible task as there are no methods of finding all the roots exactly of a polynomial equation, unless the degree of a polynomial is small (≤ 4). Hence we need to use numerical/approximate methods of finding the eigenvalues.

Next suppose one such eigenvalue λ is found by some method. To find corresponding eigenvectors, we need to solve the equation $T\mathbf{x}=\lambda\mathbf{x}$, $\mathbf{x}\neq\mathbf{0}$. This is a relatively easier task. We need to solve a homogeneous system of equations. $[T\mathbf{x}-\lambda\mathbf{x}]_A=\mathbf{0}$, that is, $[T-\lambda I]_A[\mathbf{x}]_A=\mathbf{0}$.

This can be done by usual methods of solving systems of equations. As noted above, $\mathbf{x}$ is an eigenvector corresponding to the eigenvalue λ iff $\mathbf{x}\in N(T-\lambda I)$. The null space of $T-\lambda I$, $N(T-\lambda I)$, which is a subspace of V is called an eigenspace of λ. Every non-zero vector in this eigenspace is an eigenvector of T corresponding to the eigenvalue λ. Dimension of this space is called the **Geometric multiplicity of the eigenvalue** λ. If d is the geometric multiplicity of λ, then the eigenspace of V has a basis consisting of d vectors. Now let $\lambda_1,\ldots.\lambda_k$ be distinct eigenvalues of T with the geometric multiplicities $d_1,\ldots,d_k$ respectively. No nonzero vector can belong to eigenspaces of different eigenvalues. Using this, it is easy to show that the set of $d_1+\ldots+d_k$ vectors, obtained by taking the union of basis of all the eigenspaces is linearly independent. If $d_1+d_2+\ldots+d_k=n$, then this set is a basis. Thus we have proved the following:

Theorem 7.18 Let V be a complex vector space of dimension n. Then $T: V\to V$ is diagonalizable iff the sum of geometric multiplicities of all distinct eigenvalues of T equals the dimension n of V.

Thus to decide whether T is diagonalizable, we must

1. find all the distinct eigenvalues $\lambda_1\ldots,\lambda_k$ of T.
2. find the corresponding geometric multiplicities $d_1,\ldots,d_k$.
3. check whether $d_1+\ldots+d_k=n$.

Example 7.19 Let $T: \mathbb{C}^2\to\mathbb{C}^2$ be defined by $T((z_1,z_2))=(z_1+z_2,z_2)$. The matrix of T with respect to the standard ordered basis is $\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$. Characteristic equation is $(\lambda-1)^2=0$, hence 1 is the only eigenvalue. $T(z)=z$, $z=(z_1,z_2)\in\mathbb{C}^2$ leads to the system of equation $z_1+z_2=z_1$, $z_2=z_2$. This gives $z_2=0$. Thus the eigenspace consists of all vectors of the form $(z_1, 0)$. This is of dimension 1. Hence the geometric multiplicity of 1 is 1. Since this is the only eigenvalue, the sum of geometric multiplicities is $1\neq 2$. Hence T is NOT diagonalizable.

It is clear that it is not easy to follow the above procedure unless the operator is very simple or/and dimension is very small. The situation is somewhat better in the Inner Product spaces. First of all the entries of the matrix of an operator can be expressed in a nice way using an orthonormal basis.

Let V be an inner product space and $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ be an orthonormal basis of V. Suppose $T: V \to V$ is linear and $[T]_A = [\alpha_{ij}]$. Then $T(\mathbf{a}_j)=\sum_{i=1}^{n}\alpha_{ij}\,\mathbf{a}_i, j=1,\ldots\ldots,n$. Hence

$$\langle T(\mathbf{a}_j),\mathbf{a}_k\rangle=\left\langle\sum_{i=n}^{n}\alpha_{ij}\mathbf{a}_i,\mathbf{a}_k\right\rangle=\sum_{i=1}^{n}\alpha_{ij}\langle\mathbf{a}_i,\mathbf{a}_k\rangle=\alpha_{kj} \text{ because } \langle\mathbf{a}_i,\mathbf{a}_k\rangle=1 \text{ if } i=k$$

$$=0 \text{ if } i\neq k$$

Thus $\alpha_{kj}=\langle T(\mathbf{a}_j),\mathbf{a}_k\rangle$ $k,j=1,\ldots,n$.

