REGULARITY OF BINOMIAL EDGE IDEALS OF CERTAIN BLOCK GRAPHS

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Abstract. We obtain an improved lower bound for the regularity of the binomial edge ideals of trees. We prove an upper bound for the regularity of the binomial edge ideals of certain subclass of block-graphs. As a consequence we obtain sharp upper and lower bounds for the regularity of binomial edge ideals of a class of trees called lobsters. We also obtain precise expressions for the regularities of binomial edge ideals of certain classes of trees and block graphs.

1. Introduction

Let G be a simple graph on the vertex set $[n]$. Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables, where K is a field. Then the ideal J_G generated by $\{x_iy_j - x_jy_i \mid (i, j) \text{ is an edge in } G\}$ is called the binomial edge ideal of G. This was introduced by Herzog et al., [9] and independently by Ohtani, [13]. Recently, there have been many results relating the combinatorial data of graphs with the algebraic properties of the corresponding binomial edge ideals, see [1], [2], [4], [11], [15], [16], [17]. In particular, there have been active research connecting algebraic invariants of the binomial edge ideals such as Castelnuovo-Mumford regularity, depth, betti numbers etc., with combinatorial invariants associated with graphs such as length of maximal induced path, number of maximal cliques, matching number. For example, Matsuda and Murai proved that $\ell \leq \operatorname{reg}(S/J_G) \leq n - 1$, where ℓ is the length of the longest induced path in G, [11]. They conjectured that if $\operatorname{reg}(S/J_G) = n - 1$, then G is a path of length n. Ene and Zarojanu proved this conjecture in the case of closed graphs, [5], where a graph is said to be closed if its binomial edge ideal has a quadratic Gröbner basis. Chaudhry et al. proved that if T is a tree whose longest induced path has length ℓ, then $\operatorname{reg}(S/J_T) = \ell$ if and only if T is a caterpillar, [1]. In [15], Madani and Kiani proved that for a closed graph G, $\operatorname{reg}(S/J_G) \leq c(G)$, where $c(G)$ is the number of maximal cliques in G. Later they generalized this result to the case of binomial edge ideal of a pair of a closed graph and a complete graph and proposed that if G is any graph, then $\operatorname{reg}(S/J_G) \leq c(G)$, [16]. In [10], they proved the bound for the class of generalized block graphs.

In this article, we study the regularity of binomial edge ideals of certain classes of trees and block graphs. To begin with, we obtain a lower bound for the regularity of binomial edge ideals of trees in terms of the number of internal vertices, Theorem 3.3. This is an improvement on the lower bound for the regularity given in [11].

We then obtain, in Theorem 3.7, an upper bound for the regularity of the binomial edge ideals of lobsters (see Section 2 for definition). Lobsters are well studied objects in graph theory. They are the most natural generalization of caterpillars. Lobsters occur very often in the graph theory literature, especially in the context of the well-known *graceful tree conjecture*, [12], [6]. We also obtain a precise expression for the regularity of binomial edge ideals of a subclass of lobsters, called pure lobsters, in Theorem 3.9.

In [14], Rauf and Rinaldo studied binomial edge ideals of graphs obtained by gluing two graphs at free vertices. We extend their arguments to prove that the regularity of the binomial edge ideal of a graph obtained by gluing two graphs at free vertices is equal to the sum of the regularities the binomial edge ideals of the individual graphs, Theorem 4.1. As a consequence, we obtain precise expressions for the regularities of several classes of trees and block graphs.

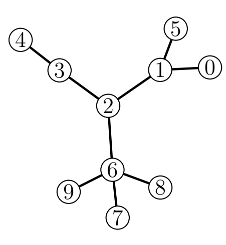
2. Preliminaries

In this section, we set up the basic definitions and notation.

Let T be a tree and $\mathrm{L}(T) = \{v \in V(T) | \deg(v) = 1\}$ be the set of all leaves of T. We say that a tree T is a *caterpillar* if $T \setminus \mathrm{L}(T)$ is either empty or is a simple path. Similarly, a tree T is said to be a *lobster*, if $T \setminus \mathrm{L}(T)$ is a caterpillar, [7]. Observe that every caterpillar is also a lobster. A longest path in a lobster is called a *spine* of the lobster. It is easy to see that given any spine, every edge of a caterpillar is incident to it. With respect to a fixed spine P, the pendant edges incident with P are called *whiskers*. It is easy to see that every non-leaf vertex u not incident on a fixed spine P of a lobster forms the center of a star ($K_{1,m}, m \geq 2$). Each such star is said to be a *limb* with respect to P. More generally, given a vertex v on any simple path P, we can attach a star ($K_{1,m}, m \geq 2$) with center u by identifying exactly one of the leaves of the star with v. Such a star is called *a limb attached to P.*

Note that the limbs and whiskers depend on the spine. Whenever a spine is fixed, we will refer to them simply as limb and whisker.