Definition 7.20 Let V be an inner product space. A linear operator T: V → V is said to be self-adjoint (or Hermitian) if $\langle T(\mathbf{x}),\mathbf{y}\rangle=\langle\mathbf{x},T(\mathbf{y})\rangle$ $\forall$ $\mathbf{x},\mathbf{y}\in V$.

It is easy to recognize self-adjoint operators from their matrix representation with respect to an orthonormal basis.

Now, let $A=\{\mathbf{a}_1,\ldots,\mathbf{a}_n\}$ be an orthonormal basis and let $M=[T]_A=[\alpha_{ij}]_{n\times n}$. Then as we have seen above, $\alpha_{kj}=\langle T(\mathbf{a}_j),\mathbf{a}_k\rangle$. If T is self-adjoint, then

$$\alpha_{kj}=\langle T(\mathbf{a}_j),\mathbf{a}_k\rangle=\langle\mathbf{a}_j,T(\mathbf{a}_k)\rangle=\overline{\langle T(\mathbf{a}_k),\mathbf{a}_j\rangle}=\bar{\alpha}_{jk} \quad \forall\, j,k=1,\ldots,n.$$

Thus if M^* denotes the conjugate transpose of M, then $M^* = M$. Such matrices are called Hermitian matrices. If V is a real inner product space, then $\alpha_{kj}=\alpha_{jk}$ for all k, j. In other words, M is a real symmetric matrix.

Theorem 7.21 Let V be a finite dimensional inner product space and T be a self-adjoint operator on V. Then there exists an orthonormal basis of V consisting of eigenvectors of T. In other words, T is diagonalizable.

Proof Let Dim (V) = n. Proof is by induction on n. Let n = 1 = Dim (V). Then $\exists\, \mathbf{a}_1\in V$ such that $\|\mathbf{a}_1\| = 1$ and V = span($\{\mathbf{a}_1\}$). $T(\mathbf{a}_1)\in V = \text{span}(\{\mathbf{a}_1\})$. Hence $\exists\lambda_1$ such that $T(\mathbf{a}_1)=\lambda_1\mathbf{a}_1$. Thus $\mathbf{a}_1$ is an eigenvector of T and $\{\mathbf{a}_1\}$ is basis of V. Thus Theorem is true for n = 1.

Next suppose Theorem is true for all vector spaces of dimension $\leq m$. Let Dim (V) = m + 1 and T be a self-adjoint operator on V. Let λ_1 be an eigenvalue of T and $\mathbf{a}_1$ be a corresponding eigenvector. Dividing by $\|\mathbf{a}_1\|$, if necessary we may assume $\|\mathbf{a}_1\| = 1$. Let $W=\{\mathbf{x}\in V:\langle\mathbf{a}_1,\mathbf{x}\rangle=0\}$. Then W is a subspace of V. Since $\mathbf{a}_1\notin W$, Dim(W) < Dim(V) = m + 1. Thus Dim(W) $\leq$m. In fact Dim(W) = m. This can be seen by applying Gram Schmidt Process to any basis of V containing $\mathbf{a}_1$.

Let $\mathbf{x} \in W$. We claim that $T(\mathbf{x}) \in W$. Consider $\langle \mathbf{a}_1, T(\mathbf{x})\rangle = \langle T(\mathbf{a}_1), \mathbf{x}\rangle = \langle \lambda_1 \mathbf{a}_1, \mathbf{x}\rangle = \lambda_1 \langle \mathbf{a}_1, \mathbf{x}\rangle = 0$. Thus $T(\mathbf{x}) \in W$. Consider the map $T|_W : W \to W$. Clearly, $\langle T(\mathbf{x}), \mathbf{y}\rangle = \langle \mathbf{x}, T(\mathbf{y})\rangle \ \forall \mathbf{x}, \mathbf{y} \in W$. Thus $T|_W$ is a self-adjoint operator on W. Hence there is an orthonormal basis $\{\mathbf{a}_2, \ldots, \mathbf{a}_{m+1}\}$ of W consisting of eigenvectors of $T|_W$. Then $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_{m+1}\}$ is an orthonormal set of eigenvector of T. To conclude the proof, we show that A is a basis of V. Since A is already orthonormal, it is enough to show that span (A) = V. Let $\mathbf{z} \in V$. Consider $\mathbf{x} = \mathbf{z} - \langle \mathbf{z}, \mathbf{a}_1\rangle \mathbf{a}_1$. Then $\langle \mathbf{x}, \mathbf{a}_1\rangle = \mathbf{0}$. Hence $\mathbf{x} \in W$. Hence $\mathbf{x} = \sum_{i=2}^{m+1} \langle \mathbf{x}, \mathbf{a}_i\rangle \mathbf{a}_i$. Then, $\mathbf{z} = \langle \mathbf{z}, \mathbf{a}_1\rangle \mathbf{a}_1 + \mathbf{x} = \langle \mathbf{z}, \mathbf{a}_1\rangle \mathbf{a}_1 + \sum_{i=2}^{m+1} \langle \mathbf{x}, \mathbf{a}_i\rangle \mathbf{a}_i \in \text{span}(\{\mathbf{a}_1, \ldots, \mathbf{a}_{m+1}\})$.

Let M be a square matrix of order n x n. The above theorem says that if M is Hermitian, that is, $M^* = M$, then M is diagonalizable.

As a special case, this says that every real symmetric matrix is diagonalizable. More generally, it can be proved that if M commutes with M^*, that is $M^* M = MM^*$, (Such a matrix is called **normal**), then M is diagonalizable.

Diagonalization of a (diagonalizable) matrix M mean finding an invertible matrix P such that $P^{-1}MP$ is a diagonal matrix. In this case, we say that the matrix P diagonalizes M. Let M be such a diagnalizable matrix of order n x n. Then it has n linearly independent eigenvectors, say $\mathbf{x}_1, \ldots, \mathbf{x}_n$ with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus we have $M\mathbf{x}_1 = \lambda_1 \mathbf{x}_1, M\mathbf{x}_2 = \lambda_2 \mathbf{x}_2, \ldots, M\mathbf{x}_n = \lambda_n \mathbf{x}_n$. Let P be the matrix where columns are $\mathbf{x}_1, \ldots, \mathbf{x}_n$, $P = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$. Since, the columns of P are linearly independent, P is invertible. Also

$$MP = M[\mathbf{x}_1, \ldots, \mathbf{x}_n] = [M\mathbf{x}_1, \ldots, M\mathbf{x}_n] = [\lambda_1 \mathbf{x}_1, \ldots, \lambda_n \mathbf{x}_n] = [\mathbf{x}_1, \ldots, \mathbf{x}_n]\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} = PD$$

Where $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Hence $P^{-1}MP = P^{-1}PD = D$. Note that the matrix P is the matrix whose columns are eigenvectors of M and the matrix D is the diagonal matrix where diagonal entries are the eigenvalues of M.

Example 7.22 Let $A=\begin{pmatrix}1 & -3 & 3\\ 0 & -5 & 6\\ 0 & -3 & 4\end{pmatrix}$. Eigenvalues of A are 1 and –2. Eigenspace corresponding to 1 has dimension 2 with a basis consisting of $\mathbf{x}_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{x}_2=\begin{pmatrix}0\\1\\1\end{pmatrix}$. The eigenspace corresponding to –2 is of dimension 1 with a basis consisting of $\mathbf{x}_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$.

Thus A has 3 linearly independent eigenvectors and is hence diagonalizable. Now let $P=\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 2\\ 0 & 1 & 1\end{pmatrix}$. Then $P^{-1}AP=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix}$.