Example 2.1. *Let G denote the given graph on* 10 *vertices:*



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In this example, G has many longest induced paths. The path induced by the vertices $\{0, 1, 2, 3, 4\}$, $\{0, 1, 2, 6, 9\}$ *are two such (there are more) paths. Let P denote the path induced by the vertices* $\{0, 1, 2, 3, 4\}$. *Then* $(1, 5)$ *is a whisker with respect to P. Also the subgraph induced by the vertices* $\{2, 6, 7, 8, 9\}$ *is a limb with respect to P. If we consider* $\{0, 1, 2, 6, 9\}$ *as spine P, then* $\{(1, 5), (6, 7), (6, 8)\}$ *are whiskers with respect to P and the path induced by* $\{2, 3, 4\}$ *is a limb.*

3. Bounds on the regularity

In this section, we obtain sharp bounds on the regularity of the binomial edge ideals of certain classes of graphs. We first obtain a lower bound for the regularity of binomial edge ideals of trees. Further, we obtain an upper bound on the regularity of lobsters.

We begin by making a general observation about Betti numbers of quotients of graded ideals.

Fact 3.1. *(Folklore)* Let I be a graded ideal of a polynomial ring $S = K[x_1, \dots, x_n]$ and f be a homogeneous element of S of degree d which is a regular element modulo I. Then we have an exact sequence

$$0 \longrightarrow S/I[-d] \xrightarrow{\mu_f} S/I \longrightarrow S/(I, f) \longrightarrow 0,$$

where μ_f denotes the multiplication by f. Correspondingly, for each $m \geq 0$, there is a graded long exact sequence of the Tor functor

$$\begin{aligned} \cdots \quad \longrightarrow \quad & \mathrm{Tor}_i^S(K, S/I)_{m-d} \xrightarrow{\mu_f} \mathrm{Tor}_i^S(K, S/I)_m \longrightarrow \mathrm{Tor}_i^S(K, S/(I, f))_m \\ \longrightarrow \quad & \mathrm{Tor}_{i-1}^S(K, S/I)_{m-d} \xrightarrow{\mu_f} \cdots . \end{aligned}$$

Since the multiplication maps on Tor are zero, for each i, we have short exact sequences

$$0 \longrightarrow \mathrm{Tor}_i^S(K, S/I)_m \longrightarrow \mathrm{Tor}_i^S(K, S/(I, f))_m \longrightarrow \mathrm{Tor}_{i-1}^S(K, S/I)_{m-d} \longrightarrow 0$$

and consequently

$$\beta_{i,m}(S/(I, f)) = \beta_{i,m}(S/I) + \beta_{i-1,m-d}(S/I).$$

As a consequence, we have the following:

Corollary 3.2. *Let G be a simple finite graph on $[n]$ with a free vertex, say n, and $G' = G \cup \{(n, n+1)\}$. Let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ and $S' = S[x_{n+1}, y_{n+1}]$. Then $\beta_{i,i+j}(S'/J_{G'}) = \beta_{i,i+j}(S/J_G) + \beta_{i-1,i+j-2}(S/J_G)$. In particular,* $\mathrm{reg}(S'/J_{G'}) = \mathrm{reg}(S/J_G) + 1$.

Proof. Since n is a free vertex in G, by Lemma 21 of [17], $x_n y_{n+1} - x_{n+1} y_n$ is a regular element on S'/J_G. Note that $\mathrm{Tor}_i^S(K, S/J_G) \cong \mathrm{Tor}_i^{S'}(K, S'/J_G)$. Hence by Fact 3.1, we get $\beta_{i,i+j}(S'/J_{G'}) = \beta_{i,i+j}(S/J_G) + \beta_{i-1,i+j-2}(S/J_G)$. Therefore it follows directly from the betti table that $\mathrm{reg}(S'/J_{G'}) = \mathrm{reg}(S/J_G) + 1$. □

Now we obtain a lower bound on the regularity of the binomial edge ideals of trees in terms of the number of internal vertices. Given a tree T, it is easy to see that one can construct T from the trivial graph by adding vertices v_i to T_{i-1} at step i to get T_i so that v_i is a leaf in in the tree T_i. Any such ordering of vertices is called a *leaf ordering*.