If M is a Hermitian matrix, then all the eigenvalues are real and as proved above, we can find an orthonormal basis $\{\mathbf{x}_1,\dots,\mathbf{x}_n\}$ consisting of eigenvectors of M. In this case, if $P=[\mathbf{x}_1,\dots,\mathbf{x}_n]$, then $P^*P=I$, that is, $P^*=P^{-1}$. Such a matrix P is called a **unitary matrix**. As a special case, if M is real and symmetric, then all the eigenvector are also real. In this case, $P^*=P^T=P^{-1}$. Such a matrix is called an **orthogonal matrix**. Thus,

Every Hermitian matrix can be diagonalized by a unitary matrix and every real symmetric matrix can be diagonalized by an orthogonal matrix.

Example 7.23 Let $M=\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\end{pmatrix}$. Note that M is real symmetric. Eigenvalues of M are –1 and 2. $\mathbf{x}_1=\begin{pmatrix}\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$ is an eigenvector corresponding to –1. $\mathbf{x}_2=\begin{pmatrix}\frac{-1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{pmatrix}$, $\mathbf{x}_3=\begin{pmatrix}\frac{-1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{2}{\sqrt{6}}\end{pmatrix}$ are

eigenvectors corresponding to 2. Note that all the eigenvalues and eigenvectors are real and

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is an orthonormal basis. Let $P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$.

Note $P^T P = I$, thus P is an orthogonal matrix. Also $P^{-1}MP = P^T MP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Proof of theorem 1.22 on page-11.

Suppose that a system of linear equations in n unknowns is consistent and assume that the augmented matrix $[A|b]$ for the system reduces to the matrix $[B|d]$ in echelon form. Thus $[B|d]$ has the form

$$[B|d] = \begin{bmatrix} b_{11} & \cdots & b_{1k_2} & \cdots & b_{1k_r} & \cdots & b_{1n} & d_1 \\ 0 & \cdots & b_{2k_2} & \cdots & b_{2k_r} & \cdots & b_{2n} & d_2 \\ \vdots & & & & & & & \\ & & & & b_{rk_r} & \cdots & b_{rn} & d_r \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & d_{r+1} \\ 0 & \cdots & 0 & \cdots & 0 & & & \\ \vdots & & & & & \cdots & 0 & d_n \\ 0 & \cdots & 0 & \cdots & 0 & & & \end{bmatrix}$$

In the above matrix, each of the entries $b_{11}, b_{2k_2} \cdots b_{rk_r}$ is the first nonzero entry in its row. If $d_{r+1} \neq 0$, then the system represented by the matrix $[B|d]$ is inconsistent, and hence so is the original system. But, we are assuming that the system is consistent, so it must be the case that $d_{r+1} = 0$. Since the matrix $[B|d]$ is in echelon form, it follows that $d_{r+1} = d_{r+2} = \cdots = d_n = 0$. Thus, r is the number of nonzero rows for the matrix $[B|d]$.

Since each of the entries $b_{11}, b_{2k_2}, \cdots b_{rk_r}$ appears in a different column of $[B|d]$ and since these entries appear in the first n columns, it follows that $r \leq n$. Further, the given system is equivalent to the reduced system

$$\left.\begin{aligned} b_{11}x_1 + \cdots + b_{1k_2}x_{k_2} + \cdots + b_{1k_r}x_{k_r} + \cdots + b_{1n}x_n &= d_1 \\ b_{2k_2}x_{k_2} + \cdots + b_{2k_r}x_{k_r} + \cdots + b_{2n}x_n &= d_2 \\ &\vdots \\ b_{rk_r}x_{k_r} + \cdots + b_{rn}x_n &= d_r \end{aligned}\right] (*)$$

In system $(*)$, the first unknown appearing in each equation can be selected as a dependent variable; that is, each of the unknowns $x_1, x_{k_2}, \cdots x_{k_r}$ can be expressed in terms of the remaining unknowns. Thus, in a reduced system of equations such as $(*)$, the number of dependent variables is equal to the number r, of equations. This leaves $n-r$ independent variales.

DEPARTMENT OF MATHEMATICS, I.I.T. MADRAS
MA 2030 Linear Algebra and Numerical Analysis

Problems Set - 1

1. Show that a set of positive real numbers forms a vector space under the operations defined by:
$x + y = xy$ and $\alpha x = x^{\alpha}$.