Theorem 3.3. *If G is a tree with m internal vertices, then* $\mathrm{reg}(S/J_G) \geq m + 1$.

Proof. Let $v_1, \dots, v_r$ be a leaf ordering of the vertices of G, and let G_i be the subgraph of G induced by $v_1, \dots, v_i$. Let m_i denote the number of internal vertices of G_i. We argue by induction on i. If $i = 2$, then G_2 is an edge and hence $\mathrm{reg}(S/J_{G_2}) = 1$. Therefore, the result holds. Assume the result for G_i. Then G_{i+1} is obtained by adding a leaf v_{i+1} to some vertex v of G_i. If v is a leaf in G_i, then v is a free vertex in G_i, and hence by Corollary 3.2, $\mathrm{reg}(S/J_{G_{i+1}}) = \mathrm{reg}(S/J_{G_i}) + 1$. Further, v becomes a new internal vertex in G_{i+1}, i.e., $m_{i+1} = m_i + 1$, and therefore the result holds. If v is an internal vertex in G_i, then $m_{i+1} = m_i$ and since G_i is an induced subgraph of G_{i+1}, $\mathrm{reg}(S/J_{G_{i+1}}) \geq \mathrm{reg}(S/J_{G_i}) \geq m_i + 1 = m_{i+1} + 1$ as required. □

We now obtain an improved upper bound on the regularity of binomial edge ideals of certain kind of block graphs.

Theorem 3.4. *Let G be the union $P \cup C_1 \cup \cdots \cup C_r \cup L_1 \cup \cdots \cup L_t$ where P is a simple path on the vertices $\{v_0, \ldots, v_\ell\}$, $C_1, \ldots, C_r$ are maximal cliques and $L_1, \ldots, L_t$ are limbs such that*

(1) $|C_i| \geq 3$ *for all* i*;*
(2) *For all* $A, B \in \{C_1, \ldots, C_r, L_1, \ldots, L_t\}$ *with* $A \neq B$*,*
   (a) $A \cap B \subset P$ *and* $|A \cap B| \leq 1$*;*
   (b) $|A \cap P| = 1$*.*

Then $\operatorname{reg}(S/J_G) \leq \ell + r + t$*.*

Proof. We prove by induction on t. If $t = 0$, then the result follows from Theorem 3.5 of [10]. Let $t \geq 1$. Let v be a leaf on a limb with $N(v) = \{u\}$ for some u. Let G' be the graph obtained by adding necessary edges to G so that $N(u) \cup \{u\}$ is a clique and G'' be the graph induced on $[n] \setminus \{u\}$. Note that G' has $r + 1$ maximal cliques of size at least three and $t - 1$ limbs. By induction, $\operatorname{reg}(S/J_{G'}) \leq \ell + (r + 1) + (t - 1) = \ell + r + t$. Moreover, G'' has r maximal cliques of size at least three and $t - 1$ limbs. Therefore, by induction $\operatorname{reg}(S/J_{G''}) \leq \ell + r + t - 1$. Therefore, by the exact sequence (1) in the proof of Theorem 1.1 of [4], we have $\operatorname{reg}(S/J_G) \leq \ell + r + t$. □

Remark 3.5. *Note that the graph considered in Theorem 3.4 is a block graph, and hence by Theorem 3.5 of* [10]*, one has* $\operatorname{reg}(S/J_G) \leq \ell + r + 2t$*. This shows that Theorem 3.4 is an improvement when* $t \geq 1$*.*

Corollary 3.6. *If G is a lobster with spine P of length ℓ, t limbs and without any whiskers attached to the spine, then* $\operatorname{reg}(S/J_G) \leq \ell + t$*.*

Proof. Apply Theorem 3.4 with $r = 0$. □

Now we prove an upper bound for the regularity of binomial edge ideals of lobsters.