2. In each of the following parts (a),(b),(c) , a set V is given and some operations are defined. Check whether V is a vector space with these operations. Justify your answers.
(a) $V = \mathbb{R}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
$\alpha(a_1, a_2) = (0, 0)$ if $\alpha = 0$ and $\alpha(a_1, a_2) = (\alpha a_1, a_2/\alpha)$ if $\alpha \neq 0$.
(b) $V = \mathbb{C}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{C}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$
$\alpha(a_1, a_2) = (\alpha a_1, \alpha a_2)$.
(c) $V = \mathbb{R}^2$, for (a_1, a_2), $(b_1, b_2) \in V$ and $\alpha \in \mathbb{R}$, define
$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$
$\alpha(a_1, a_2) = (a_1, 0)$

3. In each of the following a vector space V and a subset W is given. Check whether W is a subspace of V.
(a) $V = \mathbb{R}^2$; $W = \{(x_1, x_2) : x_2 = 2x_1 - 1\}$
(b) $V = \mathbb{R}^3$; $W = \{(x_1, x_2, x_3) : 2x_1 - x_2 - x_3 = 0\}$
(c) $V = C([0, 1])$; $W = \{f \in V : f$ is differentiable$\}$
(d) $V = C([-1, 1])$; $W = \{f \in V : f$ is an odd function$\}$
(e) $V = C([0, 1])$; $W = \{f \in V : f(x) \geq 0$ for all $x\}$
(f) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.
(g) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.
(h) $V = \mathbb{P}_3$; W is the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which a_0, a_1, a_2, a_3 are integers.
(i) $V = \mathbb{P}_3$; W is the set of all polynomials of the form $a_0 + a_1x + a_2x^2$.

4. Prove that the only proper subspaces of $\mathbb{R}^2$ are the straight lines passing through the origin.

5. Let V be a vector space and W, A, B be subsets of V. Prove the following statements.

(a) W is a subspace of V if and only if $span(W) = W$.
(b) If $A \subseteq B$, then $span(A) \subseteq span(B)$.
(c) $span(A \bigcup B) = span(A) + span(B)$
(d) $span(A \bigcap B) \subseteq span(A) \bigcap span(B)$

6. Let W_1 and W_2 be subspaces of a vector space V. Prove that
(a) $W_1 \bigcap W_2$ and $W_1 + W_2$ are subspaces of V.
(b) $W_1 + W_2 = W_1$ if and only if $W_2 \subseteq W_1$
(c) $W_1 \bigcup W_2$ is a subspace if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

7. Give an example of three linearly dependent vectors in $\mathbb{R}^2$ such that none of the three is a scalar multiple of another.

8. In each of the following , a vector space V and a set A of vectors in V is given. Determine whether A is linearly dependent and if it is, express one of the vectors in A as a linear combination of the remaining vectors.
(a) $V = \mathbb{R}^3$, $A = \{(1,0,-1),(2,5,1),(0,-4,3)\}$
(b) $V = \mathbb{R}^3$, $A = \{(1,2,3),(4,5,6),(7,8,9)\}$
(c) $V = \mathbb{R}^3$, $A = \{(1,-3,-2),(-3,1,3),(2,5,7)\}$
(d) $V = \mathbb{P}_3$, $A = \{x^2 - 3x + 5, \quad x^3 + 2x^2 - x + 1, \quad x^3 + 3x^2 - 1\}$
(e) $V = \mathbb{P}_3$, $A = \{-2x^3 - 11x^2 + 3x + 2, \quad x^3 - 2x^2 + 3x + 1, \quad 2x^3 + x^2 + 3x - 2\}$
(f) $V = \mathbb{P}_3$, $A = \{6x^3 - 3x^2 + x + 2, \quad x^3 - x^2 + 2x + 3, \quad 2x^3 + x^2 - 3x + 1\}$

(g) V is the set of all matrices of order 2×2, $A = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$

(i) V is the vector space of all real valued functions defined on $\mathbb{R}$. $A = \{2, \quad sin^2x, \quad cos^2x\}$
(j) V is same as in (i), $A = \{1, \quad sinx, \quad sin2x\}$.
(k) V is same as in (i), $A = \{cos2x, \quad sin^2x, cos^2x\}$.
(l)$V = C([-\pi,\pi]$, $A = \{sinx, \quad sin2x, \dots, sinnx\}$ where n is some natural number.

DEPARTMENT OF MATHEMATICS, I.I.T. MADRAS
MA 2030 Linear Algebra and Numerical Analysis

Problems Set - 2

1. Determine which of the following sets form bases for $\mathbb{P}_2$.
(a)$\{-1-x-2x^2, \quad 2+x-2x^2, \quad 1-2x+4x^2\}$
(b) $\{1+2x+x^2, \quad 3+x^2, \quad x+x^2\}$
(c) $\{1+2x+3x^2, \quad 4-5x+6x^2, \quad 3x+x^2\}$

2. Do the polynomials x^3-2x^2+1, $4x^2-x+3$ and $3x-2$ span $\mathbb{P}_3$? Justify your answer.

3. Suppose that V is a vector space with a basis $\{a, b, c\}$. Show that $\{a+b, b+c, c+a\}$ is also a basis for V.

4. Show that the set of all solutions of the system
$$x_1 - 2x_2 + x_3 = 0 \quad , \quad 2x_1 - 3x_2 + x_3 = 0$$
is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

5. Suppose $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ are subsets of a vector space V such that A is linearly independent and $span(B) = V$. Show that $n \geq m$. Using this, show that any two bases of V have the same number of elements.

6. Find bases and dimensions of the following subspaces of $\mathbb{R}^5$:
(a)$W_1 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 - x_3 - x_4 = 0\}$
(b)$W_2 = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$
(c) $W_3 = span(\{(1,-1,0,2,1), (2,1,-2,0,0), (0,-3,2,4,2), (3,3,-4,-2,-1), (2,4,1,0,1), (5,7,-3,-2,0)\})$

7. For each of the following matrix A, find a basis and dimension of the following subspaces: row space of A, column space of A, null space of $A := \{x : Ax = 0\}$, Range of $A := \{y : Ax = y \quad \text{for some} \quad x\}$.
(a) $A = \begin{bmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 2 & 4 & -1 & 3 & 2 & -1 \end{bmatrix}$

8. Find a basis and dimension of the subspace $span(\{1+x^2, -1+x+x^2, -6+3x, 1+x^2+x^3, x^3\})$ of $\mathbb{P}_3$.

9. Find a basis and dimension of each of the following subspaces of the vector space V of all thrice differentiable functions:
(a) $W_1 = \{x \in V : x'' + x = 0\}$
(b) $W_2 = \{x \in V : x'' - 4x' + 3x = 0\}$
(c)$W_3 = \{x \in V : x''' - 6x'' + 11x' - 6x = 0\}$

10. Show that every linearly independent set in a finite dimensional vector space can be extended to a basis. Using this show that if W is a subspace of V, then $Dim(W) \leq Dim(V)$.

11. Extend the set $\{1 + x^2, 1 - x^2\}$ to a basis of $\mathbb{P}_3$

12. Let W be a proper subspace of $\mathbb{R}^3$. Show that W must be a line passing through the origin or a plane passing through the origin.

13. Let V be a vector space of dimension n. Show that
(a) every subset of V containing more than n vectors is linearly dependent.
(b) no subset of V containing less than n vectors can span V.

14. Let V be a vector space of dimension n and A be a subset of V containing n vectors. Show that
(a) if A is linearly independent, then A is a basis of V,
(b)if $span(A) = V$, then A is a basis of V.

15. If W_1 and W_2 are subspaces of a vector space V and $W_1 + W_2$ is finite dimensional, then show that
$Dim(W_1 + W_1) = Dim(W_1) + Dim(W_2) - Dim(W_1 \bigcap W_2)$
Guess and prove a similar formula for three subspaces.

16. Let V be the vector space of all 2×2 matrices with real entries. Let W_1 be the set of all matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let W_2 be the set of all matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.
(a) Prove that W_1 and W_2 are subspaces of V.
(b) Find dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \bigcap W_2$.