Theorem 3.7. *If G is a lobster with spine P of length ℓ, t limbs and r whiskers attached to P, then* $\operatorname{reg}(S/J_G) \leq \ell + t + r + 2$*.*

Proof. Let G be a graph as given in the statement of the theorem. Let G' be the graph obtained from G by extending each whisker to a limb by appending a new pendant vertex. Then the spine length of G' will either be ℓ or $\ell + 1$ or $\ell + 2$, depending upon whether there is a whisker attached to penultimate vertices of each side of the spine. Note that G' can have at most two new whiskers. Then we have the following cases:

Case I: Suppose G' has no new whisker. Then G' is a graph with no whiskers and $t + r$ limbs. By Theorem 3.4, $\operatorname{reg}(S/J_{G'}) \leq \ell + t + r$. Since G is an induced subgraph of G', $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G'})$.

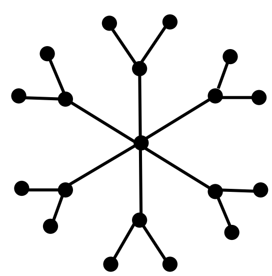
Case II: Suppose G' has only one new whisker. Then extend this whisker to a limb as done above and obtain a new graph G'' with spine length $\ell+1$, $t+r$ limbs and no whiskers.

Then by Theorem 3.4, $\operatorname{reg}(S/J_{G''}) \leq \ell+t+r+1$. Since G is an induced subgraph of G'', we have $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G''})$.

Case III: Suppose G' has two new whiskers. As done earlier, extend these two whiskers to limbs to obtain G''. Then G'' has spine length $\ell+2$, $t+r$ limbs and no whiskers. Therefore by Theorem 3.4, $\operatorname{reg}(S/J_{G''}) \leq \ell+t+r+2$. As G is an induced subgraph of G'', we have $\operatorname{reg}(S/J_G) \leq \operatorname{reg}(S/J_{G''})$.

Hence $\operatorname{reg}(S/J_G) \leq \ell+t+r+2$. □

Example 3.8. *This is an example of a lobster which achieves the upper bound given in Theorem 3.7.*



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This graph G has many different longest induced paths. Fixing any one of them, one can see that G has spine length $\ell = 4$, $t = 4$ limbs and $r = 2$ whiskers attached to the spine. A computation in Macaulay2 shows that the $\operatorname{reg}(S/J_G) = 12 = \ell+t+r+2$.

Below, we obtain a precise expression for the regularity of the binomial edge ideal of a subclass of lobsters. For a lobster, a limb of the form $K_{1,2}$ is called a *pure limb*. A lobster with only pure limbs and no whiskers is called a *pure lobster*. In the following, we show that pure lobsters attain the lower bound given in Theorem 3.3.

Theorem 3.9. *If G is a pure lobster with spine length ℓ and t pure limbs attached to the spine, then* $\operatorname{reg}(S/J_G) = \ell+t$.

Proof. Let G' be the caterpillar graph obtained by deleting all the leaves of G which are not on the spine P of G. Then $\operatorname{reg}(S/J_{G'}) = \ell$, [1]. Let m be a leaf of G and $(m, m+1)$ be an edge in G. Let G'' denote the graph $G' \cup \{(m, m+1)\}$. Then by Corollary 3.2, $\operatorname{reg}(S/J_{G''}) = \ell+1$. Observe that G is obtained by iterating the above procedure of attaching a vertex to a leaf t-times. Since a vertex is attached to a leaf, the regularity increases exactly by one at each step. Therefore, $\operatorname{reg}(S/J_G) = \ell+t$. □

Corollary 3.10. *Let G be a lobster with spine P of length ℓ, t limbs and r whiskers. Then* $\ell+t \leq \operatorname{reg}(S/J_G) \leq \ell+t+r+2$.

Proof. The upper bound is proved in Theorem 3.7. To prove the lower bound, note that G has a subgraph G' with spine P, t pure limbs and without any whiskers as an induced subgraph. By Theorem 3.9, $\operatorname{reg}(S/J_{G'}) = \ell+t$ as required. □

4. Regularity via gluing

In this section, we obtain precise expressions for the regularities of the binomial edge ideals of certain classes of graphs. Let G be a graph and v be a cut vertex in G. Let $G_1, \dots, G_k$ be the components of $G \setminus \{v\}$ and $G_i' = G[V(G_i) \cup \{v\}]$, the subgraph of G induced by $V(G_i) \cup \{v\}$. Then, $G_1', \dots, G_k'$ is called the *split* of G at v and we say that G is obtained by *gluing* $G_1, \dots, G_k$ at v.

Theorem 4.1. *Let G_1 and G_2 be the split of a graph G at v. If v is a free vertex in both G_1 and G_2, then*

$$\operatorname{reg}(S/J_G) = \operatorname{reg}(S/J_{G_1}) + \operatorname{reg}(S/J_{G_2}).$$

Proof. Let G_1 and G_2 be graphs on the vertices $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$ respectively. Assume that n is a free vertex in G_1 and $n+m$ is a free vertex in G_2. Let G be the graph obtained by identifying vertices n and $n+m$ in $G_1 \cup G_2$, i.e., $v = n = n+m$. Let $G' = G_1 \cup G_2$ and $S' = K[x_1, \dots, x_{n+m}, y_1, \dots, y_{n+m}]$. Then it can be easily seen that $S/J_G \cong S'/(J_{G'} + (x_n - x_{n+m}, y_n - y_{n+m}))$. The following claim completes the proof of the theorem.

Claim: $(x_n - x_{n+m}, y_n - y_{n+m})$ is a regular sequence on $S'/J_{G'}$.

The above claim is proved in the proof of Theorem 2.7 in [14]. We present a simpler proof here. Let H be a simple graph on $[n]$. For each subset $T \subset [n]$, let $H_1, \dots, H_c$ denote the connected components of the induced subgraph on $[n] \setminus T$. Let $\tilde{H}_i$ be the complete graph on the vertex set of H_i. Let $P_T(H)$ be the ideal generated by $\left\{\cup_{i \in S}\{x_i, y_i\},\ J_{\tilde{H}_1}, \dots, J_{\tilde{H}_c}\right\}$. Then by Theorem 3.2 of [9], $J_H = \cap_{T \subset [n]} P_T(H)$. Also, $J_H \subset P_T(H)$ is a minimal prime if and only if either $T = \emptyset$ or $T \neq \emptyset$ and $c(T \setminus \{i\}) < c$ for each $i \in T$, [14, Proposition 2.1]. Since n and $n+m$ are free vertices of G', $x_n - x_{n+m} \notin P_T(G')$ for any minimal prime $P_T(G')$ of $J_{G'}$. Therefore $x_n - x_{n+m}$ is a regular element on $S'/J_{G'}$.

Now we show that $y_n - y_{n+m}$ is not contained in any minimal prime of $J_{G'} + (x_n - x_{n+m})$. Let $\mathfrak{p}$ be a minimal prime of $J_{G'} + (x_n - x_{n+m})$. Since $J_{G'} \subset \mathfrak{p}$, there exists a minimal prime $P_T(G')$ such that $J_{G'} \subset P_T(G') \subset \mathfrak{p}$. Hence $J_{G'} + (x_n - x_{n+m}) \subset P_T(G') + (x_n - x_{n+m}) \subset \mathfrak{p}$. Since $P_T(G') + (x_n - x_{n+m})$ is a prime ideal and $\mathfrak{p}$ is a minimal prime, $\mathfrak{p} = P_T(G') + (x_n - x_{n+m})$. Again, since n and $n+m$ are free vertices of G', $y_n - y_{n+m} \notin P_T(G')$ for any minimal prime $P_T(G')$. Thus $y_n - y_{n+m} \notin P_T(G') + (x_n - x_{n+m})$. Therefore $y_n - y_{n+m}$ is a regular element on $S/J_{G'} + (x_n - x_{n+m})$, which completes the proof of the claim. □

As an immediate consequence, we have the following:

Corollary 4.2. *Let $G = G_1 \cup \cdots \cup G_k$ be such that*
1. *for $i \neq j$, if $G_i \cap G_j \neq \emptyset$, then $G_i \cap G_j = \{v_{ij}\}$, for some vertex v_{ij} which is a free vertex in G_i as well as G_j;*
2. *for distinct i, j, k, $G_i \cap G_j \cap G_k = \emptyset$.*

Then $\operatorname{reg} S/J_G = \sum_{i=1}^{k} \operatorname{reg} S/J_{G_i}$.

Recall that for a (generalized) block graph G, $\operatorname{reg} S/J_G \leq c(G)$, [10]. We obtain a subclass of block graphs which attain this bound.

Corollary 4.3. *If G is a block graph such that no vertex is contained in more than two maximal cliques, then* $\operatorname{reg} S/J_G = c(G)$.