17. Find dimensions of $W_1 + W_2$ and $W_1 \bigcap W_2$ for the subspaces W_1, W_2 in problems 6 and 9.

DEPARTMENT OF MATHEMATICS, I.I.T.MADRAS
MA 2030 Linear Algebra And Numerical Analysis
Problem Set-3

1. For the following $T : \mathbb{R}^2 \to \mathbb{R}^2$, state with reasons whether T is linear.
   (a) $T(a_1, a_2) = (1, a_2)$
   (b) $T(a_1, a_2) = (a_1, a_1^2)$
   (c) $T(a_1, a_2) = (sin a_1, 0)$
   (d) $T(a_1, a_2) = (|a_1|, a_2)$
   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

2. Let V be a vector space with a basis $\{a_1, \ldots, a_n\}$. Let W be a vector space and let $b_1, \ldots, b_n \in W$. Show that there is a unique linear transformation T from V to W such that $T(a_j) = b_j$ for $j = 1, \ldots, n$.

3. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and $T(1,0) = (1,4)$ and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?

4. Prove that there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(2,3)$?

5. Let $\mathbf{V}$ be an n-dimensional vector space over $\mathbb{R}$. Prove that there exist a linear transformation $T : \mathbf{V} \to \mathbb{R}^n$ such that T is bijective.

6. Is there a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

7. Let V, W be vector space over $\mathbb{R}$ and $T : V \to W$ be linear.
   (a) $N(T) = \{x \in V : Tx = 0\}$ is called the null space of T. Show that $N(T)$ is subspace of V.
   (b) $R(T) = \{Tx : x \in V\}$ is called the range of T. Show that $R(T)$ is a subspace of W.
   Dimension of $N(T)$ is called the nullity of T and Dimension of $R(T)$ is called the rank of T.

8. In the following prove that T is a linear transformation and find bases for both $N(T)$ and $R(T)$. Then compute the nullity and rank of T.
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2;\quad T(a_1, a_2) = (a_1 - a_2, 2a_3)$
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3;\quad T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

(c) $T : \mathbb{M}_{2\times 3} \to \mathbb{M}_{2\times 2}$;

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

(d) $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R})$; $T(f(x)) = xf(x) + f'(x)$.

(e) $T : \mathbb{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$;
$T(A) = tr(A)$, where $tr(A) = \sum_{i=1}^{n} a_{ii}$ and $A = (a_{ij})_{n\times n}$.

9. Let V, W be vector space over $\mathbb{R}$ and $T : V \to W$ be linear. Prove that if V is finite dimensional, then $N(T)$ and $R(T)$ are finite dimensional and dimension of V =rank of T+ nullity of T.

10. Let V, W, T be as in the last problem with $Dim(V) = Dim(W)$. Prove that T is $1-1$ if and only if T is onto.

11. Prove that row rank of a matrix A equals its column rank.

12. Let V and W be finite dimensional vector space and $T : V \to W$ be linear.

    (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.

    (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

13. (a) Give an example of distinct linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

    (b) Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$.

    (c) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let B be the standard ordered basis for $\mathbb{R}^2$, $C = \{(1,1,0), (0,1,1), (2,2,3)\}$ and $D = \{(1,2), (2,3)\}$.Compute $[T]_B^C$, $[T]_D^C$.

14. For the following parts,
let $\alpha = \{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$, $\beta = \{1, x, x^2\}$, $\gamma = \{1\}$

    (a) Define $T : \mathbb{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}_{2\times 2}(\mathbb{R})$ by $T(A) = A^t$. Compute $[T]_\alpha$.

    (b) Define $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{M}_{2\times 2}(\mathbb{R})$ by $T(f) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f'(3) \end{pmatrix}$ compute $[T]_\beta^\alpha$.

    (c) Define $T : \mathbb{M}_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = tr(A)$. Compute $[T]_\alpha^\gamma$.

    (d) Define $T : \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}$ by $T(f) = f(2)$. Compute $[T]_\beta^\gamma$.

DEPARTMENT OF MATHEMATICS, I.I.T.MADRAS
MA 2030 Linear Algebra And Numerical Analysis
Problem Set-4

1. Check whether each of the following are inner products on the given vector spaces.

   (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$

   (b) $\langle A, B\rangle = tr(A+B)$ on $\mathbb{M}_{2\times 2}(\mathbb{R})$

   (c) $\langle f, g\rangle = \int_0^1 f'(t)g(t)dt$ on $\mathbb{P}(\mathbb{R})$ where $'$ denotes the differentiation.

   (d) $\langle f, g\rangle = \int_0^{1/2} f(t)g(t)dt$ on $\mathbf{C}[0,1]$

2. Let B be a basis for a finite-dimensional inner product space. Prove that if $\langle x, y\rangle = 0$ for all $x \in B$, then $y = 0$

3. Let V be an inner product space, and suppose that x and y are orthogonal elements of V(that is $\langle x, y\rangle = 0$). Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$, where $\|x\|^2 = \langle x, x\rangle$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

4. Prove the parallelogram law in an inner product space V, that is show that $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$. What does this say about parallelograms in $\mathbb{R}^2$?

5. Let V be an inner product space. Prove that
   $|\langle x, y\rangle| \leq \|x\|\,\|y\|$ for all $x, y \in V$ (This is known as the Cauchy-Schwarz inequality.)

6. Show that an orthogonal set of nonzero vectors in an inner product space is linearly independent.

7. Let V be an inner product space and $\{x_1, x_2, \ldots, x_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k \in \mathbb{R}$.
   Prove that $\left\|\sum_{i=1}^k a_i x_i\right\|^2 = \sum_{i=1}^k |a_i|\,\|x_i\|^2$

8. Suppose $\langle .,.\rangle_1$ and $\langle .,.\rangle_2$ are two inner products on a vector space V. Prove that $\langle .,.\rangle = \langle .,.\rangle_1 + \langle .,.\rangle_2$ is another inner product on V.

9. For vectors x and y in an inner product space V, prove that $x - y$ and $x + y$ are orthogonal to each other if and only if $\|x\| = \|y\|$.

10. In each of the following parts, apply the Gram-Schmidt process to the given subset S of the inner product space V. Then find an orthonormal basis B for V.(exept in 10d)

   (a) $V = \mathbb{R}^3, S = \{(1,0,1),(0,1,1),(1,3,3)\}$
   (b) $V = \mathbb{R}^3, S = \{(1,1,1),(0,1,1),(0,1,3)\}$
   (c) $V = \mathbb{P}_2(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$
   (d) $V = \mathbf{C}([-1,1])$ with the inner product $\langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$
   (e) $V = \mathbb{C}^3$, $S = \{(1, i, 0), (1-i, 2, 4i)\}$

11. For a subset S of an inner product space V. define the orthogonal complement $S^\perp$ of S by
$S^\perp = \{x \in V : \langle x, y\rangle = 0 \quad \text{for all} \quad y \in S\}$

   (a) Show that $S^\perp$ is a subspace.
   (b) Show that $S \subseteq S^{\perp\perp}$.
   (c) Let V be finite dimensional and W be a subspace of V. Show that $Dim(W) + Dim(W^\perp) = Dim(V)$.
   (d) Let W_1 and W_2 be subspaces of V. Prove that $(W_1 + W_2)^\perp = W_1^\perp \bigcap W_2^\perp$ and if V is finite dimensional, then $(W_1 \bigcap W_2)^\perp = W_1^\perp + W_2^\perp$

12. Let $\{a_1, \ldots, a_n\}$ be an orthonormal set in an inner product space V and let $x \in V$. Show that $\sum_j |\langle x, a_j\rangle|^2 \le \|x\|^2$ (This is known as Bessel's inequality . Hint: Define $u = \sum_j \langle x, a_j\rangle a_j$, show that $x - u$ is orthogonal to u and then use Problem 3.)

13. Let V be a vector space , W be an inner product space and $T : V \to W$ be a linear transformation. Define for $x, \quad y \in V$, $\langle x, y\rangle = \langle T(x), T(y)\rangle$. What conditions must T satisfy so that this defines an inner product on V?

14. Let V be an inner product space, and suppose that $T : V \to V$ is linear and that $\|T(x)\| = \|x\|$ for all x. Prove that T is one-to-one.