Proof. We use induction on $c(G)$. If $c(G) = 1$, then G is a complete graph and hence $\operatorname{reg} S/J_G = 1$. Now assume that $c(G) > 1$. Consider any cut vertex v of G. Let G_1 and G_2 be the split of G at $\{v\}$. Then, $c(G) = c(G_1) + c(G_2)$. Now the result follows from Corollary 4.2 and induction hypothesis. □

See Figure 3 for an example of a graph discussed above. Now we generalize a result by Chaudhry et al., [1], on the regularity of binomial edge ideals of a subclass of trees.

Corollary 4.4. *Let T be a tree and $\{v_1, \ldots, v_k\}$ be a set of vertices of degree 2 in T such that each component in the split of T with respect to $v_1, \ldots, v_k$ is either a caterpillar or a pure lobster. Then* $\operatorname{reg} S/J_T$ *is one more than the number of internal vertices of T.*

Proof. By [1, Theorem 4.1], for any caterpillar T, $\operatorname{reg}(S/J_T) =$ length of the spine, which is one more than the number of internal vertices. Also, it follows from Theorem 3.9 that if T is a pure lobster, then $\operatorname{reg} S/J_T$ is one more than the number of internal vertices. Suppose $D = \{v_1, \ldots, v_k\}$ is a set of degree 2 vertices of G as given in the statement. Since v_i's are leaves and hence free vertices in the respective components of the split, the assertion now follows by induction and applying Corollary 4.2. □

An example of a tree that splits to 3 caterpillars and a pure lobster is given in Figure 4. In [17], Zafar and Zahid considered special classes of graphs called $\mathcal{G}_3$ and $\mathcal{T}_3$ and obtained the regularities of the corresponding binomial edge ideals. We generalize their results:

Corollary 4.5.
1. *Let $P_1, \ldots, P_s$ be paths of lengths $r_1, \ldots, r_s$ respectively. Let G be the graph obtained by identifying a leaf of P_i with the i-th vertex of the complete graph K_s. Then* $\operatorname{reg}(S/J_G) = 1 + \sum_{i=1}^{s} r_i$.
2. *Let $P_1, \ldots, P_k$ be paths of lengths $r_1, \ldots, r_k$ respectively. Let G be the graph obtained by identifying a leaf of P_i with the i-th leaf of the star $K_{1,k}$. Then,* $\operatorname{reg}(S/J_G) = 2 + \sum_{i=1}^{k} r_i$.

Proof. The assertions follow from observing that (1) is a special case of Corollary 4.3 and (2) is a special case of Corollary 4.4. □

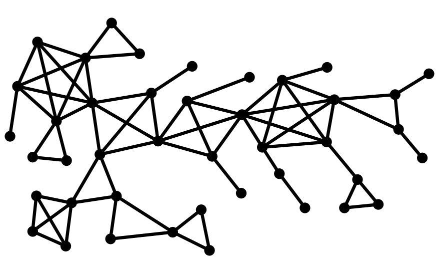

FIGURE 3. A block graph G with $\operatorname{reg}(S/J_G) = c(G) = 22$

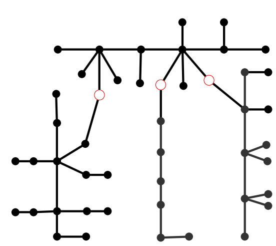

FIGURE 4. A tree T with $\operatorname{reg}(S/J_T) = \#$ intern. vert. $+ 1 = 26$

We note that the graph G depicted in Figure 5, which we call *Jewel*, is the smallest tree for which $\operatorname{reg}(S/J_G) > \#\{\text{internal vertices of } G\} + 1$. In fact, we can make the gap arbitrarily large by attaching edge disjoint copies of Jewel to leaves of any arbitrary tree. For example, Figure 2, which is two copies of the jewel superimposed together, has regularity 12, much larger than the number of internal vertices which is 7.

Observation 4.6. Suppose G is a tree containing *Jewel* as a subgraph. Then by the arguments used in the proof of Theorem 3.3, one observes that

$$\operatorname{reg} S/J_G \geq \#\{\text{internal vertices of } G\} + 2.$$

Our examples computed using SAGE and Macaulay 2 suggests that the converse is also true. Hence we conjecture:

Conjecture 4.7. *A tree T contains Jewel as a subgraph if and only if* $\operatorname{reg}(S/J_T) \geq \#\{\textit{internal vertices of } T\} + 2$.

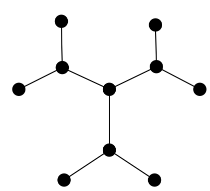

FIGURE 5. G : Jewel with regularity 6.

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